Integrable systems and Solitons

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1. Nonlinear Schrödinger systems

Equations of nonlinear Schrödinger (NLS) type are prototypical nonlinear dispersive systems of partial differential equations (PDEs) that play an important role in both mathematics and physics. NLS-type equations have been derived in such diverse fields as deep water waves, plasma physics, nonlinear fiber optics, magnetic spin waves, etc. Many dispersive, energy preserving systems give rise, in appropriate limits, to the scalar NLS equation. In other physical applications, the governing equation is the vector NLS (VNLS) system

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\nu\|\mathbf{q}\|^2\mathbf{q}\,, \qquad (1)$$

where $\mathbf{q}(x,t)$ is an N-component complex-valued vector function, $\nu = \pm 1$ denotes the focusing/defocusing cases as before, and $\|\cdot\|$ is the standard Euclidean norm. Here and in the following the boldface font is used to denote vector/matrix functions, while the regular font will be used to denote scalar functions.

Physically, VNLS systems arise under conditions similar to those giving rise to NLS, whenever there are suitable multiple wavetrains moving with nearly the same group velocity. The VNLS also models systems where the electromagnetic field has more than one nonzero component. For example, in optical fibers and waveguides, the electric field has two nonzero polarization components (which for plane waves are transverse to the direction of propagation).

The VNLS system (1) with $N = 2$ was proposed by Manakov in 1974 as an asymptotic model governing the propagation of the electric field envelope in waveguides. Accordingly, (1) with $N = 2$ is commonly referred to as the *Manakov system*. Later, the system was also derived as a model for optical fibers. In optics, the defocusing case $\nu = 1$ corresponds to the normal dispersion regime, while the focusing case $\nu = -1$ to the anomalous dispersion regime.

A number of variants of the NLS equation are also solvable by the Inverse Scattering Transform (IST) method, which is the nonlinear analogue of the Fourier transform for solving the initial value problem for linear PDEs.

The IST for NLS systems with non-zero boundary conditions (NZBCs) is much less developed than in the case of solutions which vanish rapidly at spatial infinity. In particular, even though the IST for the defocusing scalar NLS equation with NZBCs as $x \to \pm\infty$ was formulated in 1973 by Zakharov and Shabat, the development of the IST for the Manakov system with nondecaying potentials remained an open problem for over thirty years, and was only recently solved by us [1].

Already in the scalar case the IST with NZBCs is significantly more complicated than in the case of decaying potentials, due to the fact that the spectral parameter of the associated block-matrix scattering problem is an element of a two-sheeted Riemann surface. However, one still has two complete sets of analytic scattering functions, and the IST can be carried out in a standard way.
When the number of components $N > 1$, however, additional complications arise: $2(N-1)$ out of the $2(N+1)$ scattering eigenfunctions are not analytic on either sheet of the Riemann surface, and one must find a way to complete the basis. The 2-component case (Manakov system) is somehow special. In [1] we have developed the IST for the Manakov system with NZBCs using the adjoint scattering problem to construct two additional analytic eigenfunctions. The inverse scattering problem can be formulated as a generalized Riemann-Hilbert problem with poles in the upper/lower half-planes of a suitable uniformization variable. This construction allowed us to completely characterize the solitonic sector of VNLS in the normal dispersion regime (i.e., in the defocusing case).

The investigation of the soliton solutions is of particular importance. The defocusing NLS does not admit the usual "bell"-shaped soliton solutions. It does, however, possess so-called "dark solitons". For the scalar defocusing NLS with constant-amplitude BCs $|q(x,t)| \to q_0$ as $x \to \pm\infty$, these are localized dips of intensity propagating on a background field of constant, non-zero amplitude q_0. In the Manakov system with NZBCs, our study of the solitonic sector revealed vector generalizations of the aforementioned dark solitons, exhibiting dark solitonic behavior in both components, as well as novel dark-bright soliton solutions, which have one dark component and one bright component. These dark-bright soliton solutions had been previously obtained by direct methods, but had not been characterized from a spectral point of view before.

The formulation of the IST for the multi-component ($N > 2$) vector NLS system with nonzero boundary conditions was addressed most recently [2]. In this paper, we developed the IST for the defocusing vector NLS equation with an arbitrary number of components, with nonzero boundary conditions at infinity. The technique we successfully applied to the 2-component VNLS does not admit an obvious generalization to an arbitrary number of components. In order to complete the basis of analytic eigenfunctions for the general multicomponent scattering problem, in [2] we generalize the approach suggested by Beals, Deift and Tomei (1988) for general scattering and inverse scattering on the line, but developed under the assumption of vanishing boundary conditions. The key step is the introduction of a fundamental tensor family as solutions of a suitable scattering problem associated to the given one, in such a way that each tensor is sectionally analytic on the cut Riemann surface. Then we show that it is possible to algorithmically reconstruct the fundamental matrices of solutions of the scattering problem from the fundamental tensors, and to establish their analyticity properties.

2. Kadomtsev-Petviashvili equations

The theory of integrable systems in more than one spatial dimension is extremely rich and complex. One of the prototypical $(2+1)$-dimensional integrable systems is the Kadomtsev-Petviashvili:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = -3\sigma^2\frac{\partial^2 u}{\partial y^2}\,. \qquad (2)$$

Here $\sigma^2 = \mp 1$, and the two variants of the KP equation are called KPI and KPII, respectively. The KP equation, which is a generalization of the well-known KdV equation to two spatial dimensions, is a universal model for small amplitude, weakly two-dimensional waves in the long wavelength regime, and as such it arises in disparate physical settings. Like KdV, the KP equations admit rational, solitonic, periodic and quasi-periodic solutions.

The one-soliton solutions of KP are one-dimensional structures exponentially localized along a direction in the (x, y)-plane, and therefore called *line soliton*. Generalization to N-soliton solutions also exist, and have recently attracted a great deal of attention since it has been shown that the KPII solitons possess an unexpectedly rich structure. Indeed, there is a large variety of multisoliton solutions for KPII, many of which exhibit nontrivial spatial interaction patterns, resonances and web structures. In general, such solutions consist of unequal numbers of line solitons as $y \to -\infty$ and $y \to \infty$. Moreover, the directions of the line solitons as $y \to \infty$ are not necessarily the same as those along $y \to -\infty$, even when the numbers of asymptotic line solitons as $y \to \pm\infty$ coincide.

Since line solitons do not vanish at infinity along rays corresponding to the directions of the solitons, the issue of boundary conditions plays a crucial role in the study of these equations as well. Indeed, the inclusion of multi-soliton solutions of KP within the framework of the IST is a complicated, long-standing open problem on which we have worked for a number of years.

Despite their apparent similarities, the two versions of KP have markedly different properties. For instance, spatial interaction patterns as the ones described above for KPII are not found for KPI solitons. Besides, the choice of sign is critical for the stability properties of the one-dimensional line solitons with respect to small transverse perturbations: KPII solitons are stable, while KPI solitons are not. The sign of the coefficient is also critical in the development of the IST scheme for solving the IVP, due to the fact that the scattering problem associated to KPI is the nonstationary Schrödinger equation (with the transverse space variable y playing the role of time), while KPII is related to the perturbed heat equation.

Even though the KP equations have been known to be integrable for over three decades, the study of their initial value problem via the IST is not yet complete. In fact, the standard approach to the spectral theory of the scattering operators, based on integral equations for the Jost solutions, fails for potentials that are not decaying at space infinity, and therefore solitons are excluded in this approach. Our research on the subject has dealt with the development of a new mathematical technique, the so-called extended resolvent approach, which allows one to extend the spectral transform for KP equations (and possibly for other nonlinear $2+1$-dimensional equations relevant for physical applications) to the case of potentials that are not decaying along a finite number of directions in the plane (ray-type potentials). The extended resolvent proved to be an effective foundation for the proper generalization of the IST method. Indeed, in this framework we have been able to obtain some general results for the IST with ray-type potentials.

In [4], we used twisting transformations to study the existence and uniqueness of the extended resolvent in detail in the case of solutions with N line solitons as $y \to -\infty$ and one line soliton as $y \to \infty$.

The rich structure of KPII solitons has been explored so far by using a variety of approaches: dressing methods, Bäcklund transformations, twisting transformations, τ-functions, etc. All these approaches proved to be useful in order to display different properties of KPII solitons and of the eigenfunctions of the associated scattering problem. In [5], we established the explicit correspondence among different approaches for constructing multisoliton solutions of the KPII equation and elucidated some hidden invariance properties of these solutions.

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