Three-Wave and Four-Wave Interactions in the $4d$ Einstein Gauss-Bonnet (EGB) and Lovelock Theories

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We derive the symmetries and the constraints satisfied by classical vertices of the general Gauss Bonnet theory around flat space, and its $d = 4$ version (4d GB), obtained by a singular limit of the Euler-Poincarè density. Using a conformal decomposition of the metric, the theory has two versions, which are regularization dependent, a local one which is quartic in the dilaton field, and a nonlocal one, with a quadratic dilaton. The nonlocal version is derived by a finite redefinition of the GB density by an $(d-4)R^2$ correction before the singular $d \to 4$ limit. In the local version we show how the independent dynamics of the metric and of the dilaton are interwinded by a classical trace identity. Three- gravitational wave interactions are derived in the nonlocal Einstein Gauss-Bonnet (EGB) theory, obtained by combining the Einstein action with the topological GB action. The GB interactions of the nonlocal theory, on the other end, are determined by vertices satisfying classical Ward identities that we investigate at cubic and quartic level both in the $4d$ singular limit and in general, for Lovelock actions.

I. INTRODUCTION

The search for modifications of Einstein's theory of General Relativity (GR) that may explain important phenomenological aspects of current cosmology, such as inflation and dark energy, follows several directions. One of them, traditionally, contemplates the inclusion of extra scalar fields into the theory. Such a role is taken by a scalar that drives the metric inflation and couples to all of matter present around the Planck scale, finally decaying into the spectrum of particles, parents of the Standard Model ones. Another modification is the inclusion of a cosmological constant, which fits very well the CMB data within the ΛCDM model, but underscores a huge hierarchy problem.

An interesting class of modified cosmologies are those that include higher powers of the curvature of spacetime, the Riemann tensor, but in a form in which no dimensionfull coupling is present in the action and exhibiting equations of motion of the second order, as for the Einstein-Hilbert (EH) action.

A nice example of these is the Einstein Gauss-Bonnet (EGB) theory at finite GB coupling, which would be ideal for the study of some of these phenomena, were it not that, in four dimensions, the theory is topological. In string theory in $d = 10$ such quadratic corrections get combined in the GB term only in the heterotic case [1]. It was observed that they are deprived of double poles, generated by the second functional derivative of this term $(\sqrt{g}E^{(2)})^{\mu\nu\rho\sigma}$, from the quadratic metric fluctuations around flat space ($\sim O(h^2)$) [2].

We recall that topological contributions in the form of either the Einstein-Hilbert (EH) action at $d = 2$

$$V_{EH}(g,d) \equiv \mu^{\epsilon} \int d^d x \sqrt{g} R, \tag{I.1}$$

$\epsilon = d - 2$, or the Gauss Bonnet action (GB) at $d = 4$, define evanescent terms in the equations of motion of gravity. In $d = 2$ the EH action itself is metric independent.

Evanescent terms can be turned into dynamical contributions by performing a singular limit on the corresponding coupling constant, which are dimensionless. This features is commonly present and held into account in the context of conformal anomaly actions, but recently it has been reproposed in a purely classical context[3]. In the case of $d = 2$, the theory is rendered dynamical by replacing the EH action by the regulated action [4]

$$\mathcal{S}_2 = \lim_{d\to 2} \frac{\int d^d x \left(V_{EH}(g,d) - V_{EH}(\bar{g},d)\right)}{d-2} \tag{I.2}$$

where one introduces a conformal decomposition of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} e^{2\phi} \tag{I.3}$$

in terms of a dilaton (Weyl) factor ϕ and a fiducial metric $\bar{g}$, that amounts to a subtraction. The limiting procedure allows to generate special forms of dilaton gravities, which are closely related to the Weyl variant sector of a conformal anomaly action, and are of Horndeski type.

Eq. (I.2) identifies the ordinary Wess-Zumino (WZ) form of the action. A variant of the subtraction performed either in d dimensions - as in (I.2) - or at $d = 4$ allows to include extra - Weyl invariant - terms. In $d = 2$ a similar arbitrariness allows to characterize the difference between topological and non topological anomalies, or anomalies of types A and B [5].

In the case of the 4d GB theory the coupling, $g_s(d)$, becomes singular as $d \to 4$ and requires a subtraction. The result is a finite, nontopological action, whose structure depends on the subtraction. The method, well-known both in the case of 2-D gravity and of conformal anomaly actions [6–10], where the same procedure is applied to the counterterms in the dimensional regularization (DR) of the theory, has recently regained significant attention, for offering, possibly, a way to evade Lovelock' s theorem [11] in a purely classical framework [3].

The limit is purely geometrical, but induces additional scales, and borrows its features from dimensional regularization (DR). It is unrelated to the regularization of some quantum corrections, as is the case of the conformal anomaly action, derived by integrating out a conformal matter sector, but it opens the way to new interesting developments.

Lovelock's theorem states that, at $d = 4$, the only gravitational action that generates second order equations of motion is the EH action, plus a cosmological constant

$$\mathcal{S}_{EH} = \int d^dx\sqrt{g}(M_P^2 R + 2\Lambda). \tag{I.4}$$

Its generalization to higher dimensions takes the form [11]

$$\mathcal{L}^{(n)} = \frac{n!}{2^{n/2}}\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_n]}_{\nu_n} R_{\mu_1\mu_2}{}^{\nu_1\nu_2} R_{\mu_3\mu_4}{}^{\nu_3\nu_4}\cdots \cdots R_{\mu_{n-1}\mu_n}{}^{\nu_{n-1}\nu_n}, \quad n = 0, 2, 4, \dots . \tag{I.5}$$

$n = 0$ identifies the cosmological constant, $n = 2$ the EH action and $n = 4$ the GB density. Given the antisymmetrization present in its definition, this is nonzero only in specific dimensions. Once the indices are contracted, the resulting density and its coordinate dependence can be extended to any dimension. The reduction of the action to the topological dimension $d = 4$ for E (E_4) is investigated by an embedding of the metric into the extra $(d-4)$ dimensions, a procedure which is not unique [12]. The case $n = 4$, with $d = 4$, defines the GB density

$$V_E(g, d) \equiv \mu^\varepsilon \int d^dx\,\sqrt{-g}\,E, \tag{I.6}$$

where μ is a renormalization scale and E is the integrated Euler-Poincarè density

$$E = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \tag{I.7}$$

whose inclusion modifies the EH action just by boundary contributions, since in an ordinary EGB theory

$$\mathcal{S}_{EGB} = S_{EH} + g_s V_E, \tag{I.8}$$

the GB term is evanescent at $d = 4$. Its contribution to the gravitational equation of motion

$$\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}\right) + g_s(V_E(d))_{\mu\nu} = 0, \tag{I.9}$$

explicitly given by

$$V_E^{\mu\nu} \equiv \frac{\delta V_E}{\delta g_{\mu\nu}} = \sqrt{-g}\bigg(\frac{1}{2}g^{\mu\nu}E_4 - 2R^{\mu\alpha\beta\gamma}R^{\nu}_{\alpha\beta\gamma} + 4R^{\mu\alpha}R^{\nu}{}_{\alpha} + 4R^{\mu\alpha\nu\beta}R_{\alpha\beta} - 2RR^{\mu\nu}\bigg) \tag{I.10}$$

vanishes at $d = 4$ if we use (I.7). In $d > 4$, V_E is not a boundary term, and is indeed contemplated by Lovelock' s theorem as a possible modification of the EH action (see [13, 14]) We are going to investigate the conformal constraints associated with this term in d dimensions, that have not been investigated before. A similar singular limit can be performed at $d = 6$ for the topological invariant E_6, cubic in the curvature, extending the strategy discussed in this work. As we have already discussed in the introduction, such constraints are a natural consequence of the role played by such terms in the context of conformal anomaly actions in every even dimensions.

As mentioned, when expanded around a flat spacetime $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, the operators of highest derivatives $(\Box^2)$ of the GB action, contributing to the quadratic term in the action $(h\Box^2 h)$, cancel out, showing that the theory is free of ghosts. We recall that at quadratic order, the contribution to an action containing the Riemann tensor and its contractions, with arbitrary combinations, is affected by a propagator with double poles in the form

$$\int d^d\sqrt{g}\left((R_{\mu\nu\rho\sigma})^2 + a_1(R_{\mu\nu})^2 + a_2R^2\right) =$$
$$\frac{1}{4}\int d^dx\sqrt{g}\left((a_1+4)h_{\mu\nu}\Box^2 h_{\mu\nu} + (a_2-1)h\Box^2 h\right) +$$
$$O(h^3) \tag{I.11}$$

that vanish if a_1 and a_2 are chosen to reproduce the Euler-Poincarè density.
For general metric background, the analysis of the behaviour of such actions can be performed starting from as conformal decomposition

$$g_{\mu\nu} = \bar{g}_{\mu\nu}e^{2\phi} \tag{I.12}$$

and eliminating on-shell the dilaton using its expression in terms of the entire metric g. The theory becomes nonlocal if the dilaton is removed from the spectrum, as suggested for the conformal anomaly action in [15]. One relies on integrable conformal decompositions, such as the one discussed in [16], recently investigated in connection with the perturbative hierarchy of the conformal Ward identities for a specific 4-point function in [17]. Different nonlocal forms of such actions are possible, which differ - rather nontrivially- by different possible inclusions of Weyl invariant terms. As in the $d = 2$ case, in order to bring the dynamics of V_E down to $d = 4$ from $d > 4$, bypassing its evanescence, one can perform a similar singular rescaling of the coupling

$$g_sV_E \to g_s(d)V_E \qquad g_s(d) \equiv \frac{g_s\mu^\epsilon}{d-4} \tag{I.13}$$

in order to remodulate g_sV_E as a $0/0$ contribution. Obviously, the definition of the $4d$, ($d = 4$) singular limit of the GB theory, requires a specific compactification, which depends on the underlying geometry and is, in general, affected by extra Kaluza-Klein modes. We are essentially performing an infinite coupling limit ($g_s \to g_s(d)$) on the GB term, as we approach the dimension at which the GB contribution is topological.

These variant should be seen as classical modifications of the EH action that resolve the evanescence of a certain topological term.
In the Lovelock's classification of pure gravity theories yielding equations of motion of second order, this procedure is not contemplated and is essentially new. It can be performed in any even dimension in which these topological terms are defined, starting with $d = 2$. In this work we are going to provide the expression of the nonlocal EGB action expanded up to quartic order (4-graviton vertex) in the fluctuations around a classical metric background.
The result is derived elaborating on various previous analysis of the conformal anomaly actions, adapted and simplified for $4d$ EGB theories. Notice that the conformal Ward identities (CWIs) derived for the $\mathcal{S}_{WZ}$ theory carry the same structure of the anomalous CWIs characterizing the quantum anomaly action [18], where vertices of the V_E term, obtained by differentiating this functional, are constrained by the fundamental symmetries. The main difference between a classical and a quantum approach lays in the fact the constraints obtained by the procedure either apply to classical vertices - for $4d$ EGB theories - or to quantum averages of correlation functions if the analysis is performed in a quantum context. In this second case, the classical action in the functional integral $S_o(\chi_i, g)$, where the χ_i's are generic conformal field, is decomposed in terms of the two components $\bar{g}$ and ϕ, corresponding to the fiducial metric and the dilaton field. A similar decomposition can be introduced for the GB term, with equations of motion that are constrained by the "anomaly" of V_E, which is Weyl non-invariant for general d.

A. Singular rescalings and finite subtractions

The singular rescaling of the coupling can be applied to any topological term, such as E_4, E_6 and so on. In practice, the method is sufficient in order to regulate the $0/0$ limit of the $d = 4$ action, though the result, as we have mentioned, depends on the geometry of the compactification. In practical terms, the resulting actions are usually simplified, by neglecting the dependence of the metric on the extra coordinates. This is a procedure that, even if not stated explicitly in the literature, is essentially based on dimensional reduction (DRed), in the form described in [12]. Conformal anomaly effective actions, to which $4d$ GB models are related, are derived by a similar procedure, applied to the GB (V_E) and the Weyl tensor squared (V_{C^2}) counterterms. The latter, in this case, is unnecessary.
The topological evanescence of the V_E contribution is lifted by the procedure, but some ambiguities are encountered, due to the non-unique choice of the background metric against which V_E is calculated, which remains an indetermination of the method.
The derivation of the geometric affective action depends on the specific choice of the fiducial metric and of the subtraction term, here identified in the form of a Wess-Zumino action [19] via a conformal decomposition, although other subtractions are possible. A discussion of this point can be found in

[12].
The regularization of the action that results from (I.13) is not uniquely defined, since the DRed procedure is naturally affected by an integration cutoff. The singular limit of the GB term is investigated by a Weyl rescaling of this term in $d \neq 4$, which introduces a dilaton in the spectrum, and the $\epsilon = d - 4 \to 0$ expansion is performed afterwards, accompanied by the DRed procedure. We will be reviewing this point in the next section.

B. Content of this work

The goal of our work is to identify the constraints satisfied by the classical vertices of the theory, obtained by removing the dilaton from the spectrum, and resorting to a nonlocal description of the 4d GB theory. The local version of such theory, which is given by dilaton gravity, therefore, is replaced by a nonlocal theory when a finite (classical) renormalization of the GB interaction, proportional to an R^2 term, is added to the usual GB action, which is allowed by the singular limit. We identify the constraints satisfied by the vertices of such nonlocal action once its expression is expanded around flat space. These correspond to a set of Ward identities which are naturally satisfied by the Weyl-variant part of a renormalized anomaly action, due to the similarity between such action, which describes the Weyl/conformal anomaly, and the $4d$ EGB theory. We provide a description of such vertices up to quartic order in the fluctuations around a flat background. The conformal constraints satisfied by the vertex derived from $V_E(g, d)$ are valid in d dimension due to the fact that Weyl variation of this term is exactly linear in $(d-4)$. For $d = 4$ the evanescence of the term, as already pointed out, is removed by the inclusion of a subtraction, corresponding to a classical renormalization, and the conformal constraints remain valid once we replace $V_E(g, d)$ by the regulated vertex $\hat{V}_E'$, which describes the WZ form of the action.
Section 4 contains a first principle discussion of the constraints on the equations of motion found for a conformal decomposition in the previous literature, showing that they are a rigorous consequence of symmetry (II.1) that is broken by the subtraction term in the definition of $\hat{V}_E'$. Also in this case, the corresponding constraint, given in (IV.1), is naturally borrowed from the case of the conformal anomaly actions [12].

II. THE LOCAL EGB THEORY AND THE NONLOCAL ACTION

The correctly regulated theory takes the form of a Wess-Zumino (WZ) action, which depends on the regularization procedure and the treatment of the dilatonic field (ϕ). The 0/0 regularization follows closely the 2D case, where the Einstein-Hilbert term is also topological, and the limit is performed by redefining the coupling as $\alpha \to \alpha/(d-2)$.
In general, dilaton effective actions may contain solutions with the conformal factor that need to be stabilized around a certain scale f. Such a scale is the conformal breaking scale. The scale (f) is required in order to redefine the dimensionless conformal factor of the metric g in the conformal decomposition (II.1). The local shift symmetry, which allows to identify a fiducial metric and the dilaton field, via the transformation

$$\phi \to \phi - \sigma, \ \bar{g}_{\mu\nu} \to \bar{g}_{\mu\nu} e^{2\sigma} \tag{II.1}$$

with $\sigma = \sigma(x)$, is indeed broken by the regularization of the Lagrangian in the 0/0 limit. This issue is not present in the nonlocal action, since ϕ can be eliminated in terms of the entire metric, but, as we have already mentioned, one needs to perform an additional finite renormalization of the action in order to reduce the equations of motion for ϕ to a linear form.
A EGB theory is not uniquely defined in such a singular limit, due to several issues, related to the selection of the background metric and to the regularization procedure that it is invoked. The V_E (GB) term can be expanded around $d = 4$ in several ways. One possibility is defined by the ordinary DR-like procedure

$$\frac{1}{\varepsilon} V_E(g, d) = \frac{\mu^\epsilon}{\epsilon} \left(V_E(g, 4) + \epsilon V_E'(g, 4) + O(\varepsilon^2) \right), \tag{II.2}$$

in terms of a single metric g, implicitly defining the GB part of the EGB action in the form

$$V_E' = \frac{1}{\epsilon} \left(V_E(g, d) - V_E(g, 4) \right). \tag{II.3}$$

Note that the subtraction term $V_E(g, 4)$ obviously does not contribute to the dynamics, for being topological, and amounts just to a constant being added to the action, since

$$V_E(4) = \int d^4x \sqrt{g} E = 4\pi \chi_0(\mathcal{M}), \tag{II.4}$$

where $\chi_0(M)$ is the Euler-Poincarè characteristic of a manifold M. Therefore, the evanescence of the GB contribution V_E is related to the fact that at $d=4$ its variation is zero, together with all the classical vertices derived from its functional differentiation

$$V_E^{\mu_1\nu_1\ldots\mu_n\nu_n} = \frac{\delta^n V_E(4)}{\delta g_{\mu_1\nu_1}\ldots\delta g_{\mu_n\nu_n}}. \tag{II.5}$$

The finiteness of the contributions generated by the renormalized vertices $1/\epsilon V_E^{\mu_1\nu_1\ldots\mu_n\nu_n}(d)$ is therefore related to the $O(\epsilon)$ behaviour of (II.5) as $\epsilon \to 0$, as we will discuss next. This generates a finite EGB theory of the form

$$\mathcal{S}_{EGB} = S_{EH} + V_E', \tag{II.6}$$

where V_E' is bound to satisfy the constraint

$$\frac{\delta}{\delta\phi} V_E' = \sqrt{g}E, \tag{II.7}$$

as recognized in the conformal anomaly effective action. Alternatively, the finite action could be defined in the Wess-Zumino (WZ) form

$$\mathcal{S}_{WZ} \equiv \hat{V}_E' = \lim_{d\to 4}\frac{1}{\epsilon}\left(V_E(\bar{g}e^{2\phi},d) - V_E(\bar{g},d)\right) \tag{II.8}$$

that differs from (II.3) by Weyl invariant terms

$$\mathcal{S}_{EGB} = S_{E.H} + \mathcal{S}_{WZ}. \tag{II.9}$$

The different EGB actions that can be generated in the $d\to 4$ limit are all associated with the treatement of the V_E term, a procedure that should be completely defined in DR and with the choice of a specific fiducial metric $\bar{g}$. This would correspond to the choice of a specific scheme, as usually done in Minkowski space. Note that contracting (I.10) with $2g^{\mu\nu}$ gives the relation

$$2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\int d^dy\sqrt{-g}E(y) = \epsilon\sqrt{g}E(x), \tag{II.10}$$

which is at the core of (II.7), since the subtraction term $V_E(\bar{g},d)$ is Weyl independent. Such subtraction is essential in order to generate a 0/0 limit of the topological term and obtain, henceforth, a finite action. This Weyl variation is an exact property of the V_E terms, therefore valid to all orders in ϵ. We will come back to it in a next section, when discussing its implication in the context of Lovelock theory.

Its identification proceeds using the scaling relation

$$\sqrt{-g}E = \sqrt{\bar{g}}e^{(d-4)\phi}\Big\{\bar{E} + (d-3)\bar{\nabla}_\mu\bar{J}^\mu(\bar{g},\phi) + (d-3)(d-4)\bar{K}(\bar{g},\phi)\Big\}, \tag{II.11}$$

where we have defined

$$\bar{J}^\mu(\bar{g},\phi) = 8\bar{R}^{\mu\nu}\bar{\nabla}_\nu\phi - 4\bar{R}\bar{\nabla}^\mu\phi + 4(d-2)(\bar{\nabla}^\mu\phi\bar{\Box}\phi - \bar{\nabla}^\mu\bar{\nabla}^\nu\phi\bar{\nabla}_\nu\phi + \bar{\nabla}^\mu\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi), \tag{II.12}$$

$$\bar{K}(\bar{g},\phi) = 4\bar{R}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 2\bar{R}\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + 4(d-2)\bar{\Box}\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + (d-1)(d-2)(\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi)^2, \tag{II.13}$$

that allows to perform the expansion in ϵ of the form

$$V_E(g,d) = \int d^dx\sqrt{\bar{g}}\left(\bar{E} + \bar{\nabla}_M\bar{J}^M\right) + \epsilon\int d^dx\sqrt{\bar{g}}\phi\left(\bar{E} + \bar{\nabla}_M\bar{J}^M\right) + \epsilon\int d^dx\sqrt{\bar{g}}K. \tag{II.14}$$

The scheme dependence of the regularization comes as a last step, when the integrals present in (II.36) are reduced to $d=4$ from general d dimensions. This can can be obtained by introducing a cutoff (L) in the extra dimensions in the form

$$\frac{1}{d-4}V_E(g,d) = \frac{1}{\epsilon}(L\mu)^\epsilon\int d^4x\sqrt{-g}\;{}_4\bar{E} + (L\mu)^\epsilon\int d^4x\sqrt{-g}\Big[\phi_4\bar{E} - (4G^{\mu\nu}(\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi) + 2(\nabla_\lambda\phi\nabla^\lambda\phi)^2 + 4\Box\phi\nabla_\lambda\phi\nabla^\lambda\phi)\Big], \tag{II.15}$$

where all the terms in the integrands are 4-dimensional and L is a space cutoff in the $d-4$ extra dimensions. L^ϵ is the volume of the extra space. Taking the $\epsilon\to 0$ limit and the conformal separation $g_{\mu\nu} = \bar{g}_{\mu\nu}e^{2\phi}$ for the fiducial metric, we finally derive the expressions

$$\hat{V}_E'(g,\phi) \equiv \mathcal{S}_{WZ} = \frac{1}{\epsilon}\left(V_E(g,d) - V_E(\bar{g},d)\right) = \int d^4x\sqrt{-g}\Big[\phi_4 E - (4G^{\mu\nu}(\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi) + 2(\nabla_\lambda\phi\nabla^\lambda\phi)^2 + 4\Box\phi\nabla_\lambda\phi\nabla^\lambda\phi)\Big]. \tag{II.16}$$

It is easy to show that the use of the regularization in the form given above, by subtracting $V_E(\bar{g},d)$ in d

dimensions - rather than at $d=4$ -
The local action given above is quartic in ϕ, and its structure depends on the chosen fiducial metric.
In summary, it is possible to define a consistent procedure for the extraction of the effective action at $d=4$, from the singular limit of a topological term. The approach can be performed in d dimensions by a 1) rescaling of of the topological density using (II.11), with a metric which is d-dimensional. This implies, obviously, that the dilaton field carries dependence on the extra dimensions. At the last stage, 2) we dimensionally reduce the fields, by allowing only the zero mode of ϕ to survive the compactification procedure, while the extra components of the metric are assumed to be flat. The cutoff L in the size of the extra dimensions is introduced in order to guarantee the convergence of the integral V_E in the $d\to 4$ limit. Finally, 3) we subtract the same term expressed only in terns of the fiducial metric, performing the limit.
As shown above, the subtraction can be performed either as in (II.3) or as in (II.8), the difference between the two being given by Weyl invariant terms, described in [12]. The $\mathcal{S}_{WZ}$ action, which identifies a contribution that we have also called $\hat{V}_E'$ in (II.8), will define our starting action.

A. The $4d$ EGB + R^2 theory

One may proceed by introducing a finite renormalization/extension of the topological term, in order to derive a different version of $\mathcal{S}_{WZ}$, which is quadratic in ϕ, rather than quartic, as given by (IV.1). This is obtained by extending the topological term at $O(\epsilon)$ in the form

$$E_{ext}=E_4+\epsilon\frac{R^2}{2(d-1)^2} \tag{II.17}$$

and the singular limit performed on the functional

$$\tilde{V}_E=\int d^dx\sqrt{g}E_{ext}. \tag{II.18}$$

The effective action is then defined similarly to (IV.1), with $\mathcal{S}_{WZ}$ in (II.16) now redefined by the inclusion of (II.17)

$$\tilde{\mathcal{S}}_E^{WZ}=\frac{1}{\epsilon}\left(\tilde{V}_E(\bar{g}e^{2\phi},d)-\tilde{V}_E(\bar{g},d)\right) \tag{II.19}$$

induced by this additional finite modification of the action.

A direct computation, using the rescaling formula for R^2

$$\begin{aligned}\sqrt{g}R^2=\sqrt{\bar{g}}e^{\epsilon\phi}\Big(&\bar{R}-2(d-1)\bar{\Box}\phi-\\ &(d-1)(d-2)\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi\Big)^2\Big)\end{aligned} \tag{II.20}$$

gives, after an expansion at $O(\epsilon)$

$$\begin{aligned}\frac{1}{\epsilon}\int d^dx\sqrt{g}E_{ext}&=\frac{1}{\epsilon}\int d^4x\sqrt{\bar{g}}\Big(\bar{E}+\bar{\nabla}\cdot\bar{J}\Big)+\\ &\int d^4x\sqrt{\bar{g}}\phi\Big(\bar{E}+\bar{\nabla}\cdot\bar{J}\Big)\\ &+\int d^4x\sqrt{\bar{g}}\Big(\bar{K}+\frac{1}{2(d-1)^2}\Big[\bar{R}-2(d-1)\bar{\Box}\phi-\\ &(d-1)(d-2)\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi\Big]^2\Big).\end{aligned} \tag{II.21}$$

The expression can be simplified by some integration by parts and the omission of boundary terms. Explicitly, one uses

$$\begin{aligned}\phi\bar{\nabla}\cdot\bar{J}=&-8\bar{R}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi+4\bar{R}\bar{\nabla}^\mu\phi\bar{\nabla}_\mu\phi-\\ &4(d-2)\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi\bar{\Box}\phi+4(d-2)\bar{\nabla}^\mu\bar{\nabla}^\nu\phi\bar{\nabla}_\nu\phi\bar{\nabla}_\mu\phi\\ &-4(d-2)(\bar{\nabla}_\mu\bar{\nabla}^\mu\phi)^2+\text{b.t.}\end{aligned} \tag{II.22}$$

and

$$\bar{\nabla}^\mu\bar{\nabla}^\nu\phi\bar{\nabla}_\nu\phi\bar{\nabla}_\mu\phi=-\frac{1}{2}\bar{\nabla}^\mu\phi\bar{\nabla}_\mu\phi\bar{\Box}\phi+\text{b.t.} \tag{II.23}$$

(where b.t. indicates the boundary terms). This gives the modified relation

$$\delta_\phi\int d^dx\sqrt{g}E_{ext}=\epsilon\sqrt{g}(E_{ext}-\frac{2}{d-1}\Box R) \tag{II.24}$$

which can be used in (IV.1) to give

$$\frac{\delta}{\delta\phi}\frac{1}{\epsilon}\tilde{V}_E(g_{\mu\nu},d)=\sqrt{g}\left(E-\frac{2}{3}\Box R+\epsilon\frac{R^2}{2(d-1)^2}\right) \tag{II.25}$$

in (II.16), giving

$$\begin{aligned}\frac{\delta\mathcal{S}_{GB}^{(WZ)}}{\delta\phi}&=\alpha\sqrt{g}\left(E-\frac{2}{3}\Box R\right)\\ &=\alpha\sqrt{\bar{g}}\left(\bar{E}-\frac{2}{3}\bar{\Box}\bar{R}+4\bar{\Delta}_4\phi\right).\end{aligned} \tag{II.26}$$

Note that the redefinition of the GB density ($E \to E_{ext}$) allows to reobtain a rescaling of the combination $E - 2/3\Box R$ as in $d = 4$

$$\sqrt{g}\left(E-\frac{2}{3}\,\Box R\right) \;=\; \sqrt{\bar{g}}\left(\bar{E}-\frac{2}{3}\,\bar{\Box}\bar{R}+4\bar{\Delta}_4\phi\right), \quad \text{(II.27)}$$

where Δ_4 is the fourth order self-adjoint operator, which is conformal invariant when it acts on a scalar function of vanishing scaling dimensions

$$\Delta_4 = \nabla^2{+}2\,R^{\mu\nu}\nabla_\mu\nabla_\nu{-}\frac{2}{3}\,R\Box{+}\frac{1}{3}\,(\nabla^\mu R)\nabla_\mu \quad \text{(II.28)}$$

and satisfies the relation

$$\sqrt{-g}\,\Delta_4\chi = \sqrt{-\bar{g}}\,\bar{\Delta}_4\chi, \quad \text{(II.29)}$$

if χ is invariant (i.e. has scaling equal to zero) under a Weyl transformation, giving

$$\mathcal{S}^{(WZ)}_{GB} = \alpha\int\, d^4x\,\sqrt{-\bar{g}}\,\left\{\left(\overline{E}-\frac{2}{3}\bar{\Box}\overline{R}\right)\phi+2\,\phi\bar{\Delta}_4\phi\right\}. \quad \text{(II.30)}$$

The elimination of ϕ can be performed quite directly. Using the currents

$$J(x) = \bar{J}(x)+4\sqrt{g}\Delta_4\phi(x), \qquad \bar{J}(x)\equiv\sqrt{\bar{g}}\left(\bar{E}-\frac{2}{3}\bar{\Box}\bar{R}\right),$$
$$J(x)\equiv\sqrt{g}\left(E-\frac{2}{3}\Box R\right) \quad \text{(II.31)}$$

and the quartic Green function of Δ_4

$$(\sqrt{-g}\,\Delta_4)_x D_4(x,y) = \delta^4(x,y). \quad \text{(II.32)}$$

We can invert (II.27), obtaining

$$\phi(x) = \frac{1}{4}\int d^4y\,D_4(x,y)(J(y)-\bar{J}(y)). \quad \text{(II.33)}$$

The expression of ϕ in terms of the entire metric g is what defines a conformal decomposition of the metric, which in this case is integrable, in the sense that we can express ϕ covariantly.
$\mathcal{S}_{WZ}$ can be obtained by solving the equation

$$\frac{\delta\mathcal{S}^{(GB)}_{WZ}}{\delta\phi} = J, \quad \text{(II.34)}$$

clearly identified in the form

$$\mathcal{S}_{WZ} = \int d^4x\sqrt{\bar{g}}\left(\bar{J}\phi+2\phi\Delta_4\phi\right). \quad \text{(II.35)}$$

At this stage it is just matter of inserting the on-shell expression of ϕ (II.33) into this equation to obtain the WZ action, in the form

$$\mathcal{S}_{WZ} = \mathcal{S}_{anom}(g)-\mathcal{S}_{anom}(\bar{g}), \quad \text{(II.36)}$$

with

$$\mathcal{S}_{anom}(g) = \frac{1}{8}\int d^4xd^4yJ(x)D_4(x,y)J(y), \quad \text{(II.37)}$$

and a similar expression for $\mathcal{S}_{anom}(\bar{g})$. Using the explicit expression of ϕ, and including the contribution from the rescaled C^2 term, we finally find the nonlocal and covariant anomaly effective action as

$$\mathcal{S}_{\text{anom}}(g) = \frac{1}{8}\int d^4x\sqrt{-g_x}\,\left(E-\frac{2}{3}\Box R\right)_x$$
$$\int d^4x'\sqrt{-g_{x'}}\,D_4(x,x')\left[\frac{b'}{2}\,\left(E-\frac{2}{3}\Box R\right)\right]_{x'}. \quad \text{(II.38)}$$

III. CONSTRAINTS ON V_E IN d DIMENSIONS FOR LOVELOCK GRAVITY

The presence in 4d EGB of a topological term has important implications concerning the structure of such classical contributions. This point can be understood more clearly by discussing the role of the term $g_s(d)V_E$ in the context of the conformal anomaly effective action [12]. This action is naturally derived from a path integral, once we integrate out a conformal sector, and one can show that the counterterm sector - that in this case involves also the square of the Weyl tensor - separately satisfies anomalous conformal Ward identities. The derivation of such identities follows a direct pattern, that consists in writing down the conformal anomaly action in a background metric endowed with conformal Killing (CKVs) vectors. In this section we illustrate the derivation of these constraints that define, in the context of the $4d$ GB theory, the application of this method. This point can be understood geometrically in the following way.
In a local free falling frame of a curved spacetime (i.e. in tangent space), we require that a certain action is endowed with a conformal symmetry, enlarging the usual local Poincarè symmetry of Einstein's theory. Such conformal symmetry of each local frame can be gauged in the form of a general metric that allows CKVs. In the case of a conformal anomaly action, the contribution coming from V_E, as already mentioned, is paired with V_{C^2}. The latter is the only effective counterterm needed in order to remove the singularity of the quantum corrections at $d = 4$. The CWIs of the complete effective action get splitted into three separate contributions: those derived from the

renormalized quantum corrections and those associated with V_E and V_{C^2}. All the three contributions satisfy separate conservation WIs and CWIs. Those corresponding to the finite renormalized quantum corrections, once we perform the flat spacetime limit, are ordinary, while the other two hierarchies, related to V_E and V_{C^2}, are anomalous. The constraints on these functionals come from their response once we perform a variation respect to the conformal factor ϕ.

We detail the derivation.
We recall that the CKVs are solutions of the equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{2}{d} \nabla_\lambda \xi^\lambda g_{\mu\nu}. \tag{III.1}$$

To derive the CWI's in the flat limit, we need to require that the background metric allows CKVs that leave the . We start from the conservation of the conformal current

$$\int d^d x \sqrt{g}\, \nabla_\mu \left(\xi_\nu^{(K)} V_E^{\mu\nu} \right) = 0, \tag{III.2}$$

and, analogously, we can write

$$\int d^d x \sqrt{g}\, \nabla_\mu \left(\xi_\nu V_E^{\mu\nu\mu_1\nu_1} \right) = 0. \tag{III.3}$$

The identity

$$\begin{aligned} &2\delta_{\mu\nu} \eta^{\lambda\rho} \xi_\lambda \partial_\rho \phi = 2\delta_{\mu\nu} e^{2\phi} g^{\lambda\rho} \xi_\lambda \partial_\rho \phi = \\ &2\delta_{\mu\nu} e^{2\phi} \xi^\lambda \partial_\lambda \phi \end{aligned} \tag{III.4}$$

can be used in the covariant derivative

$$\begin{aligned} &\partial_\mu \xi_\nu - 2\xi_\mu \partial_\nu \phi + \partial_\nu \xi_\mu - 2\xi_\nu \partial_\mu \phi + 2\delta_{\mu\nu} \delta^{\lambda\rho} \xi_\lambda \partial_\rho \phi = \\ &\frac{2}{d} \partial_\lambda \xi^\lambda e^{2\phi} \delta_{\mu\nu} + 2\delta_{\mu\nu} e^{2\phi} \xi^\lambda \partial_\lambda \phi \end{aligned} \tag{III.5}$$

that can be writte in the form

$$\begin{aligned} &\partial_\mu \xi_\nu - 2\xi_\mu \partial_\nu \phi + \partial_\nu \xi_\mu - 2\xi_\nu \partial_\mu \phi = \\ &e^{2\phi} [\partial_\mu (e^{-2\phi} \xi_\nu) + \partial_\nu (e^{-2\phi} \xi_\mu)] = \\ &e^{2\phi} (\delta_{\nu\lambda} \partial_\mu + \delta_{\mu\lambda} \partial_\nu) \xi^\lambda, \end{aligned} \tag{III.6}$$

Substituting (III.4) and (III.6) in (III.5) we get (III.1) with the ordinary derivative replacing the covariant ones

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d} \delta_{\mu\nu} \left(\partial \cdot \xi \right), \tag{III.7}$$

Writing explicitly the action of the covariant derivative in the previous equation and taking the flat limit, the we obtain the constraint

$$0 = \int d^d x \left(\partial_\mu \xi_\nu \, V_E^{\mu\nu\mu_1\nu_1} + \xi_\nu \, \partial_\mu \, V_E^{\mu\nu\mu_1\nu_1} \right). \tag{III.8}$$

We recall that ξ_ν satisfies the conformal Killing equation in flat space and by using this equation (III.3) can be re-written in the form

$$0 = \int d^d x \left(\xi_\nu \, \partial_\mu V_E^{\mu\nu\mu_1\nu_1} + \frac{1}{d} (\partial \cdot \xi) \delta_{\mu\nu} \, V_E^{\mu\nu\mu_1\nu_1} \right). \tag{III.9}$$

We use in this previous expression the conservation and trace identities for $V_E^{\mu\nu\mu_1\nu_1}$, that are explicitly given by

$$\begin{aligned} &\partial_\mu V_E^{\mu\nu\mu_1\nu_1}(x, x_1) = \Big(\delta_\mu^{(\mu_1} \delta_\lambda^{\nu_1)} \partial^\nu \delta(x - x_1) - \\ &2\delta^{\nu(\mu_1} \delta_\mu^{\nu_1)} \partial_\lambda \delta(x - x_1) \Big) V_E^{\lambda\mu}(x), \\ &\delta_{\mu\nu} V_E^{\mu\nu\mu_1\nu_1}(x, x_1) = 2(d-4) \big[\sqrt{-g(x)} E(x) \big]^{\mu_1\nu_1}(x_1) - \\ &2\delta(x - x_1) V_E^{\mu_1\nu_1}(x) \end{aligned} \tag{III.10}$$

and the explicit expression of the Killing vector $\xi_\nu^{(C)}$ for the special conformal transformations

$$\begin{aligned} \xi_\mu^{(C)\,\kappa} &= 2x^\kappa \, x_\mu - x^2 \delta_\mu^\kappa \\ \partial \cdot \xi^{(C)\,\kappa} &= 2d \, x^\kappa \end{aligned} \tag{III.11}$$

where $\kappa = 1, \ldots, d$. By using (III.11) in the integral (III.9), we can rewrite that expression as

$$0 = \int d^d x \left[\left(2x^\kappa \, x_\nu - x^2 \delta_\nu^\kappa \right) \partial_\mu \, V_E^{\mu\nu\mu_1\nu_1} + 2 \, x^\kappa \, \delta_{\mu\nu} V_E^{\mu\nu\mu_1\nu_1} \right] \tag{III.12}$$

or

$$\begin{aligned} 0 = \int d^d x \bigg[&\left(2x^\kappa \, x_\nu - x^2 \delta_\nu^\kappa \right) \Big(\delta_\mu^{(\mu_1} \delta_\lambda^{\nu_1)} \partial^\nu \delta(x - x_1) - \\ &2\delta^{\nu(\mu_1} \delta_\mu^{\nu_1)} \partial_\lambda \delta(x - x_1) \Big) V_E^{\lambda\mu}(x) \\ &- 4 \, x^\kappa \, \delta(x - x_1) V_E^{\mu_1\nu_1}(x) \bigg], \end{aligned} \tag{III.13}$$

Integrating by parts, we obtain the expression

$$
\begin{aligned}
&\left(2d\, x_1^\kappa + 2 x_1^\kappa\, x_1^\mu \frac{\partial}{\partial x_1^\mu} + x_1^2 \frac{\partial}{\partial x_{1\kappa}}\right) V_E^{\mu_1\nu_1} + \\
&2\left(x_{1\lambda}\, \delta^{\mu_1\kappa} - x_1^{\mu_1} \delta^\kappa_\lambda\right) V_E^{\lambda\nu_1}(x_1) \\
&+ 2\left(x_{1\lambda}\, \delta^{\nu_1\kappa} - x_1^{\nu_1} \delta^\kappa_\lambda\right) V_E^{\mu_1\lambda}(x_1) = \\
&4(d-4)\int dx\, x^\kappa \left[\sqrt{-g(x)} E(x)\right]^{\mu_1\nu_1}(x_1)
\end{aligned}
\tag{III.14}
$$

that are the special CWIs for $V_E^{\mu_1\nu_1}$. This relation is trivially satisfied just because $V_E^{\mu_1\nu_1}$ in the flat limit vanishes. The non trivial cases arise when we consider the contribution from the three graviton vertex onwards and we have for $n \geq 3$

$$
\begin{aligned}
&\sum_{j=1}^n \left[2x_j^\kappa\left(d + x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right) - x_j^2\, \delta^{\kappa\alpha} \frac{\partial}{\partial x_j^\alpha}\right] \\
&V_E^{\mu_1\nu_1\ldots\mu_n\nu_n}(x_1,\ldots,x_n) + \\
&2\sum_{j=1}^n \left(\delta^{\kappa\mu_j} x_{j\,\alpha} - \delta^\kappa_\alpha x_j^{\mu_j}\right) \\
&V_E^{\mu_1\nu_1\ldots\nu_j\alpha\ldots\mu_n\nu_n}(x_1,\ldots,x_j,\ldots,x_n) + \\
&2\sum_{j=1}^n \left(\delta^{\kappa\nu_j} x_{j\,\alpha} - \delta^\kappa_\alpha x_j^{\nu_j}\right) \\
&V_E^{\mu_1\nu_1\ldots\mu_j\alpha\ldots\mu_n\nu_n}(x_1,\ldots,x_j,\ldots,x_n) = \\
&2^{n+1}(d-4)\int d^dx\, x^\kappa \Big[\sqrt{-g(x)} \\
&E(x)\Big]^{\mu_1\nu_1\ldots\mu_n\nu_n}(x_1,\ldots,x_n),
\end{aligned}
\tag{III.15}
$$

The dilatation CWI is obtained by the choice of the CKV characterising the dilatations

$$
\xi^{(D)}_\mu(x) = x_\mu, \qquad \partial\cdot\xi^{(D)} = d \tag{III.16}
$$

and equation (III.9), for the general case, becomes

$$
\begin{aligned}
0 = \int d^dx \Big\{ & x_\mu\, \partial_\nu V_E^{\mu\nu\mu_1\nu_1\ldots\mu_n\nu_n}(x,x_1,\ldots,x_n) + \\
&\delta_{\mu\nu} V_E^{\mu\nu\mu_1\nu_1\ldots\mu_n\nu_n}(x,x_1,\ldots,x_n)\Big\}.
\end{aligned}
\tag{III.17}
$$

Taking into account the conservation and trace identities satisfied by $V_E^{\mu_1\nu_1\ldots\mu_n\nu_n}$ we obtain the final expression

$$
\begin{aligned}
&\left(n\, d + \sum_{j=1}^n x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right) V_E^{\mu_1\nu_1\ldots\mu_n\nu_n}(x_1,\ldots,x_n) = \\
&2^n(d-4)\int d^dx \left[\sqrt{-g(x)}\, E(x)\right]^{\mu_1\nu_1\ldots\mu_n\nu_n}(x_1,\ldots,x_n),
\end{aligned}
\tag{III.18}
$$

that is non-trivial starting from $n = 3$.

In momentum space these equations are written as

$$
\begin{aligned}
&\left(d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right) V_E^{\mu_1\nu_1\ldots\mu_n\nu_n}(p_1,\ldots,\bar{p}_n) = \\
&2^n(d-4)\left[\sqrt{-g}\, E\right]^{\mu_1\nu_1\ldots\mu_n\nu_n}(p_1,\ldots,\bar{p}_n),
\end{aligned}
\tag{III.19}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{n-1}\left(p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_{j\alpha}} - 2p_j^\alpha \frac{\partial}{\partial p_j^\alpha \partial p_{j\kappa}}\right) V_E^{\mu_1\nu_1\ldots\mu_n\nu_n}(p_1,\ldots,\bar{p}_n) \\
&+ 2\sum_{j=1}^{n-1}\left(\delta^{\kappa\mu_j}\frac{\partial}{\partial p_{j\,\alpha}} - \delta^\kappa_\alpha \frac{\partial}{\partial p_j^{\mu_j}}\right) \\
&V_E^{\mu_1\nu_1\ldots\nu_j\alpha\ldots\mu_n\nu_n}(p_a,\ldots,p_j,\ldots\bar{p}_n) \\
&+ 2\sum_{j=1}^{n-1}\left(\delta^{\kappa\nu_j}\frac{\partial}{\partial p_{j\,\alpha}} - \delta^\kappa_\alpha \frac{\partial}{\partial p_j^{\nu_j}}\right) \\
&V_E^{\mu_1\nu_1\ldots\mu_j\alpha\ldots\mu_n\nu_n}(p_a,\ldots,p_j,\ldots\bar{p}_n) \\
&= -2^{n+1}(d-4) \\
&\left[\frac{\partial}{\partial p_{n\kappa}}\left(\left[\sqrt{-g}\, E\right]^{\mu_1\nu_1\ldots\mu_n\nu_n}(p_1,\ldots,p_n)\right)\right]_{p_n=\bar{p}_n},
\end{aligned}
\tag{III.20}
$$

where $\bar{p}_n = -\sum_{i=1}^{n-1} p_i$ for the conservation of the total momentum. These constraints are directly satisfied in $d \neq 4$ dimensions and are therefore typical of Lovelock's theories of gravity in generic $d \neq 4$ dimensions. The equations are modified by an overall factor both in the lhs and rhs for dimensional reasons, but remain identical to (III.20). In the $d \to 4$ limit the analysis of these equations requires an accurate study of the degeneracy of such such structures.

In the context of anomaly actions, their reduction to $d \to 4$ requires the inclusion of a Weyl invariant sector, which is provided by the finite quantum corrections coming from the renormalized loops of graviton vertices, which are here missing.

A. Conservation identity at $d = 4$ and trace identities at $d \neq 4$

The same vertex satisfies a hierarchy of of conservation identity at $d = 4$ in flat space, starting from a curved background. Trace Ward identities, instead, are valid for this vertex at $d \neq 4$, as we are going to show next. From

$$\nabla_\mu V_E^{\mu\nu}(x)_g = 0, \tag{III.21}$$

expanding the covariant derivative, we obtain the relation

$$\begin{aligned}
&\partial_{\nu_1} V_E^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(x_1,x_2,x_3,x_4) = \\
&-\Bigg[2\left(\frac{\delta\Gamma^{\mu_1}_{\lambda\nu_1}(x_1)}{\delta g_{\mu_2\nu_2}(x_2)}\right)_{g=\delta} \\
&V_E^{\lambda\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(x_1,x_2,x_3,x_4) + (23) + (24)\Bigg] \\
&-\Bigg[4\left(\frac{\delta^2\Gamma^{\mu_1}_{\lambda\nu_1}(x_1)}{\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\right)_{g=\delta} V_E^{\lambda\nu_1}(x_1,x_4) + \\
&(24) + (34)\Bigg],
\end{aligned} \tag{III.22}$$

where

$$\begin{aligned}
&\left(\frac{\delta\Gamma^{\mu_1}_{\lambda\nu_1}(x_1)}{\delta g_{\mu_i\nu_i}(x_i)}\right)_{g=\delta} = \frac{1}{2}\Big(\delta^{\mu_1(\mu_i}\delta^{\nu_i)}_{\nu_1}\,\partial_\lambda\delta_{x_1x_i} + \\
&\delta^{\mu_1(\mu_i}\delta^{\nu_i)}_{\lambda}\,\partial_{\nu_1}\delta_{x_1x_i} - \delta^{(\mu_i}_{\lambda}\delta^{\nu_i)}_{\nu_1}\,\partial^{\mu_1}\delta_{x_1x_i}\Big) \\
&\left(\frac{\delta^2\Gamma^{\mu_1}_{\lambda\nu_1}(x_1)}{\delta g_{\mu_i\nu_i}(x_i)\delta g_{\mu_j\nu_j}(x_j)}\right)_{g=\delta} = \\
&-\frac{\delta_{x_1x_i}}{2}\delta^{\mu_1(\mu_i}\delta^{\nu_i)\epsilon}\Big(\delta^{(\mu_j}_{\epsilon}\delta^{\nu_j)}_{\nu_1}\,\partial_\lambda\delta_{x_1x_j} + \\
&\delta^{(\mu_j}_{\epsilon}\delta^{\nu_j)}_{\lambda}\,\partial_{\nu_1}\delta_{x_1x_j} - \delta^{(\mu_j}_{\lambda}\delta^{\nu_j)}_{\nu_1}\,\partial_\epsilon\delta_{x_1x_j}\Big) + (ij),
\end{aligned} \tag{III.23}$$

that in momentum space becomes

$$\begin{aligned}
&p_{1\,\nu_1}\,V_E^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4) = \\
&= \Big[4\,\mathcal{B}^{\mu_1}{}_{\lambda\nu_1}{}^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3)V_E^{\lambda\nu_1\mu_4\nu_4}(p_1+p_2+p_3,\bar{p}_4) \\
&+ (34) + (24)\Big] \\
&+ \Big[2\,\mathcal{C}^{\mu_1}{}_{\lambda\nu_1}{}^{\mu_2\nu_2}(p_2)V_E^{\lambda\nu_1\mu_3\nu_3\mu_4\nu_4}(p_1+p_2,p_3,\bar{p}_4) + \\
&(23) + (24)\Big].
\end{aligned} \tag{III.24}$$

A similar analysis can be performed for a trace identity. We simply functionally differentiate the anomalous Weyl variation (II.10) multile times and tronasform, in this case, to momentum space, obtaining

$$\begin{aligned}
&\delta_{\mu_1\nu_1}\,V_E^{\mu_1\nu_1\ldots\mu_n\nu_n}(p_1,\ldots,p_n) = \\
&2^{n-1}(d-4)\,\left[\sqrt{-g}E\right]^{\mu_2\nu_2\ldots\mu_n\nu_n}(p_2,\ldots,p_n) \\
&-2\Big[V_E^{\mu_2\nu_2\ldots\mu_n\nu_n}(p_1+p_2,p_3,\ldots,p_n) + \\
&V_E^{\mu_2\nu_2\ldots\mu_n\nu_n}(p_2,p_1+p_3,\ldots,p_n) + \cdots + \\
&V_E^{\mu_2\nu_2\ldots\mu_n\nu_n}(p_2,p_3,\ldots,p_1+p_n)\Big].
\end{aligned} \tag{III.25}$$

This constraint is satisfied by all the vertices extracted from the E_4 term generated from the Lovelock action.

IV. CLASSICAL CONSTRAINTS ON THE EQUATIONS OF MOTION IN THE LOCAL ACTION

The constraints derived in the previous sections, as already mentioned, are obtained by performing the flat spacetime limit of the metric variations, without resorting to a conformal decomposition of the metric itself.
More general constraints are obtained if we instead perform a conformal decomposition and vary the fiducial metric and the conformal factor independently. The separation is consistent with the fact that the subtractions included in the definition of the WZ action introduce a conformal scale. This separation is scale invariant, in the sense that the resulting action is of dimension 4 and deprived of any dimensionful constant. These types of actions are typical of dilaton gravities and can be modified by the addition of extra scale invariant potentials.
In this section we investigate the consistency of the equations of motion discussing their conservation, in the local version of the theory. The same consistency will be missing once we move to the nonlocal theory, obtained by eliminating the dilaton, using the Riegert

decomposition. As we are going to illustrate in a final section, in that case we need to amend the action by Weyl invariant contributions that are necessary to derive the exact expression of the hierarchy. This, in principle, requires an analysis of the $4T$ correlator, correcting the predictions derived from the nonlocal actions with extra Weyl-invariant terms, following an approach that has been already discussed for a simpler correlator, the $TTJJ$ [17]. While this is an important point that will be discussed elsewhere, it is possible to obtain the correct hierarchies satisfied by the 4-point vertices of a 4d GB theory by resorting directly to a previous analysis of the counterterms of the same vertex. This study has been presented in [18].
Coming to the local dilaton-gravity form of the limiting theory, the two fields can be treated as independent, but their equations will be linked by the constraints coming from the anomalous variation of the Euler Poincarè density. For this reason, (II.16) defines a dilaton gravity theory in which the trace of the equations of motion of the gravity metric $\bar{g}$ and that of the conformal factor are related in the form

$$\left(2\bar{g}_{\mu\nu}\frac{\delta}{\delta\bar{g}_{\mu\nu}}-\frac{\delta}{\delta\phi}\right)\mathcal{S}_{GB}^{(WZ)}=-\alpha\sqrt{\bar{g}}\bar{E}. \tag{IV.1}$$

The derivation of this relation is discussed in [12]. Note that the regulated action is, separately, a functional of $\bar{g}$ and ϕ and one can use the relations

$$2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}V_E(g)=\frac{\delta}{\delta\phi}V_E(g)=\epsilon\sqrt{g}E, \tag{IV.2}$$

$$2\bar{g}_{\mu\nu}\frac{\delta}{\delta\bar{g}_{\mu\nu}}V_E(\bar{g})=2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}V_E(\bar{g}),=\epsilon\sqrt{\bar{g}}\bar{E} \tag{IV.3}$$

(using $\bar{g}_{\mu\nu}=g_{\mu\nu}e^{-2\phi}$) and

$$\frac{\delta}{\delta\phi}V_E(\bar{g})=0 \tag{IV.4}$$

to obtain (IV.1).

It is convenient to define the two tensors

$$T^{\mu\nu}=\frac{2}{\sqrt{\bar{g}}}\frac{\delta\mathcal{S}_{WZ}}{\delta\bar{g}_{\mu\nu}}. \tag{IV.5}$$

and

$$T_\phi=\frac{1}{\sqrt{\bar{g}}}\frac{\delta\mathcal{S}_{WZ}}{\delta\phi}. \tag{IV.6}$$

The relation can also be obtained by a direct computation using (II.16)

$$\sqrt{\bar{g}}T_\phi=\sqrt{\bar{g}}\bigg(\bar{E}+8\bar{G}_{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi + 8\bar{\Box}\phi\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi+16\bar{\nabla}_\lambda\bar{\nabla}_\mu\phi\bar{\nabla}^\lambda\phi\bar{\nabla}^\mu\phi - 8\bar{R}_{\mu\nu}\bar{\nabla}^\mu\phi\bar{\nabla}^\nu\phi+8(\bar{\Box}\phi)^2- 8\bar{\nabla}_\mu\bar{\nabla}_\nu\phi\bar{\nabla}^\mu\bar{\nabla}^\nu\phi\bigg)\equiv\sqrt{\bar{g}}\bar{g}_{\mu\nu}T^{\mu\nu} \tag{IV.7}$$

where the last equality follows from the trace of (I.10). Rescalings in the conformal decomposition are typically of the form

$$R^2_{\mu\nu\rho\sigma}=e^{-4\phi}\bigg(\bar{R}^2_{\mu\nu\rho\sigma}-8\bar{R}^{\mu\nu}\bar{\Delta}_{\mu\nu}- 4\bar{R}\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi+4(d-2)\bar{\Delta}^2_{\mu\nu}+ 4\bar{\Delta}^2+8(d-1)\bar{\Delta}\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi+ 2d(d-1)(\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi)^2\bigg) \tag{IV.8}$$

and similar ones. They can be found in [12]. The action is diffeomorphism invariant since ϕ transforms as a scalar under changes of coordinates. We can use this invariance to derive the equation satisfied by the stress energy tensor, by varying the action with respect to the fiducial metric. The Lie derivatives for the scalar field ϕ and the fiducial metric $\bar{g}_{\mu\nu}$ are

$$\begin{cases}\delta_\xi\phi=\xi^\lambda\bar{\nabla}_\lambda\phi\\ \delta_\xi\bar{g}_{\mu\nu}=\bar{g}_{\mu\lambda}\bar{\nabla}_\nu\xi^\lambda+\bar{g}_{\nu\lambda}\bar{\nabla}_\mu\xi^\lambda\end{cases}. \tag{IV.9}$$

giving the variation

$$\begin{aligned}\delta\mathcal{S}_{WZ}&=\int d^dx\left(\delta_\xi\phi\frac{\delta}{\delta\phi}+\delta_\xi\bar{g}_{\mu\nu}\frac{\delta}{\delta\bar{g}_{\mu\nu}}\right)\mathcal{S}_{WZ}\\ &=\int d^dx\left(\xi^\lambda\bar{\nabla}_\lambda\phi\frac{\delta}{\delta\phi}+(\bar{g}_{\mu\lambda}\bar{\nabla}_\nu\xi^\lambda+\bar{g}_{\nu\lambda}\bar{\nabla}_\mu\xi^\lambda)\frac{\delta}{\delta\bar{g}_{\mu\nu}}\right)\mathcal{S}_{WZ}\\ &=\int d^dx\xi^\lambda\left(\bar{\nabla}_\lambda\phi\frac{\delta}{\delta\phi}-2\bar{g}_{\mu\lambda}\bar{\nabla}_\nu\frac{\delta}{\delta\bar{g}_{\mu\nu}}\right)\mathcal{S}_{WZ}.\end{aligned} \tag{IV.10}$$

The condition to be imposed to get the invariance under diffeomorphism for a generic functional such as $\mathcal{S}_{WZ}$ is

$$\left(\bar{\nabla}_\lambda\phi\frac{\delta}{\delta\phi}-2\bar{g}_{\mu\lambda}\bar{\nabla}_\nu\frac{\delta}{\delta\bar{g}_{\mu\nu}}\right)\mathcal{S}_{WZ}=0. \tag{IV.11}$$

This relation can be verified (IV.11). From now to the rest we will omit the "bar" above all the tensor, derivatives ecc. The first variation gives

$$\nabla_\lambda\phi\frac{\delta}{\delta\phi}\mathcal{S}_{WZ} = \nabla_\lambda\phi\Big[E + 8G^{\mu\nu}\nabla_\mu\nabla_\nu\phi +$$
$$8\Box\phi\nabla_\mu\phi\nabla^\mu\phi + 16\nabla_\lambda\nabla_\mu\phi\nabla^\lambda\phi\nabla^\mu\phi$$
$$- 8R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi + 8(\Box\phi)^2 - 8\nabla_\mu\nabla_\mu\phi\nabla^\mu\nabla^\nu\phi\Big]. \tag{IV.12}$$

The second variation is given by

$$- 2g_{\mu\lambda}\nabla_\nu\frac{\delta}{\delta g_{\mu\nu}}\mathcal{S}_{WZ} =$$
$$- 2g_{\mu\lambda}\nabla_\nu\int d^4x\sqrt{-g}\frac{\delta}{\delta g_{\mu\nu}}(\phi E)$$
$$- \nabla_\lambda\phi\Big[8G^{\mu\nu}\nabla_\mu\nabla_\nu\phi + 8\Box\phi\nabla_\mu\phi\nabla^\mu\phi +$$
$$16\nabla_\lambda\nabla_\mu\phi\nabla^\lambda\phi\nabla^\mu\phi - 8R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi$$
$$+ 8(\Box\phi)^2 - 8\nabla_\mu\nabla_\mu\phi\nabla^\mu\nabla^\nu\phi\Big] +$$
$$8R_{\lambda\mu\nu\rho}\nabla^\nu\phi(\nabla^\mu\nabla^\rho\phi - \nabla^\rho\nabla^\mu\phi). \tag{IV.13}$$

Summing (IV.12) and (IV.13), recalling that $[\nabla_\mu, \nabla_\nu]\phi = 0$, we finally get

$$\left(\nabla_\lambda\phi\frac{\delta}{\delta\phi} - 2g_{\mu\lambda}\nabla_\nu\frac{\delta}{\delta g_{\mu\nu}}\right)\mathcal{S}_{WZ} = \nabla_\lambda\phi E -$$
$$2g_{\mu\lambda}\nabla_\nu\int d^4x\sqrt{-g}\frac{\delta}{\delta g_{\mu\nu}}(\phi E). \tag{IV.14}$$

The explicit form of the second term is expressed by

$$- 2g_{\mu\lambda}\nabla_\nu\int d^4x\sqrt{-g}\frac{\delta}{\delta g_{\mu\nu}}(\phi E) = -\nabla_\lambda\phi E -$$
$$4R^{\nu\mu\alpha\beta}\phi\nabla_\mu R_{\lambda\nu\alpha\beta} - 2R^{\nu\mu\alpha\beta}\phi\nabla_\lambda R_{\nu\mu\alpha\beta}$$
$$+ 4\nabla^\mu\nabla^\nu\phi(\nabla_\nu R_{\lambda\mu} - \nabla_\mu R_{\lambda\nu}) -$$
$$4(R_{\lambda\mu\nu\alpha} + R_{\lambda\nu\mu\alpha})\nabla^\alpha\nabla^\mu\nabla^\nu\phi + 4R_{\nu\mu\alpha\beta}R_\lambda{}^{\mu\alpha\beta}\nabla^\nu\phi$$
$$+ 2\phi[\nabla_\lambda, \nabla_\nu]\nabla^\nu R + 2R_{\lambda\nu}\phi\nabla^\nu R. \tag{IV.15}$$

The simplification of this expression requires some intermediate steps. Under a conformal decomposition of the metric, we can express the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ in terms of the Riemann tensor $\bar{R}^\mu{}_{\nu\rho\sigma}$

$$R^\mu{}_{\nu\rho\sigma} = \bar{R}^\mu{}_{\nu\rho\sigma} + \delta^\mu_\sigma\bar{\Delta}_{\rho\nu} - \delta^\mu_\rho\bar{\Delta}_{\sigma\nu} + \bar{g}_{\nu\rho}\bar{\Delta}^\mu_\sigma\sigma - \bar{g}_{\nu\sigma}\bar{\Delta}^\mu_\rho -$$
$$(\delta^\mu_\rho\bar{g}_{\nu\sigma} - \delta^\mu_\sigma\bar{g}_{\nu\rho})\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi, \tag{IV.16}$$

where

$$\bar{\Delta}_{\mu\nu} = \bar{\nabla}_\mu\bar{\nabla}_\nu\phi - \bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi. \tag{IV.17}$$

Since the space we are working in has metric $g_{\mu\nu}$, Bianchi Identities are not granted for $\bar{R}^\mu{}_{\nu\rho\sigma}$. The first Bianchi identity reads

$$R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\sigma\nu} + R^\mu{}_{\sigma\nu\rho} = 0, \tag{IV.18}$$

hence by using (IV.16) we get

$$\bar{R}^\mu_{\nu\rho\sigma} + \bar{R}^\mu_{\rho\sigma\nu} + \bar{R}^\mu_{\sigma\nu\rho} + \delta^\mu_\sigma\bar{\Delta}_{\rho\nu} - \delta^\mu_\rho\bar{\Delta}_{\sigma\nu} +$$
$$\bar{g}_{\nu\rho}\bar{\Delta}^\mu_\sigma\sigma - \bar{g}_{\nu\sigma}\bar{\Delta}^\mu_\rho + \delta^\mu_\nu\bar{\Delta}_{\sigma\rho} - \delta^\mu_\sigma\bar{\Delta}_{\nu\rho} +$$
$$\bar{g}_{\rho\sigma}\bar{\Delta}^\mu_\nu\nu - \bar{g}_{\rho\nu}\bar{\Delta}^\mu_\sigma + \delta^\mu_\rho\bar{\Delta}_{\nu\sigma} - \delta^\mu_\nu\bar{\Delta}_{\rho\sigma} +$$
$$\bar{g}_{\sigma\nu}\bar{\Delta}^\mu_\rho\rho - \bar{g}_{\sigma\rho}\bar{\Delta}^\mu_\nu - (\delta^\mu_\rho\bar{g}_{\nu\sigma} - \delta^\mu_\sigma\bar{g}_{\nu\rho} +$$
$$\delta^\mu_\nu\bar{g}_{\sigma\rho} - \delta^\mu_\rho\bar{g}_{\sigma\nu} + \delta^\mu_\sigma\bar{g}_{\rho\nu} -$$
$$\delta^\mu_\nu\bar{g}_{\rho\sigma})\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi = 0. \tag{IV.19}$$

Invoking symmetry of $\bar{g}_{\mu\nu}$ and $\bar{\Delta}_{\mu\nu}$ we can cancel out all the terms involving ϕ and obtain the Bianchi Identities of the fiducial Riemann tensor

$$\bar{R}^\mu{}_{\nu\rho\sigma} + \bar{R}^\mu{}_{\rho\sigma\nu} + \bar{R}^\mu{}_{\sigma\nu\rho} = 0. \tag{IV.20}$$

Thanks to the above equation, using the Bianchi Identities, the property of the Riemann tensor and

$$[\nabla_\nu, \nabla_\mu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma, \tag{IV.21}$$

it is quite easy to show (IV.11). A similar computation can be performed to derive (IV.1) from (II.16).
It is quite obvious that the singular procedure that takes to a regulated $4d$ EGB action is consistent. This analysis becomes rather nontrivial as the dilaton is removed from the spectrum. As already mentioned, in that case the nonlocal action needs to be amended by extra terms. This will occur at the level of the classical 4-graviton vertex. The 3-graviton vertex, instead, can be handled directly with the nonlocal action. Results of this analysis are given below.

V. 3- AND 4-WAVE INTERACTION IN THE NONLOCAL THEORY

The nonlocal structure of the $4d$ EGB theory results from an iterative solution of the equations of motion in which the dilaton is expressed in terms of the full original metric g, as shown in [20]. One can rewrite the nonlocal action in the form

$$\mathcal{S}_{\rm anom}(g,\varphi) \equiv -\frac{1}{2}\int d^4x\,\sqrt{-g}\,\Big[(\Box\varphi)^2 -$$
$$2\big(R^{\mu\nu} - \tfrac{1}{3}Rg^{\mu\nu}\big)(\nabla_\mu\varphi)(\nabla_\nu\varphi)\Big]$$
$$+\ \frac{1}{2}\int d^4x\,\sqrt{-g}\,\Big[\big(E - \tfrac{2}{3}\Box R\big)\Big]\,\varphi, \tag{V.1}$$

that can be varied with respect to ϕ, giving

$$\sqrt{-g}\,\Delta_4\,\varphi = \sqrt{-g}\left[\frac{E}{2} - \frac{\Box R}{3}\right]. \qquad (V.2)$$

Three-wave interactions can be derived by expanding perturbatively in the metric fluctuations in the form

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} + \dots \equiv \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots \qquad (V.3a)$$

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \dots \qquad (V.3b)$$

The expansion above should be interpreted as a collection of terms generated by setting

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu} \qquad (V.4)$$

having reinstated the coupling expansion κ, with h of mass-dimension one, and collecting all the higher order terms in the functional expansion of (V.1) of the order h^2, h^3 and so on. A similar expansion holds for φ if we redefine $\varphi^{(1)} = \kappa\bar{\varphi}^{(1)}, \varphi^{(2)} = \kappa^2\bar{\varphi}^{(2)}$ and so on. At cubic level the vertex is given by

$$\mathcal{S}^{(3)}_{\text{anom}} = -\frac{b'}{18}\int d^4x\left\{R^{(1)}\frac{1}{\overline{\overline{\Box}}}\left(\sqrt{-g}\Box^2\right)^{(1)}\frac{1}{\overline{\overline{\Box}}}R^{(1)}\right\} + \frac{b'}{9}\int d^4x\left\{\partial_\mu R^{(1)}\frac{1}{\overline{\overline{\Box}}}\left(R^{(1)\mu\nu} - \frac{1}{3}\eta^{\mu\nu}R^{(1)}\right)\frac{1}{\overline{\overline{\Box}}}\partial_\nu R^{(1)}\right\} - \frac{1}{6}\int d^4x\left(b'\,E^{(2)}\right)\frac{1}{\overline{\overline{\Box}}}R^{(1)} + \frac{b'}{9}\int d^4x\,R^{(1)}\frac{1}{\overline{\overline{\Box}}}\left(\sqrt{-g}\Box\right)^{(1)}R^{(1)} + \frac{b'}{9}\int d^4x\,R^{(2)}R^{(1)} \qquad (V.5)$$

where the suffixes (1), (2) denote the order of the expansion in the fluctuations around flat space ($g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$). From (V.5) we can extract the expressions of the classical 3-wave gravitational interactions in this effective theory by differentiating three times with respect to $h_{\mu\nu}$.

A. 3-wave interactions in momentum space

The GB term in the equations of motion induces interactions of higher orders exhibiting specific features, that we are going to identify in this and in the next section. Obviously, cubic and quartic interactions share close similarities with those identified in the nonlocal anomaly action, since they are obtained from those by some direct modifications. At cubic level, the most convenient way to organize such contributions is to transform the expressions to momentum space. For this purpose, we define

$$\int d^4x\,e^{-ip\cdot x}\,R^{(1)}_{\mu\alpha\nu\beta}(x) \equiv \left[R^{(1)}_{\mu\alpha\nu\beta}\right]^{\mu_1\nu_1}(p)\,\tilde{h}_{\mu_1\nu_1}(p) \qquad (V.6)$$

for the linear expansion of the Ricci tensor

$$R^{(1)}_{\mu\alpha\nu\beta} = \frac{1}{2}\left\{-\partial_\alpha\partial_\beta h_{\mu\nu} - \partial_\mu\partial_\nu h_{\alpha\beta} + \partial_\alpha\partial_\nu h_{\beta\mu} + \partial_\beta\partial_\mu h_{\alpha\nu}\right\}, \qquad (V.7)$$

which in momentum space becomes

$$\left[R^{(1)}_{\mu\alpha\nu\beta}\right]^{\mu_1\nu_1}(p) = \frac{1}{2}\left\{\delta_\alpha^{(\mu_1}\,\delta^{\nu_1)}{}_\beta\,p_\mu\,p_\nu + \delta_\mu^{(\mu_1}\,\delta_\nu^{\nu_1)}\,p_\alpha\,p_\beta - \delta_\beta{}^{(\mu_1}\,\delta_\mu^{\nu_1)}\,p_\alpha\,p_\nu - \delta_\alpha^{(\mu_1}\,\delta_\nu^{\nu_1)}\,p_\beta\,p_\mu\right\}. \qquad (V.8)$$

We also require the squared contractions

$$\left[R^{(1)}_{\mu\alpha\nu\beta}R^{(1)\mu\alpha\nu\beta}\right]^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2) \equiv \left[R^{(1)}_{\mu\alpha\nu\beta}\right]^{\mu_1\nu_1}(p_1)\left[R^{(1)\mu\alpha\nu\beta}\right]^{\mu_2\nu_2}(p_2) = (p_1\cdot p_2)^2\,\eta^{\mu_1(\mu_2}\eta^{\nu_2)\nu_1} - 2\,(p_1\cdot p_2)\,p_1{}^{(\mu_2}\eta^{\nu_2)(\nu_1}p_2{}^{\mu_1)} + p_1^{\mu_2}\,p_1^{\nu_2}\,p_2^{\mu_1}\,p_2^{\nu_1} \qquad (V.9)$$

and

$$\left[R^{(1)}_{\mu\nu}R^{(1)\mu\nu}\right]^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2) \equiv \left[R^{(1)}_{\mu\nu}\right]^{\mu_1\nu_1}(p_1)\left[R^{(1)\mu\nu}\right]^{\mu_2\nu_2}(p_2) = \frac{1}{4}\,p_1^2\left(p_2^{\mu_1}\,p_2^{\nu_1}\,\eta^{\mu_2\nu_2} - 2\,p_2{}^{(\mu_1}\eta^{\nu_1)(\nu_2}p_2{}^{\mu_2)}\right) + \frac{1}{4}\,p_2^2\left(p_1^{\mu_2}\,p_1^{\nu_2}\,\eta^{\mu_1\nu_1} - 2\,p_1{}^{(\mu_1}\eta^{\nu_1)(\nu_2}p_1{}^{\mu_2)}\right) + \frac{1}{4}\,p_1^2\,p_2^2\,\eta^{\mu_1(\mu_2}\eta^{\nu_2)\nu_1} + \frac{1}{4}\,(p_1\cdot p_2)^2\,\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} + \frac{1}{2}\,p_1^{(\mu_1}\,p_2^{\nu_1)}\,p_1^{(\mu_2}\,p_2^{\nu_2)} + \frac{1}{2}\,(p_1\cdot p_2)\left(p_1{}^{(\mu_1}\,\eta^{\nu_1)(\nu_2}p_2{}^{\mu_2)} - \eta^{\mu_1\nu_1}\,p_1^{(\mu_2}\,p_2^{\nu_2)} - \eta^{\mu_2\nu_2}\,p_1^{(\mu_1}\,p_2^{\nu_1)}\right). \qquad (V.10)$$

We also use the expressions

$$\left[(R^{(1)})^2\right]^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2) \equiv \left[R^{(1)}\right]^{\mu_1\nu_1}(p_1)\left[R^{(1)}\right]^{\mu_2\nu_2}(p_2) = p_1^2\,p_2^2\,\pi^{\mu_1\nu_1}(p_1)\,\pi^{\mu_2\nu_2}(p_2), \qquad (V.11)$$

and re-express the third order classical vertices of the GB_s action and its contribution S_3 to the three-point correlator in momentum space in the form

$$S_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) = \frac{8}{3}\alpha\Big\{\pi^{\mu_1\nu_1}(p_1) \Big[E^{(2)}\Big]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) + (\text{cyclic})\Big\} - \frac{16\alpha}{9}\Big\{\pi^{\mu_1\nu_1}(p_1)\, Q^{\mu_2\nu_2}(p_1,p_2,p_3)\,\pi^{\mu_3\nu_3}(p_3) + (\text{cyclic})\Big\} + \frac{16\alpha}{27}\,\pi^{\mu_1\nu_1}(p_1)\,\pi^{\mu_2\nu_2}(p_2)\,\pi^{\mu_3\nu_3}(p_3) \Big\{p_3^2\, p_1\cdot p_2 + (\text{cyclic})\Big\}, \tag{V.12}$$

where we have defined

$$\pi^{\mu\nu}(p) \equiv \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \tag{V.13}$$

and

$$Q^{\mu_2\nu_2}(p_1,p_2,p_3) \equiv p_{1\mu}\,[R^{\mu\nu}]^{\mu_2\nu_2}(p_2)\,p_{3\nu} = \frac{1}{2}\Big\{(p_1\cdot p_2)(p_2\cdot p_3)\,\delta^{\mu_2\nu_2} + p_2^2\, p_1^{(\mu_2}\, p_3^{\nu_2)} - (p_2\cdot p_3)\,p_1^{(\mu_2}\,p_2^{\nu_2)} - (p_1\cdot p_2)\,p_2^{(\mu_2}\,p_3^{\nu_2)}\Big\}. \tag{V.14}$$

We have defined with

$$\Big[E^{(2)}\Big]^{\mu_i\nu_i\mu_j\nu_j} = \Big[R^{(1)}_{\mu\alpha\nu\beta}R^{(1)\mu\alpha\nu\beta}\Big]^{\mu_i\nu_i\mu_j\nu_j} - 4\Big[R^{(1)}_{\mu\nu}R^{(1)\mu\nu}\Big]^{\mu_i\nu_i\mu_j\nu_j} + \Big[\big(R^{(1)}\big)^2\Big]^{\mu_i\nu_i\mu_j\nu_j} \tag{V.15}$$

the second functional derivative of the topological density in flat space, after Fourier transform. One can check that

$$\delta_{\alpha_1\beta_1}\, S_3^{\alpha_1\beta_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3)\Big|_{p_3=-(p_1+p_2)} = 8\alpha\,\Big[E^{(2)}\Big]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3), \tag{V.16}$$

together with the conservation identities

$$p_{2\mu_2}\, Q^{\mu_2\nu_2}(p_1,p_2,p_3) = 0 \tag{V.17a}$$

$$p_{2\mu_2}\,\Big[E^{(2)}\Big]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) = 0. \tag{V.17b}$$

Double tracing (V.12) of the nonlocal theory one obtains

$$\delta_{\alpha_1\beta_1}\delta_{\alpha_3\beta_3}\, S_3^{\alpha_1\beta_1\mu_2\nu_2\alpha_3\beta_3}(p_1,p_2,p_3)\Big|_{p_3=-(p_1+p_2)} = 8\alpha\,\delta_{\alpha_3\beta_3}\Big[E^{(2)}\Big]^{\mu_2\nu_2\alpha_3\beta_3}(p_2,p_3) = 16\alpha\, Q^{\mu_2\nu_2}(p_1,p_2,p_3)\Big|_{p_3=-(p_1+p_2)} + 8\alpha\, p_2^2\,\left(p_1^2 + p_1\cdot p_2\right)\pi^{\mu_2\nu_2}(p_2). \tag{V.18}$$

Notice that the expression above is purely polynomial since $p_2^2\pi^{\mu_2\nu_2}(p_2)$ is a local term. Thus in the first line of (V.12) we may substitute (V.16) and in the second line use (V.18) to eliminate the $Q^{\mu_2\nu_2}$ terms and its three cyclic permutations. The triple trace gives

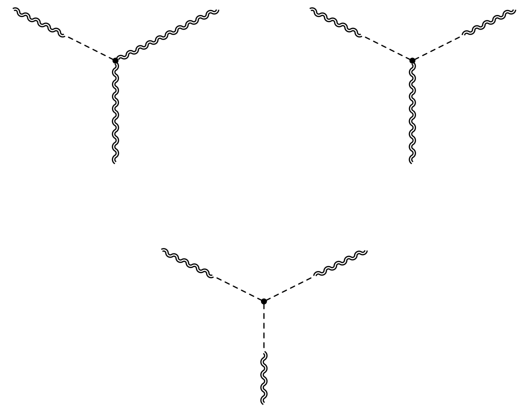

FIG. 1 Mixing in three-point functions

$$\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\, S_3^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3)\Big|_{p_3=-(p_1+p_2)} = 16\alpha\left[p_1^2\, p_2^2 - (p_1\cdot p_2)^2\right]. \tag{V.19}$$

We use these expressions to derive the structure of the 3-wave interaction in the form

$$S_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = \frac{1}{3}\,\pi^{\mu_1\nu_1}(p_1)\,\delta_{\alpha_1\beta_1}\, S_3^{\alpha_1\beta_1\mu_2\nu_2\mu_3\nu_3} + \frac{1}{3}\,\pi^{\mu_2\nu_2}(p_2)\,\delta_{\alpha_2\beta_2}\, S_3^{\mu_1\nu_1\alpha_2\beta_2\mu_3\nu_3} + \frac{1}{3}\,\pi^{\mu_3\nu_3}(p_3)\,\delta_{\alpha_3\beta_3}\, S_3^{\mu_1\nu_1\mu_2\nu_2\alpha_3\beta_3} - \frac{1}{9}\,\pi^{\mu_1\nu_1}(p_1)\,\pi^{\mu_3\nu_3}(p_3)\,\delta_{\alpha_1\beta_1}\delta_{\alpha_3\beta_3}\, S_3^{\alpha_1\beta_1\mu_2\nu_2\alpha_3\beta_3} - \frac{1}{9}\,\pi^{\mu_2\nu_2}(p_2)\pi^{\mu_3\nu_3}(p_3)\,\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\, S_3^{\mu_1\nu_1\alpha_2\beta_2\alpha_3\beta_3} - \frac{1}{9}\,\pi^{\mu_1\nu_1}(p_1)\pi^{\mu_2\nu_2}(p_2)\delta_{\alpha_1\beta_1}\,\delta_{\alpha_2\beta_2}\, S_3^{\alpha_1\beta_1\alpha_2\beta_2\mu_3\nu_3} + \frac{1}{27}\,\pi^{\mu_1\nu_1}(p_1)\pi_2^{\mu_2\nu_2}(p_2)\pi^{\mu_3\nu_3}(p_3)\,\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\, S_3^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}. \tag{V.20}$$

The nonlocal EGB_s theory has a structure that at trilinear level in the fluctuations, similarly to the case of the nonlocal anomaly actions, can be depicted as in Fig. 1. The vertex of the 3-wave is organized in terms of longitudinal insertions of massless states on each of the weavy lines, in a sequence of single, double and triple insertions. The dark blob at the center denotes the polynomial contributions coming from the functional derivatives of the Euler-Poincarè density. Each of the $\pi^{\mu\nu}$ projectors introduces a massless pole

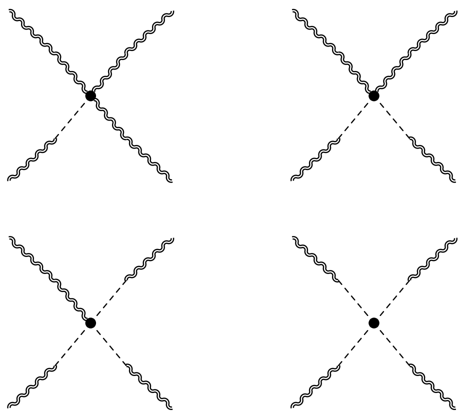

FIG. 2 Mixing in four-point functions

in momentum space, identified from the expression

$$\pi^{\mu\nu} = \frac{1}{p^2}\hat{\pi}^{\mu\nu} \qquad \hat{\pi}^{\mu\nu} = \left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right) \qquad \text{(V.21)}$$

which induce nonlocal corrections on each of the external gravitational metric fluctuations $h_{\mu\nu}$. This picture gets modifed when we move to the case of 4-wave interactions. In that case the nonlocal action and henceforth Fig. 2 does not provide the correct expression of the vertices and one has to resort to a perturbative expansion. While the specific features of these vertices can always be obtained by brute force expansion of the V_E term, there are uncommon. For instance, we have seen that at cubic level, the use of the expansion of the classical vertex, once that we borrow the formalism of the conformal anomaly action to investigate it, it allows us to identify the bilinear mixings on the external wave lines that otherwise would have been left unnoticed. The organization of the vertex in terms of bilinear insertions is quite remarkable.

VI. 4-WAVE INTERACTIONS

The identification of the structure of the 4-wave interactions is far more involved.
Defining the action

$$\mathcal{S} = \lim_{d\to 4}\frac{1}{(d-4)}\left[V_E(g,d) - V_E(g,4)\right] = \lim_{d\to 4}\frac{1}{(d-4)}\left[\int d^dx\,\mu^{d-4}\,E_4 - \int d^4x\,E_4\right], \qquad \text{(VI.1)}$$

one obtains the contribution to the 4-point function as

$$\mathcal{S}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4) = \mathcal{S}_{pole}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4) + \mathcal{S}_{0-trace}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4). \qquad \text{(VI.2)}$$

It is worth mentioning that the subtraction in (VI.1) is necessary to have a finite functional variation with respect to the metric fluctuation in the limit $d \to 4$, as also pointed out in [21, 22]. This fact is reflected in the presence of 0/0 piece, due to Lovelock tensor identities, that can be eliminated once the subtraction (VI.1) is taken into account. There is a difference in the handling expressions such as (VI.1) and performing functional derivatives with respect to the metric if the indices are contracted or not. For instance, a differentiation with respect to the ϕ can commute with the limit, and indeed reproduces the anomaly contribution, but a differentiation with open indices, followed by the flat Minkowski limit $g \to \delta$, needs special care. It can be computed either by performing the limit as a first step and identifying the finite expression of the functional, which is given by
In (VI.2) the zero trace part has the property

$$\delta_{\mu_i\nu_i}\mathcal{S}_{0-trace}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4) = 0, \quad i = 1,2,3,4, \qquad \text{(VI.3)}$$

and it is explicitly written as

$$\begin{aligned}
&\mathcal{S}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4)_{0-trace} = \\
&= \Big\{\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\mathcal{I}^{\mu_3\nu_3}_{\alpha_3}(p_3) \\
&\mathcal{I}^{\mu_4\nu_4}_{\alpha_4}(p_4)\,p_{1\beta_1}\,p_{2\beta_2}\,p_{3\beta_3}\,p_{4\beta_4} \\
&\quad + \Big[\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\mathcal{I}^{\mu_3\nu_3}_{\alpha_3}(p_3) \\
&\mathcal{I}^{\mu_4\nu_4}_{\alpha_4}(p_4)\,p_{2\beta_2}\,p_{3\beta_3}\,p_{4\beta_4} + (\text{perm.})\Big] \\
&\quad + \Big[\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\mathcal{I}^{\mu_3\nu_3}_{\alpha_3}(p_3) \\
&\mathcal{I}^{\mu_4\nu_4}_{\alpha_4}(p_4)\,p_{3\beta_3}\,p_{4\beta_4} + (\text{perm.})\Big] \\
&\quad + \Big[\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3) \\
&\mathcal{I}^{\mu_4\nu_4}_{\alpha_4}(p_4)\,p_{4\beta_4} + (\text{perm.})\Big]\Big\} \\
&\mathcal{S}_4^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}(p_1,p_2,p_3,\bar{p}_4),
\end{aligned} \qquad \text{(VI.4)}$$

where the contractions are expressed in terms of the contribution of the third and second functional derivatives as

$$p_{1\,\beta_1}\,\mathcal{S}_4^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}(p_1,p_2,p_3,\bar{p}_4) =$$
$$= \Big[4\,\mathcal{B}^{\alpha_1}{}_{\lambda_1\beta_1}{}^{\alpha_2\beta_2\alpha_3\beta_3}(p_2,p_3)$$
$$\mathcal{S}_2^{\lambda_1\beta_1\alpha_4\beta_4}(p_1+p_2+p_3,\bar{p}_4) + (34) + (24)\Big]$$
$$+\Big[2\,\mathcal{C}^{\alpha_1}{}_{\lambda_1\beta_1}{}^{\alpha_2\beta_2}(p_2)$$
$$\mathcal{S}_3^{\lambda_1\beta_1\alpha_3\beta_3\alpha_4\beta_4}(p_1+p_2,p_3,\bar{p}_4) + (23) + (24)\Big]. \quad \text{(VI.5)}$$

We have defined the transverse traceless projector

$$\Pi^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\pi^\mu_\alpha\pi^\nu_\beta + \pi^\mu_\beta\pi^\nu_\alpha\right) - \frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta}, \quad \text{(VI.6)}$$

and the tensor

$$\mathcal{I}^{\mu\nu}_\alpha = \frac{1}{p^2}\left(p^\mu\delta^\nu_\alpha + p^\nu\delta^\mu_\alpha - \frac{p_\alpha}{d-1}(\delta^{\mu\nu} + (d-2)\frac{p^\mu p^\nu}{p^2}\right). \quad \text{(VI.7)}$$

On the other hand, the pole part is then explicitly given as

$$\mathcal{S}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4)_{poles} =$$
$$= \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\delta_{\alpha_1\beta_1}\mathcal{S}_4^{\alpha_1\beta_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4) +$$
$$(perm.) - \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}$$
$$\mathcal{S}_4^{\alpha_1\beta_1\alpha_2\beta_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4) + (perm.)$$
$$+\frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\,\delta_{\alpha_2\beta_2}\,\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}$$
$$\mathcal{S}_4^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4) + (perm.)$$
$$-\frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\,\frac{\pi^{\mu_4\nu_4}(p_4)}{3}$$
$$\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4}\mathcal{S}_4^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}(p_1,p_2,p_3,\bar{p}_4), \quad \text{(VI.8)}$$

where the single trace is expressed in terms of the third functional derivative

$$\delta_{\alpha_1\beta_1}\,\mathcal{S}_4^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}(p_1,p_2,p_3,p_4) =$$
$$8\,\left[\sqrt{-g}E\right]^{\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}(p_2,p_3,p_4)$$
$$-2\Big[\mathcal{S}_3^{\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}(p_1+p_2,p_3,p_4) +$$
$$\mathcal{S}_3^{\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}(p_2,p_1+p_3,p_4) +$$
$$\mathcal{S}_3^{\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}(p_2,p_3,p_1+p_4)\Big], \quad \text{(VI.9)}$$

and more trace involve lower orders of functional derivatives.

VII. CONCLUSIONS

In this work we have investigated the nonlinear constraints emerging in a nonlocal 4d EGB theory for the 3- and 4-point classical interactions present in its fundamental action. They are hierarchical and directly linked to to the topological properties of the vertex V_E and its subtracted expression $\hat{V}'_E$, valid in $d \neq 4$ and $d = 4$ dimensions respectively.
The analysis is essentially built on several previous studies of conformal anomaly actions, that allows to identify several nontrivial features of this specific theory. Among these, the presence of bilinear mixing in their external legs of graviton vertices and of exra, traceless contributions, which are derived from the decomposition of such interactions and are not predicted by the nonlocal action. in terms of transverse-traceless, longitudinal and trace contributions.
The nonlocal version of such a theory, as pointed out, is derived by a finite renormalization of the topological density, that allows to remove the dilaton from the spectrum. Three-wave interactions are naturally derived from the nonlocal action, but the hierarchical constraint of the four-wave interactions requires a different approach, given the limitations of such actions in reproducing the correct flat spacetime limit. These constraint are satisfied also in the case of Lovelock actions. Indeed it is possible to generalize them to cases involving topological invariants of higher orders in the Riemann tensors extending the approach of this work. We hope to return to this point in future work.

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