On the high–energy stability of the non linear modal solutions for the Fermi–Pasta–Ulam β system

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In the last fifty years, the Fermi–Pasta–Ulam (FPU) [1] system has represented a privileged model for investigating problems in nonlinear dynamics (see [2], [3] for recent reviews).

The FPU [1] β chain with a number N of oscillators and periodic boundary conditions possesses exact one–mode solutions (OMSs) [4–7] corresponding to the values of the mode number

$$n = \frac{N}{4},\ \frac{N}{3},\ \frac{N}{2},\ \frac{2}{3}N,\ \frac{3}{4}N \tag{1}$$

such that, if only one of these modes is initially excited, it evolves without transferring energy to any other mode. The stability of the π–mode ($N/2$ mode) has been extensively studied in several papers [7,8,4,9,10].

The theoretical analysis of the stability of the $\pi/2$–mode (N/4 mode) is much more challenging than that of the π–mode. This last case is simpler because the differential equations of the perturbed modes are all decoupled and can be studied separately. For the $\pi/2$–mode, on the contrary, one needs to study a system of coupled linear differential equations with periodic coefficients.

In [11] the stability of this mode has been studied both numerically and analytically by means of a Floquet analysis. The main result is that the product $\mid \beta \mid \epsilon_t$ (where ϵ_t is the energy density threshold and β is the nonlinearity parameter) decreases asymptotically with N as $2\pi^2/3N^2$; when $\beta > 0$ the product $\beta\ \epsilon_t\ N^2$ decreases with N and converges asymptotically to the value $2\pi^2/3$; for $\beta < 0$, $\mid \beta \mid \epsilon_t\ N^2$ increases with N towards the same value.

In a recent paper [12] we have focused on the statistical properties of the π–mode solution for a large range of values of energy density, from a regime where the solution is stable to a regime where is unstable, first weakly and then strongly chaotic. For carrying out this analysis, we have introduced the ratio ρ (when it can be defined) between the standard deviation and the first moment of a given probability distribution. This global indicator estimates the deviation of a generic assigned distribution from the Gaussian behaviour for any value of the excitation energy density. It is a function of the dynamical variables of the configuration space only and its usefulness relies on the fact that is model–independent. We have shown numerically that the transition between weak and strong chaos can be interpreted as the symmetry breaking of a set of suitable dynamical variables.

Motivated by these results, we have extended to the $\pi/2$–mode the analysis performed for the π–mode. The main results of this analysis are the following [13].

a) Differently from the π–mode, for the $\pi/2$–mode the region of strong chaos is unexpectedly followed, when increasing the energy density ϵ, by a region where *the nonlinear mode solution becomes again stable*. The transition from the irregular to the stable behaviour is abrupt, i.e. is sensitive to a variation of one unit on the fifth decimal digit of the value of ϵ. The stability of the $\pi/2$–mode solution, above this second threshold ϵ_{st}, has been verified for very long integration times of the differential equations of motion.

From our numerical analysis, we have not found any evidence of the existence of an additional stochasticity threshold, marking the evolution to another irregular behaviour, although it appears natural to postulate the relaxation to a fully chaotic regime for extremely long times. Therefore we prefer to use the terminology *quasi–stable state* to denote this long time stable behaviour of the $\pi/2$–mode above the second threshold.

b) The value of the second threshold depends on N and tends asymptotically to a well defined value, approximately equal to 0.14780, for large values of N. We have verified this thermodynamic limit up to $N = 4096$.

c) As for the π–mode, also for the $\pi/2$–mode the transition between weak and strong chaos can be interpreted as the symmetry breaking of a set of dynamical variables.

d) We have also analyzed the OMS corresponding to $n = N/3$ and in this case as well we have verified the existence of two energy thresholds of stability of the mode.

The quasi–stable state we have found possesses some similarities with the quasi–stationary states observed in the literature, although the contexts are quite different. For instance, for the Hamiltonian Mean Field model, introduced in [14] and describing a system of classical coupled rotors, it has been found numerical evidence of the existence of quasi–stationary states out–of–equilibrium, having a lifetime that increases with the number of particles N of the system [15]. A related analysis has been performed in [16]. Also, in [17] this kind of states has been recognized in weakly chaotic regimes of several multi–dimensional hamiltonian dynamics.

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