# Moment graphs
# &
# Localisation of Wakimoto flags

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Representation Theory in Venice

**Ultimate goal**: realisation of a combinatorial category encoding information about rep theory category ("modules with Wakimoto flags") for an affine LA

truncation

$${}^{S}\mathcal{O}^{\mathrm{WF}, \leq \xi}_{\chi} \longrightarrow \mathrm{Sh}\left(\mathcal{G}^{\frac{\infty}{2}, \leq \xi}_{\chi}\right)^{\mathrm{VF}}$$

deformation

integral neg. level

analogue of Soergel functor

# Classical story (Soergel Funktor $\mathbb{V}$)

$$\mathcal{O}_0 \xrightarrow{\ \mathbb{V}\ } H^\bullet(G^\vee/B^\vee)\text{-mods}$$

fully faithful on proj's

$$\cong \quad \mathrm{End}(P)\text{-mods}$$

"big projective"

principal block of category $\mathcal{O}$ for simple f.dml LA $\mathfrak{g}$

$$M \longmapsto \mathrm{Hom}(P, M)$$

$$L(x\cdot 0) \twoheadleftarrow \Delta(x\cdot 0) \twoheadleftarrow P(x\cdot 0)$$

Via $\mathbb{V}$, get a handle ~~on~~ multiplicities

$$(P(y\cdot 0) : \Delta(xy\cdot 0)) \quad \left(= [\Delta(y\cdot 0) : L(xy\cdot 0)]\right)$$

# Moment graph version (Fiebig)

(more detailed description to come ...)

$$\mathcal{O}_0 \longrightarrow \mathcal{S}h(\mathcal{G})$$

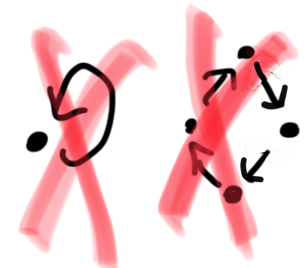
category of sheaves on the Bruhat graph of $\mathfrak{g}$

Key fact: $H^{\vee}_{T}(G^{\vee}/B^{\vee})$ can be realised as the "space of global sections" of an appropriate sheaf on $\mathcal{G}$.

# Moment graphs

Let $Y \cong \mathbb{Z}^r$

**Defn** A moment graph $\mathcal{G}$ on $Y$ is given by:

- $\mathcal{V}$ vertices
- $\mathcal{E}$ oriented edges (NO oriented cycles :  )
- $l : \mathcal{E} \to Y \setminus \{0\}$ label

**Exple** $\mathfrak{g}$ simple fin. dml LA $Y = \mathbb{Z}\Phi$

$\rightsquigarrow$ $\mathcal{V} \leftrightarrow W$ = Weyl gp $= \langle s_\alpha \mid \alpha \in \Phi^+ \rangle$

$\mathcal{E} \leftrightarrow x \xrightarrow{\alpha^\vee} s_\alpha x$ , $x \le s_\alpha x$

Bruhat graph

$\mathfrak{sl}_2$ $\quad W = \mathfrak{S}_2 = \langle s_\alpha \rangle$

$$e \xrightarrow{\alpha^\vee} s_\alpha$$

## Sheaves on $\mathcal{G}$

Fix $\mathcal{G}$ moment graph on $Y$

$S := \mathrm{Sym}(Y \otimes_{\mathbb{Z}} \mathbb{C})$ (graded with $Y \subset S_{(2)}$)

**Defn** A sheaf $\mathcal{F}$ on $\mathcal{G}$ is given by

- $\mathcal{F}^{x} \in S\text{-mod}^{\mathbb{Z}}$ $\forall x \in \mathcal{V}$
- $\mathcal{F}^{x\xrightarrow{\gamma}y} \in S\text{-mod}^{\mathbb{Z}}$ $\forall x \xrightarrow{\gamma} y \in \mathcal{E}$ s.t. $\gamma \cdot \mathcal{F}^{x\xrightarrow{\gamma}y} = (0)$
- $$\begin{tikzcd} \mathcal{F}^{x} \arrow[dr, "\rho_{x,x\to y}"'] & & \mathcal{F}^{y} \arrow[dl, "\rho_{y,x\to y}"] \\ & \mathcal{F}^{x\xrightarrow{\gamma}y} & \end{tikzcd}$$ degree 0 maps

**Exple** $\underset{e}{\bullet} \xrightarrow{\alpha^\vee} \underset{s_\alpha}{\bullet}$ $\quad S = \mathbb{C}[\alpha^\vee]$ $\quad \left( \overset{S}{\bullet} \longrightarrow \overset{S}{\bullet} \right)$, $\quad S \twoheadrightarrow S/\alpha^\vee S \twoheadleftarrow S$

**Rmk** Can always define structure sheaf with $\mathcal{Z}^{x} = S$, $\mathcal{Z}^{x\xrightarrow{\gamma}y} = S/\gamma S$ + can. quot. maps.

# Spaces of sections

**Defn** Let $\mathcal{F}$ be a sheaf on $\mathcal{G}$. Let $I \subset \mathcal{V}$.

$$\Gamma(I, \mathcal{F}) = \Big\{ (f_x) \in \bigoplus_{x \in I} \mathcal{F}^x \;\Big|\; \begin{array}{l} \rho_{x, x\to y}(f_x) = \rho_{y, x \to y}(f_y) \\ \forall x, y \in I \quad x \to y \in E \end{array} \Big\}$$

"space of sections"

**Notation** $\Gamma(\mathcal{F}) = \Gamma(\mathcal{V}, \mathcal{F})$ global sections

**Expl** $\Gamma\left( \begin{tikzcd} \overset{S}{\bullet} \arrow[r] \arrow[loop below] & \overset{S}{\bullet} \arrow[loop right] \\ & S/\alpha^\vee S \end{tikzcd} \right) = \{ (f, g) \in \mathbb{C}[\alpha^\vee]^{\oplus 2} \mid f \equiv g \bmod \alpha^\vee \}$

$= S(1,1) \oplus S(0, \alpha^\vee) \quad \left( \underset{S\text{-mod}}{\simeq} H_T^*(\mathbb{P}^1) \right)$

**Rmk.** $\mathcal{G}$ coming from $\mathfrak{g}$ fin. dml LA

$$Z := \Gamma(\mathcal{Z}) \simeq H_T^*(G^\vee / B^\vee)$$

<u>Rmk</u> $\Gamma(\mathcal{F})$ has always a structure of $\mathcal{Z}$-module.

Fiebig: $$\mathcal{Z}\text{-mods} \underset{\mathcal{L}}{\overset{\Gamma}{\leftrightarrows}} \mathrm{Sh}(\mathcal{G})$$

"localisation functor" (left adjoint to $\Gamma$)

If $\mathcal{G}$ is associated with $\mathfrak{g}$, then

$$\begin{tikzcd}
{}^{s}\mathcal{O}_0 \arrow[r] & \mathrm{Sh}(\mathcal{G}) \\
P^{s}(y\cdot 0) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \mathcal{B}(y)
\end{tikzcd}$$

"Braden-MacPherson sheaf"

$$\boxed{\mathrm{rk}\ \mathcal{B}(y)^{x} = [P(y\cdot 0) : \Delta(x\cdot 0)]}$$

# Verma flags (modules vs sheaves on MGs)

**Defn** A $\mathcal{U}(\mathfrak{g})$-mod $M$ admits a Verma flag (VF) if
$\exists$ finite filtration $0 \subset M_1 \subset \dots \subset M_n = M$
s.t. $\forall i = 1, \dots, n$ $\exists \lambda_i$ with $M_i / M_{i-1} \simeq \Delta(\lambda_i)$

**Rmk** orientation of edges $\Rightarrow$ poset structure on $\mathcal{V}$ $\left(\begin{smallmatrix} x \to y \\ \Downarrow \\ x \leq y \end{smallmatrix}\right)$

**Defn** A sheaf $\mathcal{F}$ on $\mathcal{G}$ admits a VF if

- $\Gamma(\mathcal{F}) \twoheadrightarrow \Gamma(\{\geq x\}, \mathcal{F})$ $\forall x \in \mathcal{V}$
- $\mathrm{Ker}\left(\Gamma(\{\geq x\}, \mathcal{F}) \to \Gamma(\{> x\}, \mathcal{F})\right)$
  is free fin gen'd $S$-mod.

Thm (Fiebig)

There is an equivalence of categories:

$${}^{s}\mathcal{O}_0^{VF} \xrightarrow{\sim} Sh(\mathcal{G}_0)^{VF,gl}$$

Bruhat graph

$\mathcal{F} \cong \mathcal{L}\circ\Gamma(\mathcal{F})$

Rmk Previous result extends to the symmetrisable KM setting

$${}^{s}\mathcal{O}_{\chi}^{VF,\leq\xi} \xrightarrow{\sim} Sh(\mathcal{G}_{\chi}^{\leq\xi})^{VF,gl}$$

non-critical

$(2(\lambda+\rho,\beta)\in\mathbb{Z}(\beta,\beta)\ \forall\beta\in\Phi^{im})$

singular (truncated)

Bruhat graph: $\mathcal{V}\leftrightarrow W\cdot\chi\cap\{\leq\mu\}$

# Affine Category $\mathcal{O}$

$\mathring{\mathfrak{g}}$ = finite dml simple LA / $\mathbb{C}$ $\supset \mathring{\mathfrak{b}} \supset \mathring{\mathfrak{h}}$

Borel Cartan

$$\rightsquigarrow \mathfrak{g} \simeq \mathring{\mathfrak{g}} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

central elt derivation op.

$$\cup$$

$$\mathfrak{b} = \mathring{\mathfrak{b}} \oplus \mathring{\mathfrak{g}} \otimes t\mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

$$\cup$$

$$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D$$

$= \{ m \in M \mid h.m = \lambda(h) m \ \forall h \in \mathfrak{h} \}$

$$\mathcal{O} = \left\{ M \in \mathcal{U}(\mathfrak{g})\text{-mod} \;\middle|\; \begin{array}{l} \bullet\ M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \\ \bullet\ \dim(\mathcal{U}(\mathfrak{b}).m) < \infty \ \forall m \in M \end{array} \right\}$$

# Affine / extended Weyl group

$$\mathfrak{h}^* \simeq \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$$

$\Big( \Lambda_0(K) = \delta(D) = 1$

$\Lambda_0(\mathring{\mathfrak{h}} \oplus \mathbb{C}D) = \delta(\mathring{\mathfrak{h}} \oplus \mathbb{C}K) = 0 \Big)$

$$W = \text{Weyl group of } \mathfrak{g} = \mathring{W} \ltimes \mathbb{Z}\mathring{\Phi}^\vee \subseteq \mathring{W} \ltimes X^\vee =: W^{ext}$$

Weyl gp of $\mathring{\mathfrak{g}}$

coroot lattice of $\mathring{\mathfrak{g}}$

coweight lattice of $\mathring{\mathfrak{g}}$

$(= \{\lambda \in \mathring{\mathfrak{h}} \mid \alpha(\lambda) \in \mathbb{Z} \ \forall \alpha \in \mathring{\Phi}\})$

$\curvearrowright \mathfrak{h}^*$

extends the action of $\mathring{W} \curvearrowright \mathring{\mathfrak{h}}^*$

$w(\Lambda_0) = \Lambda_0 \quad w(\delta) = \delta \quad \forall w \in \mathring{W}$

$t_\gamma(\lambda) = \lambda + \lambda(K)\gamma - \left(\lambda(\gamma) + \frac{(\gamma|\gamma)}{2}\lambda(K)\right)\delta \qquad \lambda \in \mathfrak{h}^*$

$\gamma \in \mathbb{Z}\mathring{\Phi}^\vee$ or $X^\vee$

<u>dot-action</u> : $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\rho(\alpha_i^\vee) = 1 \ \forall \alpha_i$ simple (affine) root

"Weyl vector"

# Twisting functors (Arkhipov)

Let $x \in W^{ext}$. Want to define $T_x : \mathcal{O} \to \mathcal{O}$

- $\varphi_x : \mathfrak{g} \to \mathfrak{g}$ Tits lift of $x$
- $\mathfrak{n}_x := \mathfrak{n}_- \cap x^{-1}(\mathfrak{n}_+)$ where $\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Phi^{\pm}} \mathfrak{g}_\alpha$

$N_x = \mathcal{U}(\mathfrak{n}_x)$

$\rightsquigarrow R_x := \mathcal{U}(\mathfrak{g}) \otimes_{N_x} N_x^*$ ← has a structure of $\mathcal{U}(\mathfrak{g})$-bimod

$\rightsquigarrow T_x(M) = \varphi_x(R_x \otimes_{\mathcal{U}(\mathfrak{g})} M)$

Rmk If $sx > x \Rightarrow T_{sx} = T_s \circ T_x$.
[simple refln

# Twisted Vermas & Wakimotos (Arkhipov)

Let $x \in W^{ext}$.

$$\Delta^{x}(\lambda) := T_x(\Delta(x^{-1} \cdot \lambda))$$ Twisted Verma

$\lambda$ non-critical $\quad \mathring{\rho}^{\vee} := \frac{1}{2} \sum_{\alpha \in \mathring{\Phi}^+} \alpha^{\vee} \in X^{\vee}$

$\rightsquigarrow \gamma_n := t_{-n\mathring{\rho}^{\vee}} \in W^{ext} \quad \forall n \in \mathbb{Z}_{\geq 0}$

There are maps $f_n : \Delta^{t_{\gamma_n}}(\lambda) \to \Delta^{t_{\gamma_{n+1}}}(\lambda)$

$\rightsquigarrow$ get inductive system of $\mathcal{U}(\mathfrak{g})$-mods

$$W(\lambda) = \varinjlim_{n} \Delta^{t_{\gamma_n}}(\lambda)$$ Wakimoto module

# $\mathcal{G}_x^{\frac{\infty}{2}}$ enters into the picture

$\mathcal{G}_x^{\frac{\infty}{2}}$ MG on $\mathbb{Z}\Phi^\vee$

regular antidominant

affine coroot lattice

$\nu \longleftrightarrow W \cdot x$

$\mathcal{E} \longleftrightarrow x \cdot x \xrightarrow{\alpha^\vee} s_{\alpha}x \cdot x$

$t_{n\rho^\vee}x < t_{n\rho^\vee}s_\alpha x$ for $n \gg 0$

$\frac{\infty}{2}$ Bruhat order (Lusztig's generic order)

**Thm** (Arakawa-L)

$$Z({}^{S}\mathcal{O}_x^{WF}) \simeq Z(\mathcal{G}_x^{\frac{\infty}{2}})$$

categorical centre

Wakimoto flag

## $\frac{\infty}{2}$ - analogue of $\mathbb{V}$ (Arakawa-L)

Recall: $\chi$ antidominant

In an appropriate truncated version $\mathcal{O}_\chi^{\leq \xi}$

$\exists\ P(\lambda) \twoheadrightarrow W(\lambda)$ projective cover $\in \mathcal{O}_\chi^{\leq \xi, \mathrm{WF}}$

<u>Thm</u> There is an equivalence of categories

$$ {}^{s}\mathcal{O}_\chi^{\leq \xi, \mathrm{WF}} \longrightarrow \mathrm{Sh}\left(\mathcal{G}^{\frac{\infty}{2}, \leq \xi}_{J_\chi}\right)^{\mathrm{VF}, \mathrm{gl}} $$

$$ \overset{\Psi}{W}(x \cdot \chi) \longmapsto \left(\mathbb{V}(x)^y = \begin{cases} 0 & y \neq x \\ S & y = x \end{cases}, \ \mathbb{V}(x)^{y \to z} = (0)\right) $$

$$ P(y \cdot \chi) \longmapsto \text{BMP-sheaf } \mathcal{B}^{\frac{\infty}{2}}(y) $$

# Multiplicities

<u>Thm</u> (Arakawa-L)

$$\left( P(y\cdot x) : W(x\cdot x) \right) = \mathrm{rk}\, \mathcal{B}^{\frac{\infty}{2}}(y)^{x}$$

<u>Thm</u> (L15)

$$\underline{\mathrm{rk}}\, \mathcal{B}^{\frac{\infty}{2}}(y)^{x} = m_{x,y}$$

graded rank $\in \mathbb{Z}_{\geq 0}[v^2]$

Lusztig generic poly (= limit of KL-polynomial)

<u>Cor</u> $\left( P(y\cdot x) : W(x\cdot x) \right) = m_{x,y}(1)$.

## Rmks

- hypothesis of regularity for $\chi$ NOT needed:
  there is a singular version of $\mathcal{G}_\chi^{\frac{\infty}{2}}$
  whose set of vertices is $W \cdot \chi$
  and everything works the same

- hypothesis of integrality for $\chi$ NOT needed:
  enough to replace $W$ by the integral
  Weyl group

- positive level: should be obtained by
  applying Soergel tilting functor

Thank you!

Happy birthday to Corrado!