

ABELIAN VARIETIES AS AUTOMORPHISM GROUPS OF PROJECTIVE VARIETIES

DAVIDE LONGARDO & ANDREA RUFFI

Let X be a proj var.

$\text{Aut}(X)$ it has a sub. struct.

- $\text{Aut}(X) \times X \rightarrow X$ is regular
- $\text{Aut}(X)^\circ$ is a variety.

PROBLEM Given an algebraic group G find X
s.t. $\text{Aut}(X) \cong G$. smooth
normal

- DEMAZURE 1977 $\text{Aut}(G/P)$
If G is a group of adj type then $\exists X: \text{Aut}(X) = G$.
- $G = \text{SL}(n)$ the problem is open.
- BRION 2014 If G is connected then $\exists X$
such that $\text{Aut}(X)^\circ = G$
- LEISIEUTRE 2018 X s.t. $\text{Aut}(X)$ is discrete
but it is not f.g.
DINH-DIGUISSO 2019

$G = A$ is an abelian variety.

THEOREM $\exists X$ smooth and projective s.t. $\text{Aut}(X) = A$
 if and only if $\text{Aut}_{\text{gr}}(A)$ is finite.

- smooth or normal
- FLORENCE in positive char.
- BRION \Rightarrow

$$1 \rightarrow G_{\text{eff}} \rightarrow G \rightarrow A \rightarrow 1$$

$$\text{Aut}(X) = G. \quad \left\{ \varphi \in \text{Aut}_{\text{gr}}(G) : \varphi|_{G_{\text{eff}}} = \text{id} \right\} \text{ is finite.}$$

\Rightarrow If A acts faithfully on X then $\exists n > 0$ and $X' \subset X$ stable under the action of $A[n]$ s.t.

$$X \cong \frac{A \times X'}{A[n]}$$

Suppose $\# \text{Aut}_{\text{gr}}(A) = +\infty$ then $\exists \varphi \neq \text{id} \quad \varphi|_{A[n]} = \text{id}.$

$$\varphi[a, x] = [\varphi(a), x]$$

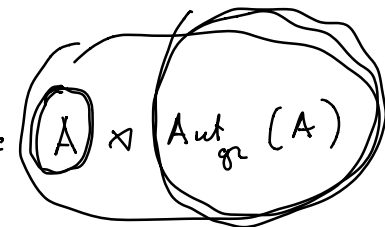
φ well define $\varphi \notin A$

#

\Leftarrow A on elliptic curve.

$$\mathbb{Z}/2 \quad \mathbb{Z}/4 \quad \mathbb{Z}/6$$

$$X = A$$

$$\text{Aut}(A) = \underbrace{A}_{\text{circle}} \rtimes \underbrace{\text{Aut}_{\text{gr}}(A)}_{\text{circle}}$$


RMK. $\varphi \in \text{Aut}(A) \quad \varphi \neq \text{id.}$

$$A^\varphi = \emptyset \quad (\Rightarrow) \quad \varphi \in A.$$

• $\varphi \notin A \quad \#(A^\varphi) = 1, 2, \textcircled{4}$

$\pi: X \longrightarrow Y \quad A\text{-principal bundle}$

• π is not trivial

• $\pi_1(Y) = \textcircled{\mathbb{Z}/p} \quad p > 4$

$\Rightarrow \text{Alb}(Y)$ is trivial

$\Rightarrow \varphi: Y \rightarrow A$ is constant.

FACT

If $\varphi: X \rightarrow X \quad \pi \circ \varphi = \pi$ then $\varphi \in A.$

• $X^\varphi \neq \emptyset \quad \pi^\varphi: X^\varphi \rightarrow Y$ is a covering with fibers of $n \leq 4$.
 $\Rightarrow \pi^\varphi$ is trivial $\Rightarrow \pi$ is trivial.

• $X^\varphi = \emptyset \quad \varphi|_{X_y} = e_y \quad e: Y \rightarrow A$
 $\varphi \neq e \in A \quad \neq$

$$Y = \textcircled{S} / \mathbb{Z}/p$$

CONSTRUCTION OF S $\lambda \in \mathbb{C} \quad p \geq 7$

$$S_\lambda \subset \mathbb{CP}^3 \quad 0 = x_1^p + x_2^p + x_3^p + x_4^p + \lambda \left(x_1^2 x_2^{p-4} x_3^2 + x_1^4 x_2^{p-6} x_4^2 \right)$$

- λ generic S_λ is smooth
- $\pi_1(S_\lambda) = 1$
- $K_{S_\lambda} = \mathcal{O}(p-4)|_{S_\lambda}$ is ample

• \mathbb{Z}/p acts on S

$$i \cdot (x_1, x_2, x_3, x_4) = (x_1, \zeta^i x_2, \zeta^{2i} x_3, \zeta^{3i} x_4)$$

THEOREM For λ generic. $\text{Aut}(S_\lambda) = \mathbb{Z}/p$

$$S = S_\lambda.$$

$$Y = S / \mathbb{Z}/p$$

- Y is smooth and proj $\mathcal{O}(1)_S/p$ is ample
- $K_Y = K_S/p$ is ample
- $\pi_1(Y) = \mathbb{Z}/p$
- $\text{Aut}(Y)$ is trivial

CONSTRUCTION OF X

$$q \in A \quad pq = 0 \quad q \neq 0.$$

$$X = \frac{A \times S}{\pi} \quad i(q, s) = (a + iq, i \cdot s)$$

$$\downarrow \pi$$

$$Y = S/p = Y$$

π is an A -principal bundle.

is not trivial.

CLAIM. If $\varphi \in \text{Aut}(X)$ then $\pi \circ \varphi = \pi$.

Let $C \subset Y$ a curve $X_C = \pi^{-1}(C)$

$$\varphi_C: X_C \xrightarrow{(\hat{\varphi})} X \xrightarrow{(\pi)} Y$$

$$\boxed{\varphi_C: X_C \rightarrow Y}$$

$$\dim X_C = \dim Y = 2.$$

$$\dim(\text{Im } \varphi_C) \neq 0$$

$$\dim(\text{Im } \varphi_C) \neq 2$$

K_Y is ample

φ_C is generically finite.

$$R + d_C \cdot K_Y = K_{X_C}$$

X_C is of genl typ

impossible.

$$\begin{array}{c} X_C \\ \downarrow \\ C \end{array} \leftarrow \begin{array}{l} \text{fibers} \\ \text{over } A. \end{array}$$

$\text{Im } \varphi_C$ is a curve.

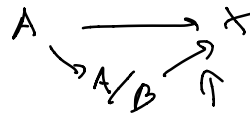
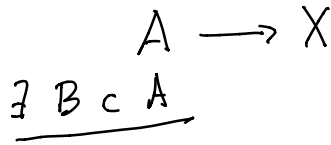
$$P \in Y \quad P \in \underline{C} \cap \underline{D} \quad \text{Im}$$

$$\varphi(X_P) = X_{\psi(P)}$$

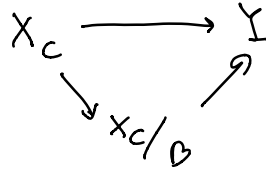
$\psi: Y \rightarrow Y$ an automorphism.

$$\varphi(X_P) = X_P.$$

LEMMA Demaily Hway Petersonell



$\exists B$ s.t.



If A is simple and $|\text{Aut}_R(A)| < +\infty$

Then for $\varphi \in \text{Id}$ $|A^\varphi|$ is bounded.

If $|\text{Aut}_R(A)| < +\infty$

$A \xrightarrow{\cong} A_1 \times \dots \times A_n$
 isos. $i \neq j$

$\text{Aut}(A_i) < +\infty$

$\text{Hom}(A_i, A_j) = 0$.

$$\frac{A \times \gamma}{\mathbb{Z}(P)}$$