ABELIAN VARIETIES AS AUTOMORPHISM GROUPS OF PROJECTIVE VARIETIES

DAVIDE LOMBARDO & ANDREA MAFFEI

Let X be a proj var.

$Aut(X)$ it has a sch. structure.

- $Aut(X) \times X \longrightarrow X$ is regular
- $Aut(X)^0$ is a variety.

PROBLEM Given an algebraic group G find X (smooth normal) s.t. $Aut(X) \simeq G$.

- DEMAZURE 1977 $Aut(G/P)$
  If G is a group of adj type then $\exists X : Aut(X) = G$.
- $G = SL(n)$ the problem is open.
- BRION 2019 If G is connected then $\exists X$ such that $Aut(X)^0 = G$
- LEISIEUTRE 2018 X s.t. $Aut(X)$ is discrete but it is not f.g.

  DINH - OGUISO 2019

$G = A$ is an abelian variety.

THEOREM $\exists\, X$ smooth and projective s.t. $\mathrm{Aut}(X) = A$ if and only if $\mathrm{Aut}_{gr}(A)$ is finite.

- smooth or normal
- FLORENCE in positive char.
- BRION $\Rightarrow$

$$1 \to G_{aff} \to G \to A \to 1$$

$\mathrm{Aut}(X) = G.$ $\quad \{\varphi \in \mathrm{Aut}_{gr}(G) : \varphi|_{G_{aff}} = id\}$ is finite.

$\boxed{\Rightarrow}$ If A acts faithfully on X then $\exists\, n>0$ and $X' \subset X$ stable under the action of $A[n]$ s.t.

$$X \simeq \frac{A \times X'}{A[n]}$$

Suppose $\# \mathrm{Aut}_{gr}(A) = +\infty$ then $\exists\, \varphi \neq id$ $\varphi|_{A[n]} = id$.

$$\psi[a, x] = [\varphi(a), x]$$

ψ well defined $\quad \psi \notin A$

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$\boxed{\Leftarrow}$ A an elliptic curve.

$\mathbb{Z}/2 \quad \mathbb{Z}/4 \quad \mathbb{Z}/6$

$X = A \qquad \mathrm{Aut}(A) = A \rtimes \mathrm{Aut}_{gr}(A)$

RMK. • $\varphi \in \mathrm{Aut}(A) \quad \varphi \neq id.$

$$A^\varphi = \emptyset \iff \varphi \in A.$$

• $\varphi \notin A \quad \#(A^\varphi) = 1, 2, 4.$

$\pi : X \longrightarrow Y$ A-principal bundle

• π is not trivial

• $\pi_1(Y) = \mathbb{Z}/p \quad p > 4$

$\Rightarrow \mathrm{Alb}(Y)$ is trivial

$\Rightarrow \psi : Y \longrightarrow A$ is constant.

FACT

If $\varphi : X \to X \quad \pi \circ \varphi = \pi$ then $\varphi \in A$.

• $X^\varphi \neq \emptyset \quad \pi^\varphi : X^\varphi \longrightarrow Y$ is a covering with fibers of cr ≤ 4,

$\Rightarrow \pi^\varphi$ is trivial $\Rightarrow \pi$ is trivial.

• $X^\varphi = \emptyset \quad \varphi|_{X_y} = a_y \quad a : Y \longrightarrow A$

$\varphi = a \in A$

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$Y = S / \mathbb{Z}/p$

CONSTRUCTION OF S $\quad \lambda \in \mathbb{C} \quad p \geq 7$

$$S_\lambda \subset \mathbb{P}^3 \qquad 0 = x_1^p + x_2^p + x_3^p + x_4^p + \lambda\left(x_1^2 x_2^{p-4} x_3^2 + x_1^4 x_2^{p-6} x_4^2\right)$$

- λ generic $\quad S_\lambda$ is smooth

- $\pi_1(S_\lambda) = 1$
- $K_S = \mathcal{O}(p-4)|_S$ is ample

- $\mathbb{Z}/p$ acts on S

$$i \cdot (x_1, x_2, x_3, x_4) = (x_1, \zeta^i x_2, \zeta^{2i} x_3, \zeta^{3i} x_4)$$

THEOREM For λ generic. $\mathrm{Aut}(S_\lambda) = \mathbb{Z}/p$

$S = S_\lambda$.

$$Y = S/\mathbb{Z}/p$$

- Y is smooth and proj $\quad \mathcal{O}(1)_S/p$ is ample
- $K_Y = K_S/p$ is ample
- $\pi_1(Y) = \mathbb{Z}/p$
- $\mathrm{Aut}(Y)$ is trivial

CONSTRUCTION OF X $\qquad q \in A \quad pQ = 0 \quad Q \neq 0.$

$$X = \frac{A \times S}{p} \qquad i(a,s) = (a + iQ, \, i \cdot s)$$

$$\downarrow \pi$$

$$Y = S/p = Y$$

π is an A-principal bundle.

is not trivial.

CLAIM. If $\varphi \in Aut(X)$ then $\pi \circ \varphi = \pi$.

Let $C \subset Y$ a curve $X_C = \pi^{-1}(C)$

$$\varphi_C : X_C \xrightarrow{\varphi} X \xrightarrow{\pi} Y$$

$$\boxed{\varphi_C : X_C \longrightarrow Y}$$

$\dim X_C = \dim Y = 2$.

$\dim(\mathrm{Im}\, \varphi_C) \neq 0$

$\dim(\mathrm{Im}\, \varphi_C) \neq 2$

K_Y is ample

φ_C is generically finite.

$$R + \varphi_C^* K_Y = K_{X_C}$$

X_C is of general type

impossible.

$X_C \downarrow C$ fibers are A.

$\mathrm{Im}\, \varphi_C$ is a curve.

$p \in Y$ $\quad p \in C \cap D$ $\quad$ Im

$$\varphi(X_p) = X_{\psi(p)}$$

$\psi : Y \to Y$ an automorph.

$$\varphi(X_p) = X_p.$$

LEMMA Demailly Hwang Peternell

$$A \longrightarrow X$$

$\exists\, B \subset A$

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & X \\ & A/B \arrow[ur] & \end{tikzcd}$$

$\exists\, B$ s.t.

$$\begin{tikzcd} X_c \arrow[rr] \arrow[dr] & & Y \\ & X_c/B \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} X_c \arrow[d] \\ C \end{tikzcd}$$

If A is simple and $|\mathrm{Aut}_{gr}(A)| < +\infty$

Then for $\varphi \in$ ld $|A^{\varphi}|$ is bounded.

If $|\mathrm{Aut}_{gr}(A)| < +\infty$

$$A \underset{\text{isog.}}{\simeq} A_1 \times \dots \times A_n \qquad \mathrm{Aut}(A_i) < +\infty$$

$$i \neq j \quad \mathrm{Hom}(A_i, A_j) = 0.$$

$$\frac{A \times T}{\mathbb{Z}_p}$$