Splines, Representation theory and Geometry : September 2019

Series of articles on splines

• I.J. Schoenberg (1973) : *Cardinal Space Interpolation I,II,III,...*

• Dahmen-Micchelli (1985) :
On the solution of certain systems of partial difference equations and linear dependance of translates of Box Splines.

• Many articles (C. De Concini+C.Procesi+MV) around 2008
Vector partition functions and generalized Dahmen-Micchelli spaces,
Vector partition functions and index of transversally elliptic operators,
. . .

• Loizides-Paradan-Vergne (2019)
Semi-classical analysis of piecewise quasi-polynomial functions

Functions on a lattice and difference equations

V real vector space of dimension d, with lattice $\Lambda \subset V$
$\mathcal{F}(\Lambda)$: the space of $\mathbb{Z}$-valued functions on Λ.

Difference operator

$\nabla_\alpha(f)(\lambda) = f(\lambda) - f(\lambda - \alpha)$

Periodic functions on Λ : functions on $\Lambda / D\Lambda$, for some $D > 0$.

Quasi polynomial functions

$\mathcal{QP}(\Lambda)$ the algebra generated by polynomials functions on Λ and periodic functions on Λ.

Example $\Lambda = \mathbb{Z}$:

$$f(n) = \frac{n}{2} + \frac{3}{4} + (-1)^n \frac{1}{4}$$

Dahmen-Micchelli space

$\Phi = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ list of vectors in Λ spanning V
$A \subset \Phi$ is called short if A does not generate V.
$B \subseteq \Phi$ is called long if $\Phi - B$ does not generate V.
$\nabla_B = \prod_{\alpha \in B} \nabla_\alpha$

Definition

$DM(\Phi)$ is the space of $\mathbb{Z}$-valued functions f on Λ which are solutions of the system of difference equations :

$$\nabla_B(f) = 0$$

for all long subsets of Φ.

Theorem : Dahmen-Micchelli (1985)

The space $DM(\Phi)$ is free over $\mathbb{Z}$ of finite rank, and consists of quasi-polynomials.

Examples

$\Phi = [\omega, \omega]$ in $\Lambda = \mathbb{Z}\omega$.

One equation $\nabla_\omega^2 \cdot f = 0$
Basis of $DM(\Phi)$ over $\mathbb{Z}$:

$$\{f_1(n) = n + 1, \quad f_2(n) = 1\}.$$

$\Phi = [\omega, 2\omega]$

One equation $\nabla_\omega \nabla_{2\omega} \cdot f = 0$
Basis of $DM(\Phi)$ over $\mathbb{Z}$:

$$\{g_1(n) = \frac{1}{2}n + \frac{3}{4} + \frac{1}{4}(-1)^n,\ g_2(n) = \frac{1}{2} - \frac{1}{2}(-1)^n,\ g_3(n) = 1\}$$

Geometry : $M(\Phi) = \oplus_{\alpha\in\Phi}\mathbb{C}_\alpha$

T torus with lattice of characters $\Lambda \subset V$: $Lie(T) = V^*$;
$M(\Phi) = \{m = \sum_{\alpha\in\Phi} z_\alpha e_\alpha, z_\alpha \in \mathbb{C}\}$: a complex vector space ;

$$T\text{action}: \; X \in V^* \; : \exp(X)\cdot m = \sum_{\alpha\in\Phi} z_\alpha e^{i\alpha(X)} e_\alpha;$$

Moment map $J : M(\Phi) \to V$

$$J(m) = \sum_{\alpha\in\Phi} |z_\alpha|^2 \alpha$$

Image of J :

$$\text{Cone}(\Phi) = \{\sum_{\alpha\in\Phi} t_\alpha \alpha, t_\alpha \geq 0\}$$

Equivariant topological K-theory of $M_f(\Phi)$

Open subset $M_f(\Phi)$ of $M(\Phi)$ where T acts almost freely.

$$M_f(\Phi) = M(\Phi) \setminus \cup_{A \text{ short subsets}} M(A)$$

complement of an arrangement of vector spaces.

Theorem (DPV 2008)

$s = dimV$. The space $K_T^s(M_f(\Phi))$ is isomorphic to $DM(\Phi)$.

The isomorphism is via equivariant index theory. More later :

Example $\Phi = [\omega, \omega]$, $M(\Phi) = \mathbb{C}^2$, and $f_1(n) = (n+1)$

$$\{|z_1|^2 + |z_2|^2 = 1\}.$$

The tangential Cauchy-Riemann $\overline{\partial}$ operator has index

$$\text{index}(E_\alpha)(e^{i\theta}) = \sum_{n \in \mathbb{Z}} (n+1) e^{in\theta}.$$

Vector partition functions

Consider the case where Φ generates an acute cone and $\lambda \in \Lambda$.

The Vector Partition function

$$\mathcal{K}(\Phi)(\lambda) = \mathrm{Cardinal}(\{\sum_{\alpha \in \Phi} x_\alpha \alpha = \lambda\}); x_\alpha \text{ non negative integers}$$

Kostant Partition function when $\Phi = \Delta_{\geq 0}(\mathfrak{k}, \mathfrak{t})$ positive root system.

Examples $\Lambda = \mathbb{Z}\omega$.

$$\mathcal{K}[\omega](n) = 1, n \geq 0,$$

,

$$\mathcal{K}([2\omega])(n) = \frac{1}{2} - \frac{1}{2}(-1)^n, n \geq 0,$$

$$\mathcal{K}[\omega, \omega](n) = n + 1, n \geq 0,$$

$$\mathcal{K}([\omega, 2\omega])(n) = \frac{1}{2}n + \frac{3}{4} + \frac{1}{4}(-1)^n, n \geq 0$$

Representation theory

Action of T on $\mathrm{Sym}(M(\Phi))$: polynomials functions on the complex space $M(\Phi)^*$:

$$Tr_{\mathrm{Sym}(M(\Phi))}(t) = \sum_{\lambda} \mathcal{K}(\Phi)(\lambda) t^{\lambda}$$

So $\mathcal{K}(\Phi)(\lambda)$ is the multiplicity of the character t^{λ} in $\mathrm{Sym}(M(\Phi))$.
Morally :

$$Tr_{\mathrm{Sym}(M(\Phi))}(t) = \frac{1}{\prod_{\alpha \in \Phi}(1 - t^{\alpha})}$$

The Partition function is a locally quasi-polynomial function

V_{reg} :the set of regular values of the moment map $J : M \to V$.
Each connected component τ is the interior of a convex polyhedral cone $\overline{\tau}$.

$$V_{reg} = \cup_{\tau}\tau$$

union over its connected components τ.

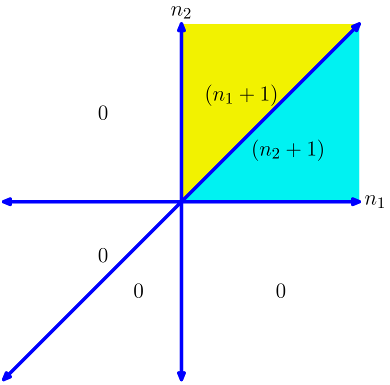
Theorem (Dahmen-Micchelli, Brion-V, Szenes-V, ...)

$\mathcal{K}(\Phi)(\lambda)$ coincide on $\overline{\tau} \cap \Lambda$ with a Dahmen-Micchelli polynomial belonging to $DM(\Phi)$.

So $\mathcal{K}(\Phi)(\lambda)$ is a piecewise quasi polynomial function **and** is "continuous" on $\mathrm{Cone}(\Phi) \cap \Lambda$.

Example : $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

$$\Phi = [\omega_1, \omega_2, \omega_1 + \omega_2]$$



Zeroes of the moment map

Consider now ANY set $\Delta \subset \Lambda$ without assuming that Δ spans an acute cone.
For example $\Delta = \Phi \cup -\Phi$, so $M(\Delta) = T^*M(\Phi)$.
$Z = J^{-1}(0)$ the set of zeroes of the moment map is a convex cone, and Z/T has the structure of a "stratified" symplectic space.

Example $\Delta = [\omega, -\omega]$

$$Z = \{|z_1|^2 - |z_2|^2 = 0\}$$

$$Z/T = \{0\} \cup T^*S^1.$$

A remarkable space of $\mathbb{Z}$-valued functions on Λ

Consider the space $\mathcal{S}(\Delta)$ of functions on Λ generated by all functions $\mathcal{K}(A)$ for $A \subset \Delta$ generating an acute cone, and all their translates.

Conjecture of Boutet de Monvel

The equivariant theory of $K^0_T(Z)$ is isomorphic to $\mathcal{S}(\Delta)$

Evident if Δ generates an acute cone.

Theorem DPV

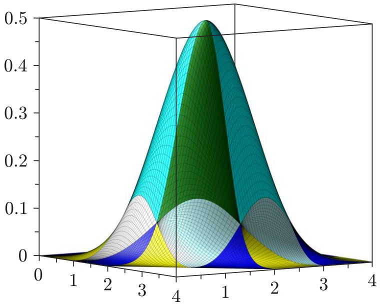
Assume that $\Delta = \Phi \cup -\Phi$. Then this is true.

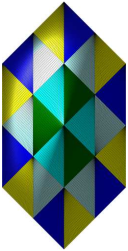
Proof via transversally elliptic equivariant index theory.

$$\sum_{n \in \mathbb{Z}} (n+1) z^n = \text{"}\frac{1}{(1-z)^2}\text{"} - \text{"}\frac{1}{(1-z)^2}\text{"}.$$

Piecewise quasi polynomial functions and asymptotics

Many multiplicity functions in representation theory lead to piecewise quasi-polynomials. Geometry leads to locally polynomial functions such as Duistermaat-Heckman measures. Relation via asymptotics.





A space of quasi-polynomial functions on cones and depending of a parameter $k \geq 1$

We now consider $E = V \oplus \mathbb{R}$, with $\Lambda \oplus \mathbb{Z}$. P : a **rational closed polyhedron** in V, $\sigma \in V$ rational.

$$C(P, \sigma) := \{[tv + \sigma, t], v \in P; t \geq 0\}.$$

$[C(P, \sigma)]$: the characteristic function of $C(P, \sigma) \cap (\Lambda \oplus \mathbb{Z})$.

Definition

The space $\mathcal{L}(\Lambda)$ consists of functions m on $\Lambda \oplus \mathbb{Z}_{>0}$ of the form

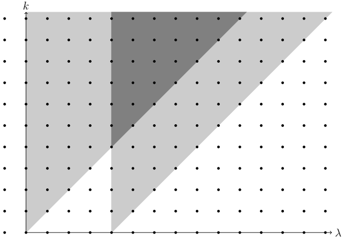
$$m(\lambda, k) = \sum_{P,\sigma} q_{P,\sigma}(\lambda, k)[C(P, \sigma)](\lambda, k)$$

where $q_{P,\sigma}$ are quasi polynomial functions on $\Lambda \oplus \mathbb{Z}$

We allow locally finite sums, but in this talk I restrict myself to finite sums.

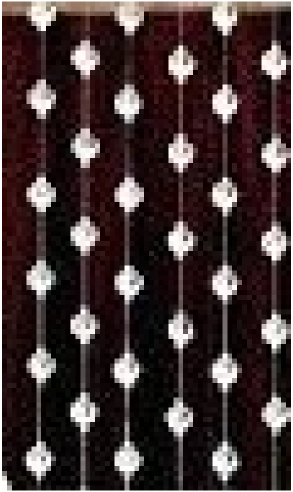
$\Lambda = \mathbb{Z}\omega$, $P = [0, 1]$, $\sigma = 0, 4$

Two cones : we can take any quasi polynomial function, each on one of these cones.



$\Lambda = \mathbb{Z}\omega$, $P = [0]$, $\sigma = 1, 2, 3, 4, 5, 6$

A constant function c_1, c_2, c_3, c_4, c_6 on each of the 6 vertical lines.



Examples motivated by geometry

K compact group acting on M is compact complex, $\mathcal{E}$ K-equivariant holomorphic vector bundle and $\mathcal{L} \to M$ a holomorphic line bundle.
V_λ^K irreducible representation of K with highest weight λ.
$m(\lambda, k)$ the multiplicity of V_λ^K in

$$\sum_{j=0}^{\dim M} (-1)^j H^j(M, \mathcal{E} \otimes \mathcal{L}^k)$$

is in our space $\mathcal{L}(\Lambda)$.
More generally quantization of M spin manifold with line bundle and proper moment map $J : M \to \mathfrak{k}^*$.
Example $M = M(\Phi)$ and $m(\lambda, k) = \mathcal{K}(\Phi)(\lambda)$.

Asymptotics

$m(\lambda, k)$ a function on $\Lambda \oplus \mathbb{Z}_{>0}$
φ : a C^∞ function of compact support on V :

$$\langle \Theta(m; k), \varphi \rangle = \sum_{\lambda \in \Lambda} m(\lambda, k) \varphi(\lambda / k)$$

A not surprising result

If $m \in \mathcal{L}(\Lambda)$, when $k \to \infty$, the family of distributions $\Theta(m; k), k \geq 1$ admits an asymptotic expansion

$$\mathcal{A}(m)(k) = \sum_{j \geq j_0} k^{-j} \theta_j(k)$$

in powers of k^{-1} where the distributions θ_j may be periodic in k (different formulae for k modulo some integer)

The beaded curtain

$$m(2k, 1, 3, 5) = [c_1, c_3, c_5],$$

$$m(2k+1, 2, 4, 6) = [c_2, c_4, c_6].$$

When $k \to \infty$, k even :

$$\langle \Theta(m)(k), \varphi \rangle = c_1\varphi(1/k) + c_3\varphi(3/k) + c_5\varphi(5/k)$$

$$\equiv (c_1 + c_3 + c_5)\varphi(0) + \frac{1}{k}(c_1 + 3c_3 + 5c_5)\varphi^{(1)}(0)$$

$$+\frac{1}{2k^2}(c_1 + 3^2 c_3 + 5^2 c_5)\varphi^{(2)}(0) + \cdots$$

Series of distributions supported at 0.
Similar formula for k odd.

A more surprising result

Theorem (Loizides-Paradan-V)

If $m \in \mathcal{L}(\Lambda)$, the function m is determined by its asymptotic $A(m)$

Example : the beaded curtain.
When $k \to \infty$, k even :

$$\langle A(m)(k), \varphi \rangle = (c_1 + c_3 + c_5)\varphi(0) + \frac{1}{k}(c_1 + 3c_3 + 5c_5)\varphi^{(1)}(0)$$
$$+ \frac{1}{2k^2}(c_1 + 3^2 c_3 + 5^2 c_5)\varphi^{(2)}(0) + \cdots$$

With 3 terms of the asymptotic expansion, and the Vandermonde determinant, we can determine c_1, c_3, c_5.

Applications to geometric quantization

G torus (or a compact connected group) acting in an Hamiltonian way on a symplectic manifold (M, Ω) ;
$J : M \to Lie(G)^*$ the moment map. G torus (or more generally a compact connected group) acting in an Hamiltonian way on M symplectic, and $\mathcal{L} \to M$ a Kostant line bundle.
Then if M is compact, one can define a finite dimensional representation of G : $Q^G(M, \mathcal{L})$. If M is Kahler :

$$Q^G(M, \mathcal{L}) = \sum_{j=0}^{\dim M} (-1)^j H^j(M, \mathcal{O}(L)).$$

Our aim : define $Q^G(M, \mathcal{L})$ when M is not necessarily compact, and give character formulae. Example $M = M(\Phi)$, $Q^G(M, \mathcal{L}) = Sym(M)$.

Equivariant cohomology and Duistermaat-Heckman measure

$X \in Lie(G) : \Omega(X) = \langle J, X \rangle + \Omega$ the equivariant symplectic form
φ test function on $Lie(G)^*$, $\hat{\varphi}$ its Fourier transform. Then

$$\int \int_{M \times Lie(G)} e^{i\Omega(X)} \hat{\varphi}(X) dX = \langle DH, \varphi \rangle$$

where *DH* is the Duistermaat-Heckman measure, and is piecewise locally polynomial.

Twisted Duistermaat-Heckman distributions

$H^*_G(M)$, the equivariant cohomology ring.
More generally, if $\eta \in H^*_G(M)$, then

$$\int\int_{M \times Lie(G)} e^{i\Omega(X)} \wedge \eta(X)\hat{\varphi}(X)dX = \langle DH(\eta), \varphi$$

where $DH(\eta)$ is a distribution on $Lie(G)^*$ obtained as a derivatives of piecewise locally polynomial measures.

Quantizing a symplectic manifold with proper moment map

G torus (or more generally a compact connected group) acting in an Hamiltonian way on M symplectic, and $\mathcal{L} \to M$ a Kostant line bundle. Then if the moment map is proper one can associate to it (Formal quantization) a representation of G :

$$Q^G(M, \mathcal{L}^k) = \sum_{\lambda \in \hat{G}} m(\lambda, k) t^\lambda.$$

with the following formula when M is Kahler : consider $M_\lambda = J^{-1}(\lambda)/G_\lambda$. This is a Kahler manifold (orbifold), when λ is a regular value of J. Then define

$$m(\lambda, k) = \sum_j (-1)^j H^j(M_\lambda, \mathcal{O}(L_\lambda)).$$

If the set of critical point of the norm square of the moment map is compact, the function m belongs to $\mathcal{L}(\Lambda)$

The infinitesimal equivariant Riemann-Roch formula

Consider the equivariant Todd class $Todd(X, M)$. If M is compact, one has the equivariant Riemann-Roch formula (for X small)

$$Tr_{Q^G(M,\mathcal{L}^k)}(exp(X)) = \int_M e^{ik\Omega(X)} Todd(X, M)$$

Now M not necessarily compact, but with proper moment map $J : M \to Lie(G)^*$: Write the equivariant Todd class $Todd$ as $Todd = \sum_{j=0}^{\infty} T_j$ in the graded equivariant cohomology ring $H_G^*(M)$

Theorem (V)

$$\sum m(\lambda, k)\varphi(\lambda/k) \sim \sum_{j=0}^{\infty} k^{-\infty}\langle DH(T_j), \varphi\rangle$$

Morally this is the Riemann-Roch formula for X/k

$$Tr_{Q^G(M,\mathcal{L}^k)}(exp(X/k)) = \int_M e^{ik\Omega(X/k)} Todd(X/k, M)$$

Formal quantization is determined by its asymptotics

Theorem : The above infinitesimal formula (interpreted as asymptotic series) :

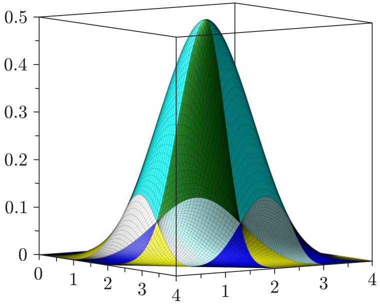
$$\int_{M\times Lie(G)} e^{ik\Omega(X/k)} Todd(X/k, M)\hat{\varphi}(X)dX$$

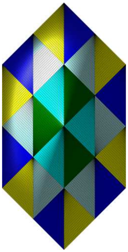
determines $Q^G(M, \mathcal{L})$.

The right hand side is a series of twisted Duistermaat distributions on $Lie(G)^*$. It is possible to recover $m(\lambda, k)$ from this formula.
Application (Loizides) : functoriality of formal quantization.

EXAMPLE : recovering multiplicities from asymptotics

F : Flag manifold for $SU(3)$: The Duistermaat Heckman measure for $O(k\rho) \times O(k\rho)$ and the diagonal action of the torus T of $SU(3)$ on $F \times F$:





EXAMPLE : recovering multiplicities from asymptotics

F : Flag manifold for $SU(3)$ We quantize $O(k\rho)$ as the representation with highest weight $(k-1)\rho$.. For $k=1$, multiplicity should be 0 every where except at $\lambda = 0$...