Dynamics of noncommutative monopoles

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In the last decade the interest in the dynamical systems, both classical and quantum, designed as non-commutative, grew out of the idea of non vanishing commutation relations among the position operators. This assumption is ascribed to Heisenberg himself, in order to introduce a fundamental length, which may control the short distance singularities of quantum field theory, as the Planck's constant does in the classical phase space. This approach had its mathematical foundations in the so-called "Noncommutative Geometry" developed in the '70 by Connes [1] (see [2] for a recent review). The procedure has a classical counterpart, which was formalized in [3] by dealing with the so-called *exotic* mechanical model in the plane, the kinematics of which is obtained by the Coadjoint Orbit method for the (2+1)-Galilei group endowed with a 2-fold central extension: the usual mass m and the "exotic" parameter θ, describing the non-commutativity of Galilean boost generators

$$\{K_1, K_2\} = -\theta m^2.$$

It can be seen as a "non-relativistic shadow of the spin" for relativistic particles in the plane [4]. In condensed matter physics, θ can be identified with a constant Berry curvature acting on electron wave-packets [5,6], but it is a function of the crystal quasi-momentum.

Coupling such a system to an external electromagnetic field, the main features are: i) the existence of an effective mass $m^* = m\,(1 - e\,\theta\,B)$, ii) the anomalous velocity term $-em\theta\,\epsilon_{ij}E_j$, then $\dot{\vec{r}} \nparallel \vec{p}$, iii) $\frac{d\vec{p}}{dt}$ is still determined by the Lorentz force. When the effective mass vanishes for a critical value of the magnetic field, the system becomes singular. Then, the Symplectic Reduction procedure leads to a two-dimensional system characterized by the remarkable Poisson structure $\{x_1, x_2\} = \theta$. Thus, the symplectic plane plays the role both of configuration and phase space. Moreover, in the quantization of the reduced system, not only the position operators no longer commute, but the quantized equation of motions yields the Laughlin wave functions [7], which are the ground states in the Fractional Quantum Hall Effect (FQHE), thus the classical counterpart of the *anyons* are in fact the *exotic* particles. In our review article [8] several examples of 2-dimensional models, generalizing the above settings, have been discussed, taken from very different physical contexts.

Recently, looking for generalizations of the above approach [6,10–12], we [9] proved that the Lagrange-Souriau 2-form in three dimensions

$$\begin{aligned}\sigma &= [(1+\mu_{ii})\,dp_i - e\,E_i\,dt] \wedge (dr_i - g_i\,dt) + \\ &\quad \tfrac{1}{2}\,e\,B_k\,\epsilon_{kij}\,dr_i \wedge dr_j + \\ &\quad \tfrac{1}{2}\,\kappa_k\,\epsilon_{kij}\,dp_i \wedge dp_j - \mu_{ij}\,dr_i \wedge dp_j\end{aligned}$$

is the exterior derivative of the Cartan 1-form,

$$\lambda = \left(\vec{p} - e\vec{\mathcal{A}}\right) \cdot d\vec{r} - \vec{\mathcal{R}} \cdot d\vec{p} - \mathcal{T}\,dt,$$

where one introduced a "dual" potential $\vec{\mathcal{R}}\,(\vec{r}, \vec{p}, t)$, the local mass fluctuations $\mu_{ij} = \partial_{r_i}\mathcal{R}_j$ and the mass flow $g_i = \frac{\partial_{p_i}\mathcal{T} - \partial_t\mathcal{R}_i}{1+\mu_{ii}}$. This formula generalizes the expression (12.46) in [13], where it was assumed to be the basic law of Mechanics, implying then its differential consequences $d\sigma \equiv 0$, which are in fact the Maxwell equations for the E.M. field and their analogous for $\vec{\mathcal{R}}$ and its strength fields κ_k , g_i and μ_{ij} in the $(\vec{p}, t)$ variables. For non trivial second cohomology group of the evolution space only locally defined potentials $\vec{\mathcal{A}}$ nor $\vec{\mathcal{R}}$ exist, but the Lagrange - Souriau form will be globally defined and, moreover, if $\partial_t\vec{\mathcal{A}} = \partial_t\vec{\mathcal{R}} \equiv 0$ it is possible to split it in the hamiltonian form $\sigma = \omega - d\mathcal{T} \wedge dt$.

A particular case of such a construction is dual monopole model $\vec{\kappa} = \theta\frac{\vec{p}}{|\vec{p}|^3}$, introduced [13,14] to describe classical relativistic particles with non-vanishing helicity in the 0-mass limit. On the other hand, It appears consistent with the experimental data in Anomalous Hall Effect [15] and in Spin Hall Effect [16] and dynamics in its field was discussed in [11]. Thus, it is natural to consider a charged particle simultaneously interacting with a Dirac magnetic monopole and a dual monopole, the equations of motion of which are

$$\begin{aligned}M^*\dot{r}_i &= \left(p_i - e\theta\frac{r_i}{|\vec{p}||\vec{r}|^3}\right)|\vec{r}|^3|\vec{p}|^3, \\ M^*\dot{p}_i &= e\varepsilon_{ijk}p_j r_k|\vec{p}|^3, \\ M^* &= |\vec{r}|^3|\vec{p}|^3 - e\theta\,\vec{r}\cdot\vec{p}.\end{aligned}$$

The vanishing of the effective mass M^* will provide an anholonomic constraint to the dynamics.

Looking for the conservation laws, other than the energy H, the angular momentum is

$$\vec{j} = \vec{r} \times \vec{p} - \frac{\theta}{|\vec{p}|}\vec{p} - \frac{e}{|\vec{r}|}\vec{r}, \tag{0.1}$$

However, the double breaking of the Jacobi identities $\frac{1}{2}\varepsilon_{ijk}\,[p_i,[p_j,p_k]] = e\,\delta\,(\vec{r})$ and $\frac{1}{2}\varepsilon_{ijk}\,[r_i,[r_j,r_k]] = \theta\,\delta\,(\vec{p})$ is an obstruction to the complete integrability of the model. Then, the $(\vec{r},\vec{p})$ phase space has a nontrivial topological structure, its second cohomoogy group being $\mathbf{R}\otimes\mathbf{R}\otimes\mathbf{R}\otimes\mathbf{R}$. Consequently a coherent gauge theory requires the quantization both of magnetic charge $e = N_m/2$ and of the dual one $\theta = N_d/2$.

The motion on the critical invariant submanifold $\mathcal{M} = \{(\vec{r},\vec{p}) : M^* = 0\}$ has been analyzed by the symplectic reduction. In particular, for any initial data $\{\vec{r}_0,\,\vec{p}_0\}$ such that $|\vec{j}_0| = |\theta| + |e|$ and $H|_{\mathcal{M}} = E_0$, there exists the first contact position $\vec{r}_{cr} = -\text{sign}\,(e)\,\frac{\sqrt{|e\theta|}}{|\theta|+|e|}\,\frac{\vec{j}_0}{\sqrt{2\,E_0}}$ with the critical sub-manifold, after that a restricted uniform motion dynamics follows, realizing a sort of reductions to Hall motions in 3D, with a very similar mechanism based on the interplay of magnetic field and its dual counterpart.

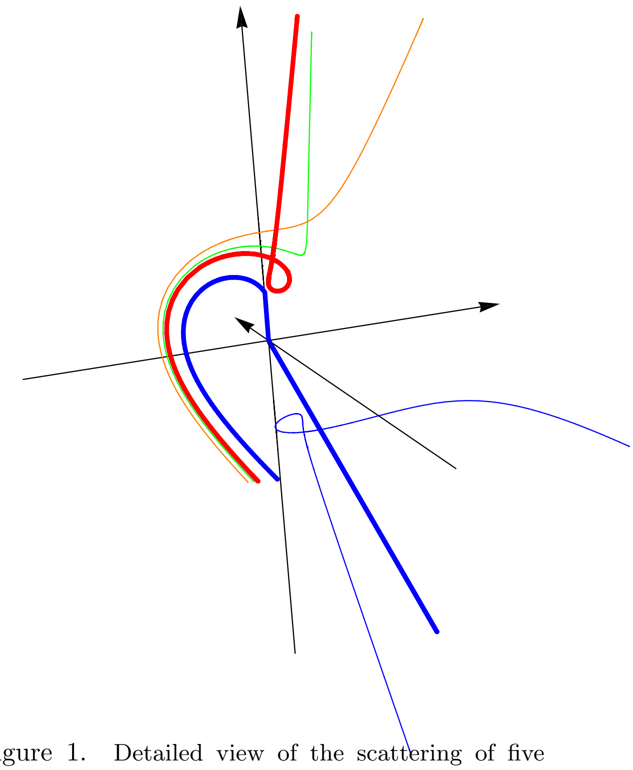

Figure 1. Detailed view of the scattering of five charges, of equal energy, which collide on a double monopole, starting from coplanar points far away. So they differ only for the angular momentum. In particular, with the thick red curve corresponds to the case critico $|\vec{j}\,| = |\theta| + |e|$, in comparison with two others initially close trajectories of greater angular momentum. The thick blue curve corresponds to a particle of $||\theta| - |e|| < |\vec{j}\,| < |\theta| + |e|$. The thicker curves intersect a point such that $M^* = 0$, after that the motion becomes uniform. Finally, the thin curve on the right has the same critical angular momentum, but starting from a specularly symmetric point.

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