Nonlinear Schrödinger systems with nonzero boundary conditions

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Equations of nonlinear Schrödinger (NLS) type are prototypical nonlinear dispersive systems of partial differential equations (PDEs) that play an important role in both mathematics and physics. NLS-type equations have been derived in such diverse fields as deep water waves, plasma physics, nonlinear fiber optics, magnetic spin waves, etc. Many dispersive, energy preserving systems give rise, in appropriate limits, to the scalar NLS equation. In other physical applications, the governing equation is the vector NLS (VNLS) system

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\nu\|\mathbf{q}\|^2\mathbf{q}\,, \tag{1}$$

where $\mathbf{q}(x,t)$ is an N-component complex-valued vector function, $\nu = \pm 1$ denotes the focusing/defocusing cases as before, and $\|\cdot\|$ is the standard Euclidean norm. Here and in the following the boldface font is used to denote vector/matrix functions, while the regular font will be used to denote scalar functions.

Physically, VNLS systems arise under conditions similar to those giving rise to NLS, whenever there are suitable multiple wavetrains moving with nearly the same group velocity. The VNLS also models systems where the electromagnetic field has more than one nonzero component. For example, in optical fibers and waveguides, the electric field has two nonzero polarization components (which for plane waves are transverse to the direction of propagation).

The VNLS system (1) with $N = 2$ was proposed by Manakov in 1974 as an asymptotic model governing the propagation of the electric field envelope in waveguides. Accordingly, (1) with $N = 2$ is commonly referred to as the *Manakov system*. Later, the system was also derived as a model for optical fibers. In optics, the defocusing case $\nu = 1$ corresponds to the normal dispersion regime, while the focusing case $\nu = -1$ to the anomalous dispersion regime.

A number of variants of the NLS equation are also solvable by the Inverse Scattering Transform (IST) method, which is the nonlinear analogue of the Fourier transform for solving the initial value problem for linear PDEs.

The IST for NLS systems with non-zero boundary conditions (NZBCs) is much less developed than in the case of solutions which vanish rapidly at spatial infinity. In particular, even though the IST for the defocusing scalar NLS equation with NZBCs as $x \to \pm\infty$ was formulated in 1973 by Zakharov and Shabat, the development of the IST for the Manakov system with nondecaying potentials remained an open problem for over thirty years, and was only recently solved by us [1].

Already in the scalar case the IST with NZBCs is significantly more complicated than in the case of decaying potentials, due to the fact that the spectral parameter of the associated block-matrix scattering problem is an element of a two-sheeted Riemann surface. However, one still has two complete sets of analytic scattering functions, and the IST can be carried out in a standard way.
When the number of components $N > 1$, however, additional complications arise: $2(N-1)$ out of the $2(N+1)$ scattering eigenfunctions are not analytic on either sheet of the Riemann surface, and one must find a way to complete the basis. The 2-component case (Manakov system) is somehow special. In [1] we have developed the IST for the Manakov system with NZBCs using the adjoint scattering problem to construct two additional analytic eigenfunctions. The inverse scattering problem can be formulated as a generalized Riemann-Hilbert problem with poles in the upper/lower half-planes of a suitable uniformization variable. This construction allowed us to completely characterize the solitonic sector of VNLS in the normal dispersion regime (i.e., in the defocusing case).

The investigation of the soliton solutions is of particular importance. The defocusing NLS does not admit the usual “bell”-shaped soliton solutions. It does, however, possess so-called “dark solitons”. For the scalar defocusing NLS with constant-amplitude BCs $|q(x,t)| \to q_0$ as $x \to \pm\infty$, these are localized dips of intensity propagating on a background field of constant, non-zero amplitude q_0. In the Manakov system with NZBCs, our study of the solitonic sector revealed vector generalizations of the aforementioned dark solitons, exhibiting dark solitonic behavior in both components, as well as novel dark-bright soliton solutions, which have one dark component and one bright component. These dark-bright soliton solutions had been previously obtained by direct methods, but had not been characterized from a spectral point of view before.

The formulation of the IST for the multi-component ($N > 2$) vector NLS system with nonzero boundary conditions was addressed most recently [2]. In this paper, we developed the IST for the defocusing vector NLS equation with an arbitrary number of components, with nonzero boundary conditions at infinity. The technique we successfully applied to the 2-component VNLS does not admit an obvious generalization to an arbitrary number of components. In order to complete the basis of analytic eigenfunctions for the general multicomponent scattering problem, in [2] we generalize the approach suggested by Beals, Deift and Tomei (1988) for general scattering and inverse scattering on the line, but developed under the assumption of vanishing boundary conditions. The key step is the introduction of a fundamental tensor family as solutions of a suitable scattering problem associated to the given one, in such a way that each tensor is sectionally analytic on the cut Riemann surface. Then we show that it is possible to algorithmically reconstruct the fundamental matrices of solutions of the scattering problem from the fundamental tensors, and to establish their analyticity properties.

In [3] we then used the IST machinery to investigate soliton interactions in 2-component VNLS. We have determined the long-time behavior of dark-dark and dark-bright solutions before and after any interactions, and obtained the shifts in the phases and in the soliton centers associated to the interactions.

The results will be relevant from the point of view of physical applications, and they will provide a valuable insight in the study of the interaction of vector solitons with more than 2-components, which we are currently pursuing, as well as in the effort of extending the IST to more general nonzero boundary conditions, which we plan to address in the near future.

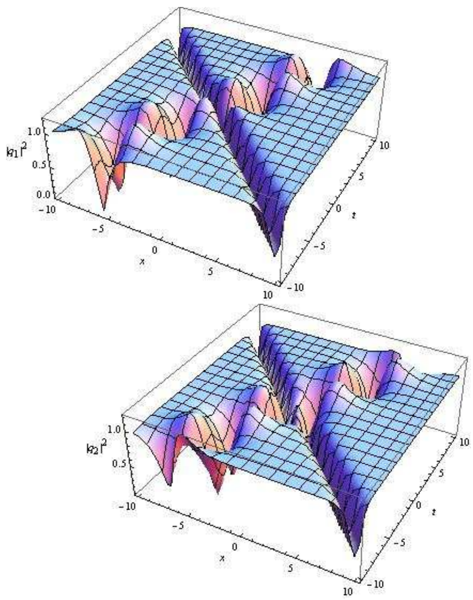

Figure 1. One dark-dark + one dark-bright solitons: $|\mathbf{q}_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). Here the dark and bright parts of the solitons are not separated in different components, i.e., both components of $\mathbf{q}_+$ are nonzero.

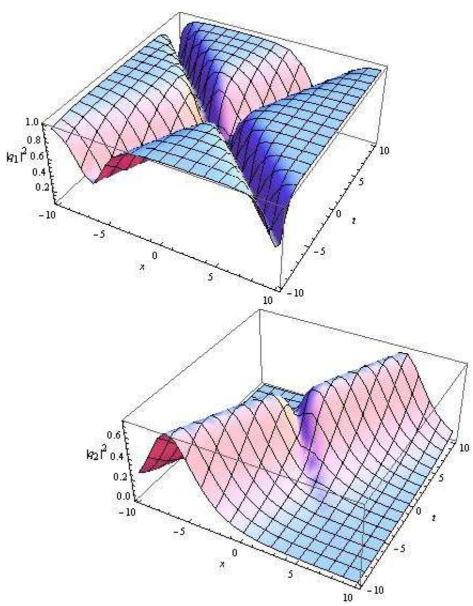

Figure 2. One dark-dark + one dark-bright solitons: $|\mathbf{q}_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). Here the dark and bright parts of the solitons are separated in different components, e.g., $\mathbf{q}_+ = (q_0, 0)^T$.

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tions in the two-component defocusing nonlinear Schrdinger equation", *Applic. Anal.* DOI:10.1080/00036811.2011.618126 (2011).