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Graviton vertices and the mapping of anomalous correlators to momentum space for a general conformal field theory

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Abstract: We investigate the mapping of conformal correlators and of their anomalies from configuration to momentum space for general dimensions, focusing on the anomalous correlators TOO, TVV — involving the energy-momentum tensor (T) with a vector (V) or a scalar operator (O) — and the 3-graviton vertex TTT. We compute the TOO, TVV and TTT one-loop vertex functions in dimensional regularization for free field theories involving conformal scalar, fermion and vector fields. Since there are only one or two independent tensor structures solving all the conformal Ward identities for the TOO or TVV vertex functions respectively, and three independent tensor structures for the TTT vertex, and the coefficients of these tensors are known for free fields, it is possible to identify the corresponding tensors in momentum space from the computation of the correlators for free fields. This works in general d dimensions for TOO and TVV correlators, but only in 4 dimensions for TTT, since vector fields are conformal only in $d = 4$. In this way the general solution of the Ward identities including anomalous ones for these correlators in (Euclidean) position space, found by Osborn and Petkou is mapped to the ordinary diagrammatic one in momentum space. We give simplified expressions of all these correlators in configuration space which are explicitly Fourier integrable and provide a diagrammatic interpretation of all the contact terms arising when two or more of the points coincide. We discuss how the anomalies arise in each approach. We then outline a general algorithm for mapping correlators from position to momentum space, and illustrate its application in the case of the VVV and TOO vertices. The method implements an intermediate regularization — similar to differential regularization — for the identification of the integrands in momentum

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space, and one extra regulator. The relation between the ordinary Feynman expansion and the logarithmic one generated by this approach are briefly discussed.

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Contents

1 Introduction

The analysis of correlation functions in d-dimensional quantum field theory possessing conformal invariance has found widespread interest over the years (see [19] for an overview). Given the infinite dimensional character of the conformal algebra in 2-dimensions, conformal field theories (CFT's) in 2-dimensions have received the most attention, although 4-dimensional conformal theories have also been studied (see for instance [31, 32]). In d dimensional CFT's the structure of generic conformal correlators is not entirely fixed just by conformal symmetry, but for 2- and 3-point functions the situation is rather special and these can be significantly constrained, up to a small number of constants.

In several recent works [3, 4, 21] certain correlation functions describing the interaction between a gauge theory and gravity with massless fields in the internal loop and related therefore to the axial and trace anomalies in these theories have been analyzed. The interesting property that such anomalous amplitudes contain massless poles in 2-particle intermediate states has been exposed in these investigations. In particular this has been demonstrated in the TVV amplitude in massless QED and QCD, characterized by the insertion of the energy momentum tensor (T) on 2-point functions of vector gauge currents (V). This amplitude gives the leading order contribution to the interaction between a gauge theory and gravity, mediated by the trace anomaly.

The complete evaluation of this amplitude in the Standard Model [12] confirms the conclusion of [21], namely the presence of an effective massless scalar "dilaton-like" degree of freedom in intermediate 2-particle states intimately connected with the trace anomaly, in the sense that the non-zero residue of the pole is necessarily proportional to the coefficient of the anomaly. The perturbative results of [3, 4, 21] are also in agreement with the anomaly-induced gravitational effective action in 4 dimensions whose non-local form was found in [30], and whose local covariant form necessarily implies effective massless scalar degree(s)

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of freedom [24–26]. This is the 4-dimensional analog of the anomaly-induced action in 2-dimensional CFT's coupled to a background metric generated by the 2-dimensional trace anomaly and related to the central term in the infinite dimensional Virasoro algebra [28]. The anomaly-induced scalar in the 2-dimensional case is the Liouville mode of non-critical string theory on the 2-dimensional world sheet of the string.

In even dimensions greater than 2 it is important to recognize that the anomaly-induced effective action discussed in [24–26, 30] is determined only up to Weyl invariant terms. The full quantum effective action is not determined by the trace anomaly alone, and hence only when certain anomalous contributions to the TVV or other amplitudes are isolated from their non-anomalous parts should any comparison with the anomaly-induced effective action be made. The non-anomalous components are dependent upon additional Weyl invariant terms in the quantum effective action and thus even in the CFT limit need not agree with the anomaly-induced action, without contradicting its validity for determining the anomalous terms [24]. On the other hand these additional Weyl invariant terms for simple amplitudes such as TVV can be determined in principle by the Ward identities of $\mathrm{SO}(d, 2)$ conformal invariance, together with those of gauge invariance for the vector currents. Other triangle amplitudes in 4 dimensions such as the graviton-fermion-antifermion vertex function, for which similar considerations should apply have been investigated primarily for phenomenological reasons [14], although this amplitude is anomaly-free.

From the CFT side some important information is available [10, 18, 27]. These results concern the TOO — with O denoting a generic scalar — TVV and TTT vertices, which are determined by applying the conformal Ward identities in Euclidean position space. Some of the vertices, such as the TTT, for $d = 4$ are shown in the analysis of [18, 27] to be expressible in terms of three linearly independent tensor structures. Imposing the conformal Ward identities and identifying these tensor structures directly in momentum space turns out the be technically quite involved. The main goal of the present work is to initiate a systematic study enabling comparison of general results of 4-dimensional CFT's based on position space analysis such as [18, 27] with explicit realizations of anomalous 3-point vertices in free field theory, most commonly expressed in momentum space. Recent results of studies of three- and four- point functions in $d = 3$ in the context of the ADS_4/CFT_3 correspondence are contained in [9, 23, 29].

For general d dimensions and, specifically, in $d = 4$, rather than trying to identify these tensor structures directly in momentum space, which is quite cumbersome, it is much simpler to calculate explicitly the TTT correlator for specific free-field theories of scalars, spinors, and vectors in one-loop Lagrangian perturbation theory, thereby identifying the three linearly independent tensor structures *a posteriori* with the general CFT analysis of [18, 27]. A similar method works for the TVV, VVV and TOO vertices for any dimension, while in the TTT case the contribution coming from the exchange of a spin 1 field in the loop diagrams is conformally invariant only in $d = 4$.

While the imposition of the conformal Ward identities is technically simpler in position space, the appearance of massless poles associated with anomalies is very much obscured. Indeed conformal anomalies necessarily arise quite differently in momentum space and in

Euclidean position space, where the only possibility for anomalous terms lies in appearance of ultralocal divergences proportional to delta functions or derivatives thereof at cooincident spatial points. Thus a very careful regularization procedure is required to determine these anomalous ultralocal contributions which are absent for any finite point separation. The special strategy followed in determining these anomalous ultralocal contributions in position space, developed in [27], merits some comments for its peculiarity. In [18, 27] the Ward identities are solved in each case by combining a homogeneous solution — obtained for separate (non-coincident) points of the correlator — with inhomogenous terms, identified via a regularization of the same correlator in the coincidence limit and with the inclusion of contact terms. The contact terms proportional to delta functions and derivatives thereof determine the anomalies. Such a separation, based on homogeneous and inhomogeneous terms in the Ward identities cannot be easily carried out in momentum space. Moreover in the approach of [21] the origin of the conformal anomaly as an *infrared* effect (rather than a result of any UV regularization procedure) following from the imposition of all non-anomalous Ward identities and the spectral representation of the amplitude was emphasized. In this approach massless anomaly poles at $k^2 = 0$ play an essential role. At first glance this appears to be quite different than the ultralocal delta function terms obtained in the position space approach of [18, 27]. Thus the relationship of the several approaches requires some clarification, and this is a principal motivation for the present work. The eventual agreement of the two approaches may seem less surprising if it is remembered that cooincident point singularities in Euclidean position space become light cone singularities in Minkowski spacetime, and these lightcone singularities are associated with the propagation of massless fields, which generally have long range infrared effects.

Our work is composed of two main parts. In the first part, building on the results of [18, 27], we compute the complete structure of the 3-point correlators in configuration and in momentum space for a general CFT. In particular we generalize our previous studies of the TVV correlator, formally studied by us in 4 dimensions [3, 4, 21] in QED and QCD, to d dimensions and for any CFT. We also study the TTT vertex and perform a complete investigation of this correlator by the same approach. The analysis is performed in perturbation theory and the result is secured by a successful test of all the Ward identities satisfied by this vertex, outlining their derivation and their perturbative implementation, and using a symbolic manipulation program written by us. Both for the TVV and TTT cases our computations have been performed under the most general (off-shell) conditions, but the remarkable complexity of the general result allows us to present here, in a compact form, only the expression for the 2-particle on-shell case. We give particular emphasis to the discussion of the connection between the general approach of [27] and the perturbative picture. In particular, we give a diagrammatic interpretation of the various contact terms introduced by Osborn and Petkou in order to solve the Ward identities for generic positions of the points of the correlators. This allows to close a gap between the bootstrap method of [27], our previous investigations of the TVV [3, 4, 21], and the current study of the TTT vertex. We show that the perturbative analysis in momentum space in dimensional regularization is in complete agreement with their results.

It should be remarked that, in general, the momentum space formulation of the correlators of a CFT remains largely unexplored, since in many cases there is no Lagrangian description which may justify such an effort, and the spacetime formulation remains the only significant one. The use of symmetry principles to infer the general solution to conformal Ward identities from some specific correlation functions computed in momentum space perturbation theory, allows to collect information about a conformal theory even when a Lagrangian formulation of the same correlators is not readily found or may not exist at all.

This brings us to the second part of our work, contained in section 8, where we discuss a general and very efficient procedure to map to momentum space any massless correlator, not necessarily related to a Lagrangian description. This part is motivated by the attempt of transforming to momentum space any massless correlator given in position space, independently from whether this is Fourier integrable or not.

The investigation of these correlators in momentum space reveals, in general, some specific facts, such as the presence of single and multi-logarithmic integrands which, in general, can't be re-expressed in terms of ordinary master integrals, typical of the Feynman expansion. To address these points, one has to formulate an alternative and general approach to perform the transforms, *not directly linked to the free-field realization*, since in this case such representation, as we have just mentioned, may not exist.

The method that we propose combines a d-dimensional version of differential regularization, similar to the approach suggested in [18, 27]. In our case we use the standard technique of "pulling out" derivatives (via partial integration) in very singular correlators in such a way to make them Fourier integrable, i.e. expressible as integrals in momentum space which are well-defined for non-coincident points. This is combined with *the method of uniqueness* [22], here generalized to tensor structures, in order to formulate a complete and self-consistent procedure. As in [18, 27] we need an extra regulator (ω), unrelated to the dimensional regularization parameter (ϵ). Our approach is defined as a generic algorithm which can handle rather straightforwardly any massless correlator written in configuration space. The algorithm has been implemented in a symbolic manipulation program and can handle correlators of any rank.

The aim of the method is to test the Fourier integrability of a given correlator, by checking the cancellation of the singularities in the extra regulator directly in momentum space, and to provide us with the direct expression of the transform. After a few non trivial examples, we will show how to reproduce, by this method, some of the results of the conformal correlators discussed in the first part, the VVV and the TOO being two examples.

Given the large space and scope of this analysis, which is technically quite involved, we will not attempt in this work to address the issue of the presence of anomaly poles in the TTT correlator in analogy to what discussed in [3, 4, 21] for the TVV case. Although this is an important motivation for initiating this study, demonstrating the existence of the pole(s) requires additional analysis which we do not attempt in this paper. We expect to address this final point in a related work making use of the technical framework and building upon the results of the present study.

I	$\beta_a(I) \times 2880\,\pi^2$	$\beta_b(I) \times 2880\,\pi^2$	$\beta_c(I) \times 2880\,\pi^2$
S	$\frac{3}{2}$	$-\frac{1}{2}$	-1
F	9	$-\frac{11}{2}$	-6
V	18	-31	-12

Table 1. Anomaly coefficients for a conformally coupled scalar, a Dirac Fermion and a vector boson

2 Conformal correlators and the trace anomaly

2.1 Conventions and the trace anomaly equation

Before coming to a discussion of the main correlators investigated in our work we introduce here our definitions and conventions which will be used throughout.

The basic trace anomaly equation for a conformal theory in $d = 4$ is [15, 16]

$$g_{\mu\nu}(z)\,\langle T^{\mu\nu}(z)\rangle = \sum_{I=f,s,V} n_I \Big[\beta_a(I)\,F(z) + \beta_b(I)\,G(z) + \beta_c(I)\,\Box R(z) + \beta_d(I)\,R^2(z)\Big] + \frac{\kappa}{4} n_V\,F^{a\,\mu\nu}\,F^a_{\mu\nu}(z) \equiv \mathcal{A}(z,g)\,, \tag{2.1}$$

whose coefficients β_a, β_b, β_c and β_d depend on the field content of the Lagrangian (fermion, scalar, vector) and we have a multiplicity factor n_I for each particle species.[1] Actually the coefficient of R^2 must vanish identically

$$\beta_d \equiv 0 \tag{2.2}$$

since a non-zero R^2 in this basis cannot be obtained from any effective action (local or not) [2, 8, 24]. In addition, the value of β_c is regularization dependent, corresponding to the fact that it can be changed by the addition of an arbitrary local R^2 term in the effective action. In particular, the values for β_c reported in table 1 hold in dimensional regularization. Thus only β_a, β_b and κ correspond to true anomalies in trace of the stress tensor. In dimensional regularization one finds

$$\beta_c = -\frac{2}{3}\,\beta_a\,. \tag{2.3}$$

In table 1 we list the values of the coefficients for the three theories of spin $0, \frac{1}{2}, 1$ mentioned, that we are going to consider extensively throughout the paper. $\mathcal{A}(z,g)$ contains the field-strength of the background gauge field, $F^a_{\mu\nu}$, and the invariants built out of the Riemann tensor, $R^\alpha{}_{\beta\gamma\delta}$, as well as the Ricci tensor $R_{\alpha\beta}$ and the scalar curvature R. G and F in eq. (2.1) are the Euler density and the square of the Weyl tensor respectively.

All our conventions are listend in appendix A.

Eq. (2.1) plays the role of a generating functional for the anomalous Ward identities of any underlying Lagrangian field theory. These conditions are not necessarily linked

[1] Equivalent and more popular notations are $c \equiv 16\pi^2\beta_a$ and $a \equiv -16\pi^2\beta_b$.

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to any Lagrangian, since the solution of these and of the other (non anomalous) Ward identities — which typically define a certain correlator — are based on generic requirements of conformal invariance. For our purposes, all these identities can be extracted from an ordinary generating functional, defined in terms of a generic Lagrangian $\mathcal{L}$ which offers a convenient device to identify such relations. For this reason we introduce the ordinary definition of the energy-momentum tensor

$$T^{\mu\nu}(z) = -\frac{2}{\sqrt{g_z}}\frac{\delta\,\mathcal{S}}{\delta g_{\mu\nu}(z)} = g^{\mu\alpha}(z)\,g^{\nu\beta}(z)\,\frac{2}{\sqrt{g_z}}\,\frac{\delta\mathcal{S}}{\delta g^{\alpha\beta}(z)}\,, \tag{2.4}$$

in terms of the quantum action $\mathcal{S}$, so that its quantum average is

$$\langle T^{\mu\nu}(z)\rangle = \frac{2}{\sqrt{g_z}}\frac{\delta\,\mathcal{W}}{\delta\,g_{\mu\nu}(z)}\,, \tag{2.5}$$

(with $\det g_{\mu\nu}(z) \equiv g_z$) where $\mathcal{W}$ is the Euclidean generating functional of the theory[2]

$$\mathcal{W} = \frac{1}{\mathcal{N}}\int \mathcal{D}\Phi\, e^{-\,\mathcal{S}}\,, \tag{2.6}$$

where $\mathcal{N}$ a normalization factor and Φ denotes all the quantum fields of the theory.

Inserting these definitions in (2.1) and multiplying both sides by $\sqrt{g_z}$ we obtain

$$2\,g_{\mu\nu}(z)\frac{\delta\,\mathcal{W}}{\delta\,g_{\mu\nu}(z)} = \sqrt{g_z}\,\mathcal{A}(z,g)\,. \tag{2.7}$$

From (2.1) and (2.7) we can extract an identity for the anomaly for correlators involving n insertions of energy momentum tensors, by taking n functional derivatives with respect to the metric of both sides of (2.7) and setting $g_{\mu\nu} = \delta_{\mu\nu}$ at the end. In the same way, the anomalous Ward identity for the TVV can be obtained by functional differentiation of the same equation respect to the background gauge fields. In perturbation theory, however, imposing the conservation Ward identity for the energy-momentum tensor and of the Ward identity for the vector currents — whenever these are present — is sufficient to obtain the corresponding anomalous Ward identity. In the case of the TVV, for instance, this is a common practice, since only one term $(F^{a\,\mu\nu}(z)\,F^a_{\mu\nu}(z))$ can appear in the anomaly. Therefore the anomaly condition comes as a necessary consequence of the other Ward identities and can be checked at the end of the computation to correspond to the one derived from eq. (2.1). Things are far more involved for vertices with multiple insertions of gravitons, such as the TTT vertex, and a successful test of the anomalous Ward identity is crucial in order to secure the correctness of the result of the computation.

2.2 Definition of the correlators and Ward identities for the TVV and TOO vertices

We provide the basic definition of the correlators that we are going to investigate, in analogy to [27]. We start from the TVV vertex and use the Euclidean convention. We recall

[2] $\mathcal{W}$ depends, in general, from the background metric $g_{\mu\nu}(x)$, the gauge fields $A^a(x)$ and scalar sources $J(x)$ In the equations below, only those dependences which are relevant for the case at hand will be explicitly indicated.

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that in this case the functional average of the gauge current V is obtained by functional differentiation of the generating functional with respect to the background gauge field A_μ^a

$$\langle V^{a\,\mu}(x)\rangle = -\frac{1}{\sqrt{g_x}}\frac{\delta \mathcal{W}}{\delta A_\mu^a(x)}\,. \tag{2.8}$$

To construct the TVV correlator we can first perform a functional derivative with respect to the metric followed by the flat space-time limit ($g_{\mu\nu} = \delta_{\mu\nu}$) and then insert the vector currents by taking derivatives with respect to the gauge field source A

$$\begin{aligned}\left\langle T^{\mu\nu}(x_1)V^{a\,\alpha}(x_2)V^{b\,\beta}(x_3)\right\rangle &= \left\{\frac{\delta^2}{\delta A_\alpha^a(x_2)\delta A_\beta^b(x_3)}\left[\frac{2}{\sqrt{g_{x_1}}}\frac{\delta\mathcal{W}}{\delta g_{\mu\nu}(x_1)}\right]_{g=\delta}\right\}_{A=0}\\ &= \left\langle T^{\mu\nu}(x_1)V^{a\,\alpha}(x_2)V^{b\,\beta}(x_3)\right\rangle_{A=0} + \left\langle\frac{\delta T^{\mu\nu}(x_1)}{\delta A_\alpha^a(x_2)}V^{b\,\beta}(x_3)\right\rangle_{A=0}\\ &\quad + \left\langle\frac{\delta T^{\mu\nu}(x_1)}{\delta A_\beta^b(x_3)}V^{a\,\alpha}(x_2)\right\rangle_{A=0}\end{aligned} \tag{2.9}$$

where $T_{\mu\nu}$ is the energy-momentum tensor calculated in the presence of the background source A_μ^a. The first term in the previous expression represents the insertion of the three operators, while the last two are contact terms, with the topology of 2-point functions, exploiting the linear dependence of the energy-momentum tensor from the source field A.

The construction of the TOO correlator is analogous. If the scalar operator O is coupled to the source J we define

$$\langle O(x)\rangle = -\frac{1}{\sqrt{g_x}}\frac{\delta\mathcal{W}}{\delta J(x)} \tag{2.10}$$

and then the three point function is generated as

$$\begin{aligned}\langle T^{\mu\nu}(x_1)O(x_2)O(x_3)\rangle &= \left\{\frac{\delta^2}{\delta J(x_2)\delta J(x_3)}\left[\frac{2}{\sqrt{g_{x_1}}}\frac{\delta\mathcal{W}}{\delta g_{\mu\nu}(x_1)}\right]_{g=\delta}\right\}_{J=0}\\ &= \langle T^{\mu\nu}[J](x_1)O(x_2)O(x_3)\rangle_{J=0} + \left\langle\frac{\delta T^{\mu\nu}[J](x_1)}{\delta J(x_2)}O(x_3)\right\rangle_{J=0}\\ &\quad + \left\langle\frac{\delta T^{\mu\nu}[J](x_1)}{\delta J(x_3)}O(x_2)\right\rangle_{J=0}\,.\end{aligned} \tag{2.11}$$

The third correlator that we will analyze will be the VVV vertex, which is defined by the third functional derivative of the generating functional with respect to the source gauge field $A_\mu^a(x)$

$$\left\langle V^{a\,\mu}(x_1)V^{b\,\nu}(x_2)V^{c\,\rho}(x_3)\right\rangle = -\left.\frac{\delta^3\mathcal{W}|_{g=\delta}}{\delta A_\mu^a(x_1)\delta A_\nu^b(x_2)\delta A_\rho^c(x_3)}\right|_{A=0}. \tag{2.12}$$

The VVV is anomaly free, as is the TVV for general ($d \neq 4$) dimensions. To derive the non-anomalous Ward identities for general dimensions we assume that the generating functional $W[g,A]$ is invariant under diffeomorphisms

$$\mathcal{W}[g,A] = \mathcal{W}[g',A']\,, \tag{2.13}$$

where g' and A' are transformed metric and gauge field under the general infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$

$$\delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \,, \qquad \delta A^a_\mu = \epsilon^\lambda \nabla_\lambda A^a_\mu + A^{a\,\lambda} \nabla_\mu \epsilon_\lambda \,. \tag{2.14}$$

Diffeomerphism invariance and gauge invariance give the relation

$$\nabla_\mu \langle T^{\mu\nu} \rangle + \nabla^\nu A^a_\mu \langle V^{a\,\mu} \rangle + \nabla_\mu \left(A^{a\,\nu} \langle V^{a\,\mu} \rangle \right) = 0 \,, \tag{2.15}$$

$$\nabla_\mu \langle V^{a\,\mu} \rangle + f^{abc} A^b_\mu \langle V^{c\mu} \rangle = 0 \,, \tag{2.16}$$

while naive scale invariance gives the traceless condition

$$g_{\mu\nu} \langle T^{\mu\nu} \rangle = 0. \tag{2.17}$$

This last Ward identity is naive, due to the appearance of an anomaly at quantum level, after renormalization of the correlator for $d = 4$. It is however the correct identity in the TVV, TOO and TTT cases away from $d = 4$. In this respect, the functional differentiation of (2.15) and (2.17) allows to derive ordinary Ward identities for the various correlators. In the TVV case we obtain the conservation equation

$$\begin{aligned}
&\partial^{x_1}_\mu \left\langle T^{\mu\nu}(x_1) V^{a\,\alpha}(x_2) V^{b\,\beta}(x_3) \right\rangle \\
&\quad = \partial^\nu_{x_1} \delta^d(x_{12}) \left\langle V^{a\,\alpha}(x_1) V^{b\,\beta}(x_3) \right\rangle + \partial^\nu_{x_1} \delta^d(x_{31}) \left\langle V^{a\,\alpha}(x_2) V^{b\,\beta}(x_1) \right\rangle \\
&\qquad - \delta^{\nu\alpha} \partial^{x_1}_\mu \left(\delta^d(x_{12}) \left\langle V^{a\,\mu}(x_1) V^{b\,\beta}(x_3) \right\rangle \right) - \delta^{\nu\beta} \partial^{x_1}_\mu \left(\delta^d(x_{31}) \left\langle V^{a\,\alpha}(x_2) V^{b\,\mu}(x_1) \right\rangle \right)
\end{aligned} \tag{2.18}$$

and vector current Ward identities

$$\partial^{x_2}_\alpha \left\langle T^{\mu\nu}(x_1) V^{a\,\alpha}(x_2) V^{b\,\beta}(x_3) \right\rangle = 0 \,, \qquad \partial^{x_3}_\beta \left\langle T^{\mu\nu}(x_1) V^{a\,\alpha}(x_2) V^{b\,\beta}(x_3) \right\rangle = 0 \,, \tag{2.19}$$

while the naive identity (2.17) gives the non-anomalous condition

$$\delta_{\mu\nu} \left\langle T^{\mu\nu}(x_1) V^{a\,\alpha}(x_2) V^{b\,\beta}(x_3) \right\rangle = 0 \tag{2.20}$$

for $d \neq 4$.

2.3 Definitions for the TTT amplitude

For the multi-graviton vertices, it is convenient to define the corresponding correlation function as the n-th functional variation with respect to the metric of the generating functional $\mathcal{W}$ evaluated in the flat-space limit

$$\begin{aligned}
\langle T^{\mu_1\nu_1}(x_1) \dots T^{\mu_n\nu_n}(x_n) \rangle &= \left[\frac{2}{\sqrt{g_{x_1}}} \dots \frac{2}{\sqrt{g_{x_n}}} \frac{\delta^n \mathcal{W}}{\delta g_{\mu_1\nu_1}(x_1) \dots \delta g_{\mu_n\nu_n}(x_n)} \right] \Bigg|_{g_{\mu\nu} = \delta_{\mu\nu}} \\
&= 2^n \frac{\delta^n \mathcal{W}}{\delta g_{\mu_1\nu_1}(x_1) \dots \delta g_{\mu_n\nu_n}(x_n)} \Bigg|_{g_{\mu\nu} = \delta_{\mu\nu}} ,
\end{aligned} \tag{2.21}$$

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so that it is explicitly symmetric with respect to the exchange of the metric tensors. As we are going to deal with correlation functions evaluated in the flat-space limit all through the paper we will omit to specify it from now on, so as to keep our notation easy. The 3-point function we are interested in studying is found by evaluating (2.21) for $n = 3$,

$$
\begin{aligned}
\left\langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)T^{\alpha\beta}(x_3)\right\rangle = 8 \bigg[&- \left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)} \frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)} \right\rangle \\
&+ \left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\alpha\beta}(x_3)\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)} \right\rangle + \left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\rho\sigma}(x_2)\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)} \right\rangle \\
&+ \left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\rho\sigma}(x_2)\delta g_{\alpha\beta}(x_3)} \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \right\rangle - \left\langle \frac{\delta^3 \mathcal{S}}{\delta g_{\rho\sigma}(x_2)\delta g_{\alpha\beta}(x_3)\delta g_{\mu\nu}(x_1)} \right\rangle \bigg], \qquad (2.22)
\end{aligned}
$$

where the angle brackets denote the vacuum expectation value. Notice that the last term is identically zero in dimensional regularization, being proportional to a massless tadpole. The correlator

$$
\left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)} \frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)} \right\rangle, \qquad (2.23)
$$

has the diagrammatic representation of a triangle topology, while the contributions

$$
\begin{aligned}
&\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\rho\sigma}(x_2)\delta g_{\alpha\beta}(x_3)} \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \right\rangle, \quad \left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\alpha\beta}(x_3)\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)} \right\rangle, \\
&\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\rho\sigma}(x_2)\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)} \right\rangle \qquad (2.24)
\end{aligned}
$$

are interpreted in the perturbative analysis as the "k", "q" and "p" bubble respectively, also termed "T-bubbles" in [3].

In the perturbative realization of these expressions we will also establish a connection between these contributions and the extra terms generated at the 2-point coincidence limit of the general 3-point vertices discussed in [27]. For a 3-point vertex the dependence in configuration space is labelled as (x_1, x_2, x_3) with an incoming momentum (k) at x_1 and two outgoing momenta q, p at x_2 and x_3 respectively. These conventions are summarized by the transforms

$$
\begin{aligned}
&\int d^4x_1\, d^4x_2\, d^4x_3 \left\langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)T^{\alpha\beta}(x_3)\right\rangle e^{-i(k\cdot x_1 - q\cdot x_2 - p\cdot x_3)} = \\
&\qquad (2\pi)^4\, \delta^{(4)}(k-p-q) \left\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}\right\rangle(p,q)\,, \qquad (2.25)
\end{aligned}
$$

and

$$
\int d^4x_2\, d^4x_3 \left\langle T^{\rho\sigma}(x_2)T^{\alpha\beta}(x_3)\right\rangle e^{-i(q\cdot x_2 - p\cdot x_3)} = (2\pi)^4\, \delta^{(4)}(p-q) \left\langle T^{\rho\sigma}T^{\alpha\beta}\right\rangle(p)\,, \quad (2.26)
$$

for 3- and 2-point functions respectively.

2.4 General covariance Ward identities for the TTT

The requirement of general covariance for the generating functional $\mathcal{W}$ immediately leads to the master Ward identity for the conservation of the energy momentum tensor given in (2.15) (of course we disregard background gauge fields here),

$$\nabla_\nu \langle T^{\mu\nu}(x_1)\rangle = \nabla_\nu \left(\frac{2}{\sqrt{g_{x_1}}} \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}(x_1)} \right) = 0\,, \tag{2.27}$$

and expanding the covariant derivative we can write it as

$$\frac{2}{\sqrt{g_{x_1}}} \left(\partial_\nu \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}(x_1)} - \Gamma^\lambda_{\lambda\nu}(x_1) \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}(x_1)} + \Gamma^\mu_{\kappa\nu}(x_1) \frac{\delta \mathcal{W}}{\delta g_{\kappa\nu}(x_1)} + \Gamma^\nu_{\kappa\nu}(x_1) \frac{\delta \mathcal{W}}{\delta g_{\mu\kappa}(x_1)} \right) = 0, \tag{2.28}$$

where the first of the three Christoffel symbols is generated by differentiation of $1/\sqrt{g_{x_1}}$ in the definition of $T_{\mu\nu}$ together with

$$\Gamma^\alpha_{\alpha\beta}(x_1) = \frac{1}{2}\, g^{\alpha\gamma}(x_1)\, \partial_\beta\, g_{\alpha\gamma}(x_1) \tag{2.29}$$

or, equivalently, as

$$2 \left(\partial_\nu \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}(x_1)} + \Gamma^\mu_{\kappa\nu}(x_1) \frac{\delta \mathcal{W}}{\delta g_{\kappa\nu}(x_1)} \right) = 0\,. \tag{2.30}$$

By taking one and two functional derivatives of (2.30) with respect to $g_{\rho\sigma}(x_2)$ and $g_{\rho\sigma}(x_2)$ and $g_{\alpha\beta}(x_3)$ respectively, one gets, in curved space-time,

$$4 \left[\partial_\nu \frac{\delta^2 \mathcal{W}}{\delta g_{\rho\sigma}(x_2) \delta g_{\mu\nu}(x_1)} + \frac{\delta \Gamma^\mu_{\kappa\nu}(x_1)}{\delta g_{\rho\sigma}(x_2)} \frac{\delta \mathcal{W}}{\delta g_{\kappa\nu}(x_1)} + \Gamma^\mu_{\kappa\nu}(x_1) \frac{\delta^2 \mathcal{W}}{\delta g_{\mu\nu}(x_1) \delta g_{\rho\sigma}(x_2)} \right] = 0 \tag{2.31}$$

$$\begin{aligned} 8 \Bigg[\partial_\nu & \frac{\delta^3 \mathcal{W}}{\delta g_{\alpha\beta}(x_3) \delta g_{\rho\sigma}(x_2) \delta g_{\mu\nu}(x_1)} + \frac{\delta \Gamma^\mu_{\kappa\nu}(x_1)}{\delta g_{\rho\sigma}(x_2)} \frac{\delta^2 \mathcal{W}}{\delta g_{\alpha\beta}(x_3) \delta g_{\kappa\nu}(x_1)} + \frac{\delta \Gamma^\mu_{\kappa\nu}(x_1)}{\delta g_{\alpha\beta}(x_3)} \frac{\delta^2 \mathcal{W}}{\delta g_{\rho\sigma}(x_2) \delta g_{\kappa\nu}(x_3)} \\ & + \frac{\delta^2 \Gamma^\mu_{\kappa\nu}(x_1)}{\delta g_{\rho\sigma}(x_2) \delta g_{\alpha\beta}(x_3)} \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}(x_1)} + \Gamma^\mu_{\kappa\nu}(x_1) \frac{\delta^3 \mathcal{W}}{\delta g_{\rho\sigma}(x_2) \delta g_{\alpha\beta}(x_2) \delta g_{\kappa\nu}(x_1)} \Bigg] = 0\,, \end{aligned} \tag{2.32}$$

where $\delta(x_1, x_2) \equiv \delta(x_1 - x_2)$ and so on.

As we are interested in the flat space-time limit, we must evaluate 2.31 and (2.32) by letting the Christoffel symbols go to zero. Another simplification is obtained by noticing that the Green's functions

$$\left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \right\rangle = - \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}(x_1)} \tag{2.33}$$

and

$$\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\mu\nu}(x_1) \delta g_{\alpha\beta}(x_3)} \right\rangle \tag{2.34}$$

are proportional to massless tadpoles, so that we can ignore them in the following expression

$$\frac{\delta^2 \mathcal{W}}{\delta g_{\alpha\beta}(x_3) \delta g_{\mu\nu}(z)} = \left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)} \right\rangle - \left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\alpha\beta}(x_3) \delta g_{\mu\nu}(x_1)} \right\rangle = \left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)} \right\rangle . \tag{2.35}$$

So the Ward identity for the 2-point function in flat coordinate space-time is immediately seen to be

$$\partial_\nu \langle T^{\mu\nu}(x_1) T^{\rho\sigma}(x_2)\rangle = 0, \tag{2.36}$$

where, due to the vanishing of (2.34), we have set

$$\langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)\rangle \equiv 4\left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)}\frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)}\right\rangle . \tag{2.37}$$

Obviously, its form in momentum space, exploiting (2.26), is

$$p_\mu \left\langle T^{\mu\nu}T^{\rho\sigma}\right\rangle (p) = 0\,. \tag{2.38}$$

The terms surviving in (2.32) are those in the first line. In order to make them explicit, we evaluate the functional derivative of the Christoffel symbols using (A.3), (A.8) and (A.9), finding

$$\frac{\delta \Gamma^{\mu}_{\kappa\nu}(x_1)}{\delta g_{\rho\sigma}(x_2)} = \frac{1}{2}\delta^{\mu\alpha}\Big[-s^{\rho\sigma}{}_{\kappa\nu}\partial_\alpha + s^{\rho\sigma}{}_{\alpha\nu}\partial_\kappa + s^{\rho\sigma}{}_{\alpha\kappa}\partial_\nu\Big]\delta(x_1,x_2)\,, \tag{2.39}$$

where the s tensor is defined by eq. (A.9) in the appendix. Plugging this into (2.32) and using (2.37), the second term becomes

$$\begin{aligned} 8\,\frac{\delta \Gamma^{\mu}_{\kappa\nu}(x_1)}{\delta g_{\rho\sigma}(x_2)}\frac{\delta^2 \mathcal{W}}{\delta g_{\alpha\beta}(x_3)\delta g_{\kappa\nu}(x_1)} = &\bigg[\delta^{\mu\rho}\left\langle T^{\nu\sigma}(x_1)T^{\alpha\beta}(x_3)\right\rangle\partial_\nu + \delta^{\mu\sigma}\left\langle T^{\nu\rho}(x_1)T^{\alpha\beta}(x_3)\right\rangle\partial_\nu \\ &-\left\langle T^{\rho\sigma}(x_1)T^{\alpha\beta}(x_3)\right\rangle\partial^\mu\bigg]\delta(x_1,x_2)\,. \end{aligned} \tag{2.40}$$

A completely analogous relation holds for the exchanged term $\big(g_{\alpha\beta}(x_3) \leftrightarrow g_{\rho\sigma}(x_2)\big)$.

Finally, we can recast the Ward identity (2.32) in the form

$$\begin{aligned} &\partial_\nu\left\langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)T^{\alpha\beta}(x_3)\right\rangle = \\ &\bigg[\left\langle T^{\rho\sigma}(x_1)T^{\alpha\beta}(x_3)\right\rangle\partial^\mu\delta(x_1,x_2) + \left\langle T^{\alpha\beta}(x_1)T^{\rho\sigma}(x_2)\right\rangle\partial^\mu\delta(x_1,x_3)\bigg] \\ &-\bigg[\delta^{\mu\rho}\left\langle T^{\nu\sigma}(x_1)T^{\alpha\beta}(x_3)\right\rangle + \delta^{\mu\sigma}\left\langle T^{\nu\rho}(x_1)T^{\alpha\beta}(x_3)\right\rangle\bigg]\partial_\nu\delta(x_1,x_2) \\ &-\bigg[\delta^{\mu\alpha}\left\langle T^{\nu\beta}(x_1)T^{\rho\sigma}(x_2)\right\rangle + \delta^{\mu\beta}\left\langle T^{\nu\alpha}(x_1)T^{\rho\sigma}(x_2)\right\rangle\bigg]\partial_\nu\delta(x_1,x_3)\,, \end{aligned} \tag{2.41}$$

having used the definitions (2.21) and (2.22).

Fourier-transforming according to (2.25) and (2.26), we get the Ward identity in momentum space that we need, i.e.

$$\begin{aligned} k_\nu\left\langle T^{\mu\nu}T^{\alpha\beta}T^{\rho\sigma}\right\rangle(p,q) = &\, p^\mu\left\langle T^{\alpha\beta}T^{\rho\sigma}\right\rangle(q) + q^\mu\left\langle T^{\rho\sigma}T^{\alpha\beta}\right\rangle(p) \\ &-p_\nu\Big[\delta^{\mu\beta}\left\langle T^{\nu\alpha}T^{\rho\sigma}\right\rangle(q) + \delta^{\mu\alpha}\left\langle T^{\nu\beta}T^{\rho\sigma}\right\rangle(q)\Big] \\ &-q_\nu\Big[\delta^{\mu\sigma}\left\langle T^{\nu\rho}T^{\alpha\beta}\right\rangle(p) + \delta^{\mu\rho}\left\langle T^{\nu\sigma}T^{\alpha\beta}\right\rangle(p)\Big]. \end{aligned} \tag{2.42}$$

Similar Ward identities can be obtained when we contract with the momenta of the other lines. These are going to be essential in order to test the correctness of the computation once we turn to perturbation theory.

2.5 The anomalous Ward identities for the TTT

The anomalous Ward identities for the 3-graviton vertex is obtained after a lengthy computation, performing two functional variations of (2.7) and taking the flat-space limit, thereby obtaining

$$\begin{aligned}\delta_{\mu\nu}\left\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}\right\rangle(p,q) &= 4\,\mathcal{A}^{\alpha\beta\rho\sigma}(p,q)-2\left\langle T^{\alpha\beta}T^{\rho\sigma}\right\rangle(p)-2\left\langle T^{\rho\sigma}T^{\alpha\beta}\right\rangle(q)\\ &= 4\bigg[\beta_a\bigg(\left[F\right]^{\alpha\beta\rho\sigma}(p,q)-\frac{2}{3}\left[\sqrt{g}\Box\, R\right]^{\alpha\beta\rho\sigma}(p,q)\bigg)+\beta_b\left[G\right]^{\alpha\beta\rho\sigma}(p,q)\bigg]\\ &\quad -2\left\langle T^{\alpha\beta}T^{\rho\sigma}\right\rangle(p)-2\left\langle T^{\rho\sigma}T^{\alpha\beta}\right\rangle(q)\,, \end{aligned} \tag{2.43}$$

where $\mathcal{A}^{\alpha\beta\rho\sigma}(p,q)$ is generated by the anomaly. We remark, if not obvious, that all the contractions with the metric tensor in the flat spacetime limit ($\delta_{\mu\nu}$) should be understood as being 4-dimensional. This is the case for all the anomaly equations. The various contributions to the trace anomaly are given in terms of the functional derivatives of quadratic invariants in appendix C. Analogous anomalous Ward identities can be obtained by tracing the other two pairs of indices.

3 Inverse mappings: the correlators VVV, TOO and TVV in position space using the Feynman expansion

Having by now defined all the fundamental (anomalous and regular) Ward identities which allow to test the consistency of all the correlator which we are interested in, we now turn to provide the expression of these correlators in position space using their realization in free field theory.

We remind that an important result of [27] is the identification of the solution of the Ward identities in terms of a set of constants and of certain linearly independent tensor structures in (Euclidean) position space. Since these same tensor structures must occur in direct computations of the same vertex functions in free field theories in momentum space, we can use the one-loop computations of the vertex functions in momentum space to infer what those tensor structures must be, and find the exact correspondence between CFT amplitudes in position space and momentum space *a posteriori*, provided that we have enough linearly independent vertex functions for different free theories to determine the linear combinations uniquely. We call this procedure an "inverse mapping", as it allows to re-express the correlators of [27] in such a form that their Fourier integrability is explicit. This result is obtained by pulling out derivatives of the corresponding diagrams in such a way that integrability becomes trivial. More details on this procedure is contained in section 8.

We start with the VVV vertex function. The two types of diagrams contributing to the general conformal expression of the VVV in any dimensions are shown in figure (3). In [27] the VVV, as all the other correlators, are fixed by general CFT requirements. It

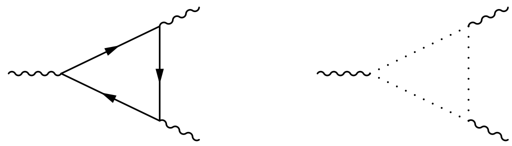

Figure 1. The fermion and the scalar sectors contributing to the conformal VVV vertex in any dimension.

takes the form [27]

$$\left\langle V^a_\mu(x_1)V^b_\nu(x_2)V^c_\rho(x_3)\right\rangle = \frac{f^{abc}}{(x^2_{12})^{d/2-1}\,(x^2_{23})^{d/2-1}\,(x^2_{31})^{d/2-1}}\Bigg\{(a-2b)\,X_{23\,\mu}\,X_{31\,\nu}\,X_{12\,\rho}$$
$$-b\left[\frac{1}{x^2_{23}}\,X_{23\,\mu}\,I_{\nu\rho}(x_{23})+\frac{1}{x^2_{31}}\,X_{31\,\nu}\,I_{\mu\rho}(x_{31})+\frac{1}{x^2_{12}}\,X_{12\,\rho}\,I_{\mu\nu}(x_{12})\right]\Bigg\}\,, \tag{3.1}$$

where f^{abc} are the structure constants of the gauge group, $I_{\mu\nu}(x)$ is the inversion operator defined as

$$I^{\mu\nu}(x)=\delta^{\mu\nu}-2\frac{x^\mu x^\nu}{x^2} \tag{3.2}$$

and

$$x_{ij}\equiv x_i-x_j\,,\qquad X_{ij}=-X_{ji}\equiv\frac{x_{ik}}{x^2_{ik}}-\frac{x_{jk}}{x^2_{jk}}\,,\quad i,j,k=1,2,3\,. \tag{3.3}$$

The correlator is Fourier integrable, although this is not immediately evident from (3.1). The simplest way to prove this point consists in showing that (3.1) can be reproduced in d-dimensions by the combination of the scalar and the fermion sectors of a free field theory. For this purpose we use two realizations of the vector current V^a_μ, using scalar and fermion fields

$$V^a_\mu=\phi^* t^a\partial_\mu\phi-\partial_\mu\phi^* t^a\phi\,,\qquad V^a_\mu=\bar\psi t^a\gamma_\mu\psi\,. \tag{3.4}$$

The diagrammatic expansion of this correlator consists of two triangle diagrams, the direct and the exchanged, both in the scalar and fermion sectors. Using the Feynman rules in coordinate space we obtain, after some manipulations

$$\left\langle V^a_\mu(x_1)V^b_\nu(x_2)V^c_\rho(x_3)\right\rangle_{\text{fermion}} =$$
$$-\frac{c_f\,f^{abc}}{(d-2)^3}\Delta_{\mu\alpha\nu\beta\rho\gamma}\partial^\alpha_{12}\partial^\beta_{23}\partial^\gamma_{31}\frac{1}{(x^2_{12})^{d/2-1}(x^2_{23})^{d/2-1}(x^2_{31})^{d/2-1}}\,, \tag{3.5}$$
$$\left\langle V^a_\mu(x_1)V^b_\nu(x_2)V^c_\rho(x_3)\right\rangle_{\text{scalar}} =$$
$$\frac{c_s\,f^{abc}}{(d-2)^2}\left(\partial^{12}_\mu+\partial^{31}_\mu\right)\left(\partial^{23}_\nu+\partial^{12}_\nu\right)\left(\partial^{31}_\rho+\partial^{23}_\rho\right)\frac{1}{(x^2_{12})^{d/2-1}(x^2_{23})^{d/2-1}(x^2_{31})^{d/2-1}} \tag{3.6}$$

where

$$\Delta_{\mu\alpha\nu\beta\rho\gamma}=\frac{1}{4}Tr\left[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\rho\gamma_\gamma\right]\,, \tag{3.7}$$

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and c_f, c_s are normalization constants whose numerical values are irrelevant here. Written in these forms, the two expressions are manifestly integrable. Tracing over the γ matrices and applying the derivatives over all the denominators, we generate the result of [27] by taking a linear combination of these two sectors

$$\left\langle V_\mu^a(x_1)V_\nu^b(x_2)V_\rho^c(x_3)\right\rangle = \left(a\, t^a_{\mu\nu\rho} + b\, t^b_{\mu\nu\rho}\right)\frac{f^{abc}}{(x_{12}^2)^{d/2-1}(x_{23}^2)^{d/2-1}(x_{31}^2)^{d/2-1}} \tag{3.8}$$

where

$$t^a_{\mu\nu\rho} = \frac{1}{d(d-2)^2}\left(\partial_\mu^{12}+\partial_\mu^{31}\right)\left(\partial_\nu^{23}+\partial_\nu^{12}\right)\left(\partial_\rho^{31}+\partial_\rho^{23}\right) - \frac{1}{d}t^b_{\mu\nu\rho}\,, \tag{3.9}$$

$$t^b_{\mu\nu\rho} = -\frac{1}{(d-2)^3}\Delta_{\mu\alpha\nu\beta\rho\gamma}\partial^\alpha_{12}\partial^\beta_{23}\partial^\gamma_{31}\,. \tag{3.10}$$

The equivalence between this expression and eq. (3.1) can be verified explicitly.

3.1 The *TOO* case

The next correlator that we are going to investigate extensively is the *TOO*. The structure of this function in coordinate space — for non coincident points — is given by [27]

$$\langle T_{\mu\nu}(x_1)\, O(x_2)\, O(x_3)\rangle = \frac{a}{(x_{12}^2)^{d/2}\,(x_{23}^2)^{\eta-d/2}\,(x_{31}^2)^{d/2}}\, h^1_{\mu\nu}(\hat{X}_{23})\,, \tag{3.11}$$

where a is a constant, η the dimension of the scalar field O and where

$$\hat{X}_\mu = \frac{X_\mu}{\sqrt{X^2}}\,, \qquad h^1_{\mu\nu}(\hat{X}) = \hat{X}_\mu\,\hat{X}_\nu - \frac{1}{d}\,\delta_{\mu\nu}\,.$$

In the short-distance limits of its external points this vertex is singular and needs regularization. In [27] the authors, in their direct solutions of the Ward identites, introduce some extra terms which are given by

$$\left[\hat{A}_{\mu\nu}(x_{12}) - A_{\mu\nu}(x_{12}) + \hat{A}_{\mu\nu}(x_{31}) - A_{\mu\nu}(x_{31})\right]\frac{1}{(x_{23}^2)^\eta}\,, \tag{3.12}$$

where

$$A_{\mu\nu}(s) = \frac{a}{s^d}\left(\frac{s_\mu s_\nu}{s^2} - \frac{1}{d}\delta_{\mu\nu}\right), \qquad \hat{A}_{\mu\nu}(s) = \frac{a}{d}\left(\frac{\partial_\mu\partial_\nu}{d-2}\frac{1}{s^{d-2}} + \frac{\eta-d+1}{\eta}S_d\delta_{\mu\nu}\delta^d(s)\right). \tag{3.13}$$

These are contact terms. In the expression above S_d denotes the volume of the d-dimensional sphere, $S_d = 2\,\pi^{\frac{1}{2}}/\Gamma(d/2)$. The delta function term in $\hat{A}$ reflects the arbitrariness typical of any regularization scheme, and its coefficient is chosen to satisfy the Ward identities.

3.1.1 Manifest integrability of the CFT result and comparisons with free field theory

Expanding the previous expression and bringing it in the derivative form we obtain

$$
\begin{aligned}
&\langle T_{\mu\nu}(x_1)\, O(x_2)\, O(x_3)\rangle = \\
&\quad \frac{a}{(d-2)^2}\left\{(\partial^{12}_\mu \partial^{31}_\nu + \partial^{12}_\nu \partial^{31}_\mu) + \frac{d-2}{d}(\partial^{12}_{\mu\nu} + \partial^{31}_{\mu\nu})\right\}\frac{1}{(x_{12}^2)^{d/2-1}(x_{23}^2)^{\eta-d/2+1}(x_{31}^2)^{d/2-1}} \\
&\quad + a\frac{x_{12}^2 x_{23}^2 + x_{31}^2 x_{23}^2 - (x_{23}^2)^2}{(x_{12}^2)^{d/2}(x_{23}^2)^{\eta-d/2+1}(x_{31}^2)^{d/2}}\frac{\delta_{\mu\nu}}{d} + a\frac{\eta-d+1}{d\eta} S_d \delta_{\mu\nu}\frac{\delta^d(x_{12}) + \delta^d(x_{31})}{(x_{23}^2)^\eta}\,,
\end{aligned} \tag{3.14}
$$

where, from now on, we set $\partial^{12}_\mu \equiv \frac{\partial}{\partial x_{12\,\mu}}$ and $\partial^{12}_{\mu\nu} \equiv \frac{\partial}{\partial x_{12\,\mu}}\frac{\partial}{\partial x_{12\,\nu}}$.

Notice that the first term of the second line proportional to $\delta_{\mu\nu}$ is not manifestly integrable. As we have already mentioned, one can use identities such as $x_{12}^2 + x_{13}^2 - x_{23}^2 = 2x_{12}\cdot x_{13}$ in order to rewrite it in the form

$$
\frac{x_{12}^2\, x_{23}^2 + x_{31}^2\, x_{23}^2 - (x_{23}^2)^2}{(x_{12}^2)^{d/2}\,(x_{23}^2)^{d/2}\,(x_{31}^2)^{d/2}} = \frac{2}{(d-2)^2}\partial^{12}_\mu \partial^{31\,\mu}\frac{1}{(x_{12}^2)^{d/2-1}(x_{31}^2)^{d/2-1}(x_{23}^2)^{\eta-d/2+1}} \tag{3.15}
$$

which shows its integrability when $\eta < d-1$.

In order to test the consistency of the result (3.11) obtained from the application of the conformal Ward identities for the TOO, we can consider a particular scalar free field theory. We suppose for instance that the scalar operator O is given by $O = \phi^2$ with dimensions $\eta = d-2$, whose energy-momentum tensor T is given by

$$
T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\,\delta_{\mu\nu}\,\partial_\alpha\phi\,\partial^\alpha\phi + \frac{1}{4}\frac{d-2}{d-1}\left[\delta_{\mu\nu}\partial^2 - \partial_\mu\,\partial_\nu\right]\phi^2 \tag{3.16}
$$

which is conserved and traceless in d dimensions.

Using the Feynman rules in coordinate space together with the expression of a scalar propagator we obtain the $T\phi^2\phi^2$ correlation function

$$
\begin{aligned}
&\left\langle T_{\mu\nu}(x_1)\phi^2(x_2)\phi^2(x_3)\right\rangle = \\
&\quad \frac{2a(d-1)}{d(d-2)^2}\Bigg[\partial^{12}_\mu\partial^{31}_\nu + \partial^{12}_\nu\partial^{31}_\mu - \delta_{\mu\nu}\partial^{12}\cdot\partial^{31} - \frac{d-2}{2(d-1)}\Big(- \partial^{12}_{\mu\nu} - \partial^{31}_{\mu\nu} + \partial^{12}_\mu\partial^{31}_\nu \\
&\quad +\partial^{12}_\nu\partial^{31}_\mu + \delta_{\mu\nu}\left(\partial^2_{12} + \partial^2_{31} - 2\partial^{12}\cdot\partial^{31}\right)\Big)\Bigg]\frac{1}{(x_{12}^2)^{d/2-1}(x_{23}^2)^{d/2-1}(x_{31}^2)^{d/2-1}} \\
&\quad - a\frac{d-1}{d(d-2)}S_d\delta_{\mu\nu}\frac{\delta^d(x_{12}) + \delta^d(x_{31})}{(x_{23}^2)^{d-2}}\,.
\end{aligned} \tag{3.17}
$$

The equivalence of this expression with the solution given in (3.11) can be explicitly checked by performing the derivative of (3.17) and expanding the result. We remark that (3.17) is clearly integrable and does not require any intermediate regularization. The first term in the previous expression comes from the triangle topology diagram while the last two, proportional to the delta functions, are contact terms with two-point topology.

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3.2 The TVV case: integrability and free field theory realization

To identify the diagrammatic structure of the TVV correlator we can proceed with an inverse mapping. In fact, we know from [27] that such solution is characterized by 2 constants when the 3 external coordinates (x_1, x_2, x_3) are separated. This homogeneous solution has to be modified by the additions of extra contact terms $(A - \hat{A})$ terms which have the topology of 2-point functions.

The homogeneous solution is then modified further by the addition of a $1/\epsilon$ counterterm — in dimensional regularization — to regulate its ultraviolet behaviour. This regularization procedure is crucial to obtaining the anomalous contribution. We will come to a discussion of this point once we move completely to momentum space. Before that let us provide a diagrammatic interpretation of the various contributions to this correlators, except for the contribution coming from the counterterm, using the information that in any dimension this can be constructed as a linear combination of two independent sectors, the fermion and the scalar. Therefore we get

$$\left\langle T_{\mu\nu}(x_1)V^a_\alpha(x_2)V^b_\beta(x_3)\right\rangle = \sum_{I=f,s}\left(\left\langle T_{\mu\nu}(x_1)V^a_\alpha(x_2)V^b_\beta(x_3)\right\rangle^I_{A=0} + \left\langle \frac{\delta T_{\mu\nu}(x_1)}{\delta A^{a\,\alpha}(x_2)}V^b_\beta(x_3)\right\rangle^I_{A=0} + \left\langle \frac{\delta T_{\mu\nu}(x_1)}{\delta A^{b\,\beta}(x_3)}V^a_\alpha(x_2)\right\rangle^I_{A=0}\right) \tag{3.18}$$

where the sum is over the fermion (f) and scalar (s) sectors. In a diagrammatic expansion, all the terms above have a diagrammatic interpretation, which will turn useful in order to derive an integrable expression of this vertex.

Using the Feynman rules in configuration space one can obtain the following parameterization of the TVV vertex for fermions within the loop,

$$\left\langle T_{\mu\nu}(x_1)V^a_\alpha(x_2)V^b_\beta(x_3)\right\rangle^f_{A=0} = \frac{c\,\delta^{ab}}{d(d-2)^3}\,\mathcal{A}_{\mu\nu\xi\eta}\,\Delta_{\xi\rho\alpha\sigma\beta\lambda}\,(\partial^{12}_\eta + \partial^{31}_\eta)\,\partial^\rho_{12}\partial^\sigma_{23}\partial^\lambda_{31} \times \frac{1}{(x^2_{12})^{d/2-1}(x^2_{23})^{d/2-1}(x^2_{31})^{d/2-1}}\,, \tag{3.19}$$

where $\Delta_{\mu\rho\alpha\sigma\beta\lambda}$ is defined in eq. (3.7) and $\mathcal{A}_{\mu\nu\rho\sigma}$ in appendix D. This contribution alone is not sufficient to satisfy all the inhomogeneous Ward identities and we must consider also the contributions coming from the contact terms. In the framework of the analysis of [27], in which the correlation functions are obtained exploiting the symmetries without any reference to their perturbative structure, this is less evident. In fact in [27] the arbitrariness in the regularization procedure is exploited in order to impose the Ward identities by hand. This is achieved by introducing the differentially regulated expressions proportional to $A - \hat{A}$, which will be given below. These terms exactly correspond to the contributions proportional to 2-point functions discussed above, as we are going to show in a moment.

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The two contact terms identified by the diagrammatic expansion are given by

$$\left\langle \frac{\delta T_{\mu\nu}(x_1)}{\delta A^{a\,\alpha}(x_2)} V^b_\beta(x_3) \right\rangle^f_{A=0} = \frac{c\,\delta^{ab}}{d(d-2)^2} S_d \delta^d(x_{12}) \Delta^{(2)}_{\mu\nu\alpha\beta\rho\sigma} \partial^\rho_{31} \frac{1}{(x^2_{31})^{d/2-1}} \partial^\sigma_{31} \frac{1}{(x^2_{31})^{d/2-1}} \tag{3.20}$$

$$\left\langle \frac{\delta T_{\mu\nu}(x_1)}{\delta A^{b\,\beta}(x_3)} V^a_\alpha(x_2) \right\rangle^f_{A=0} = \frac{c\,\delta^{ab}}{d(d-2)^2} S_d \delta^d(x_{31}) \Delta^{(2)}_{\mu\nu\beta\alpha\rho\sigma} \partial^\rho_{12} \frac{1}{(x^2_{12})^{d/2-1}} \partial^\sigma_{31} \frac{1}{(x^2_{12})^{d/2-1}} \tag{3.21}$$

with

$$\begin{aligned} \Delta^{(2)}_{\mu\nu\alpha\beta\rho\sigma} = {} & \delta_{\alpha\nu}\delta_{\beta\sigma}\delta_{\mu\rho} + \delta_{\alpha\mu}\delta_{\beta\sigma}\delta_{\nu\rho} + \delta_{\alpha\nu}\delta_{\beta\rho}\delta_{\mu\sigma} + \delta_{\alpha\mu}\delta_{\beta\rho}\delta_{\nu\sigma} - \delta_{\alpha\nu}\delta_{\beta\mu}\delta_{\rho\sigma} - \delta_{\alpha\mu}\delta_{\beta\nu}\delta_{\rho\sigma} \\ & -2\,\delta_{\mu\nu}\left(\delta_{\alpha\rho}\delta_{\beta\sigma} + \delta_{\alpha\sigma}\delta_{\beta\rho} - \delta_{\alpha\beta}\delta_{\rho\sigma}\right) . \end{aligned} \tag{3.22}$$

In the scalar sector the TVV correlation function can be recast in the manifestly integrable form as

$$\begin{aligned} \left\langle T_{\mu\nu}(x_1) V^a_\alpha(x_2) V^b_\beta(x_3) \right\rangle^s_{A=0} = {} & c\,\delta^{ab} \frac{2(d-1)}{d(d-2)^3} \bigg[\partial^{12}_\mu \partial^{31}_\nu + \partial^{12}_\nu \partial^{31}_\mu - \delta_{\mu\nu} \partial^{12} \cdot \partial^{31} \\ & - \frac{d-2}{2(d-1)} \bigg(- \partial^{12}_{\mu\nu} - \partial^{31}_{\mu\nu} + \partial^{12}_\mu \partial^{31}_\nu + \partial^{12}_\nu \partial^{31}_\mu + \delta_{\mu\nu} \left(\partial^2_{12} + \partial^2_{31} - 2 \partial^{12} \cdot \partial^{31} \right) \bigg) \bigg] \times \\ & \times \left(\partial^{12}_\alpha + \partial^{23}_\alpha \right) \left(\partial^{31}_\beta + \partial^{23}_\beta \right) \frac{1}{(x^2_{12})^{d/2-1} (x^2_{23})^{d/2-1} (x^2_{31})^{d/2-1}} . \end{aligned} \tag{3.23}$$

This contribution originates only from the triangle diagram. This term corresponds to the expression given in [27] (for non coincident points) for the same correlator. The only differences are in the ∂^2_{12} and ∂^2_{31} terms which are proportional to $\delta_{\mu\nu}$, which vanish in the non-coincident point limit and are given by

$$\begin{aligned} & - \frac{c\,\delta^{ab}}{d(d-2)^2} \delta_{\mu\nu} \left(\partial^2_{12} + \partial^2_{31} \right) \left(\partial^{12}_\alpha + \partial^{23}_\alpha \right) \left(\partial^{31}_\beta + \partial^{23}_\beta \right) \frac{1}{(x^2_{12})^{d/2-1} (x^2_{23})^{d/2-1} (x^2_{31})^{d/2-1}} \\ & = \frac{2c\,\delta^{ab}}{d(d-2)} S_d \delta_{\mu\nu} \left[\partial^{23}_\alpha \left(\partial^{31}_\beta + \partial^{23}_\beta \right) \frac{\delta^d(x_{12})}{(x^2_{23})^{d/2-1} (x^2_{31})^{d/2-1}} + \partial^{23}_\beta \left(\partial^{12}_\alpha + \partial^{23}_\alpha \right) \frac{\delta^d(x_{31})}{(x^2_{12})^{d/2-1} (x^2_{23})^{d/2-1}} \right] . \end{aligned} \tag{3.24}$$

They have the topology of 2-point functions. These terms, together with those arising from the triangle diagrams, correspond exactly to those identified as $A - \hat{A}$ [27], which have been introduced in order to satisfy the Ward identities (contact terms)

$$\begin{aligned} \left\langle \frac{\delta T_{\mu\nu}(x_1)}{\delta A^{a\,\alpha}(x_2)} V^b_\beta(x_3) \right\rangle^s_{A=0} = {} & \frac{c\,\delta^{ab}(d-1)}{d(d-2)^2} S_d \delta^d(x_{12}) \times \\ & \times \left((\partial^{23}_\mu + \partial^{31}_\mu) \delta_{\nu\alpha} + (\partial^{23}_\nu + \partial^{31}_\nu) \delta_{\mu\alpha} - \delta_{\mu\nu} (\partial^{23}_\alpha + \partial^{31}_\alpha) \right) \times \\ & \times (\partial^{23}_\beta + \partial^{31}_\beta) \frac{1}{(x^2_{31})^{d/2-1} (x^2_{23})^{d/2-1}} \end{aligned} \tag{3.25}$$

$$\begin{aligned} \left\langle \frac{\delta T_{\mu\nu}(x_1)}{\delta A^{b\,\beta}(x_3)} V^a_\alpha(x_2) \right\rangle^s_{A=0} = {} & \frac{c\,\delta^{ab}(d-1)}{d(d-2)^2} S_d \delta^d(x_{31}) \times \\ & \times \left((\partial^{23}_\mu + \partial^{12}_\mu) \delta_{\nu\beta} + (\partial^{23}_\nu + \partial^{12}_\nu) \delta_{\mu\beta} - \delta_{\mu\nu} (\partial^{23}_\alpha + \partial^{12}_\beta) \right) \times \\ & \times (\partial^{23}_\alpha + \partial^{12}_\alpha) \frac{1}{(x^2_{12})^{d/2-1} (x^2_{23})^{d/2-1}} . \end{aligned} \tag{3.26}$$

This expression is in complete agreement with the solution given in [27], to which we refer for further details

$$\left\langle T_{\mu\nu}(x_1)V^a_\alpha(x_2)V^b_\beta(x_3)\right\rangle = \frac{\delta^{ab}}{(x_{12}^2)^{d/2}\,(x_{31}^2)^{d/2}\,(x_{23}^2)^{d/2-1}}\,I_{\alpha\sigma}(x_{12})\,I_{\beta\rho}(x_{31})\,t_{\mu\nu\rho\sigma}(X_{23})$$
$$-\delta^{ab}\left[A_{\mu\nu\alpha\rho}(x_{12})-\hat{A}_{\mu\nu\alpha\rho}(x_{12})\right]\frac{I_{\rho\beta}(x_{23})}{(x_{23}^2)^{d-1}}$$
$$-\delta^{ab}\left[A_{\mu\nu\sigma\beta}(x_{31})-\hat{A}_{\mu\nu\sigma\beta}(x_{31})\right]\frac{I_{\sigma\alpha}(x_{23})}{(x_{23}^2)^{d-1}}, \tag{3.27}$$

which is expressed in terms of tensor structures whose coefficients, denoted as a, b, c and e in [27], satisfy two constraint equation, and of contact terms A and $\hat{A}$ which are given in [27]. For this reason, only 2 independent constants are left free to parameterize any conformal correlator of this type in d dimensions. In the notation of [27] $e = 0$ and hence $b = 0$, so that there is only one independent structure. A final comment concerns the issues of renormalization. These expressions are unrenormalized. The issue of renormalization will be addressed by discussing in parallel the position and the momentum space approaches, that we will do starting from the next section. For this reason we turn to specific realizations of theories containing scalars and fermions — which are conformal in any dimension — and vectors, which are conformal for $d = 4$.

4 The TTT amplitude

4.1 The correlator

Now we are ready to turn to the analysis of the 3-graviton vertex. The general structure of the $< TTT >$ correlator in momentum space is [27]

$$\left\langle T^{\mu\nu}(x_1)\,T^{\rho\sigma}(x_2)\,T^{\alpha\beta}(x_3)\right\rangle = \frac{1}{(x_{12}^2)^{d/2}\,(x_{23}^2)^{d/2}\,(x_{31}^2)^{d/2}}\,\mathcal{I}^{\mu\nu\mu'\nu'}\,\mathcal{I}^{\rho\sigma\rho'\sigma'}\,t^{\mu'\nu'\rho'\sigma'\alpha\beta}(X_{12}) \tag{4.1}$$

$$\mathcal{I}^{\mu\nu,\alpha\beta}(s) = I^{\mu\rho}(s)I^{\nu\sigma}(s){\epsilon_T}^{\rho\sigma,\alpha\beta}\,, \qquad s = x-y \tag{4.2}$$

where

$${\epsilon_T}^{\mu\nu,\alpha\beta} = \frac{1}{2}\,(\delta^{\mu\alpha}\delta^{\nu\beta}+\delta^{\mu\beta}\delta^{\nu\alpha})-\frac{1}{d}\,\delta^{\mu\nu}\delta^{\alpha\beta} \tag{4.3}$$

is the projector onto the space of symmetric traceless tensors.

We perform the computation of the 3-graviton vertex TTT in free field theory, for $d = 4$, in all its 3 relevant sectors, the conformally coupled scalar, the fermion and the vector, since in this case the general solution of the Ward identities, for any CFT, is parameterized by 3 independent constants. This corresponds to the most general anomalous solution. For $d \neq 4$ the spin-1 sector is not conformally invariant and we can't build the general expression just by superposing the scalar and the fermion sectors. However, the combination of the scalar and the fermion sectors corresponds to an anomaly-free special solution also for generic d [18].

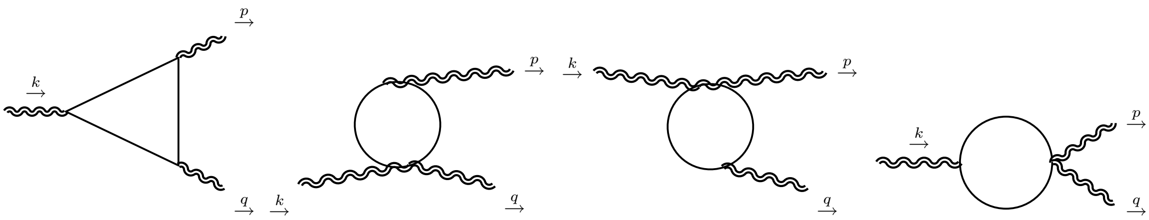

Figure 2. One loop expansion of the 3-graviton vertex. Shown here are the general topologies, i.e. the triangle and the self-energy type (T-bubble) contributions for the fermion case. The general correlator for any CFT in $d = 4$ can be obtained by adding to these diagrams similar ones where the fermion is replaced by a scalar and a photon in the loops. Ghost corrections follow the same topologies.

As we have already mentioned above, the correctness of our results has been checked first by a complete test of all the Ward identities for each case, which is already a nontrivial test to pass, given the large complexity of the computations. At the same time we will show that the counterterm introduced in [18, 27] in position space, which is extracted from the general expression of the trace anomaly when $d = 4$, coincides with that required in momentum space using dimensional regularization. The connection between the two methods will be discussed thoroughly.

4.2 Inverse mapping for the TTT amplitude

As done before for the $\langle VVV\rangle$, $\langle TOO\rangle$ and $\langle TVV\rangle$ correlators, here we check the result (4.1) building explicitly the correlator from the diagrammatic expansion in free field theory. This allows to come up with an expression for this vertex which is manifestly integrable. We will be using the Feynman rules obtained from the Lagrangian descriptions for scalars, fermions and spin 1 in configuration space, given in section 5. We start testing the non-coincident case, for which we can omit the contact terms. This corresponds to the "bulk" contribution to the correlator, which involves only the triangle topology. We give the d-dimensional expression for the scalar and the fermion cases, while — as already remarked — we have to limit our analysis to $d = 4$ for the spin-1 vector. Moreover, in the vector case the gauge-fixing and ghost parts of the amplitude have to cancel since the vertex is obviously gauge invariant. This has been explicitly verified in the computation in momentum space (see section 6.4). So, in performing our inverse mapping, we include in the interactions vertices only the Maxwell $\tilde{V}$ contributions, omitting ghosts and gauge-fixing terms. We have

$$\left\langle \frac{\delta\mathcal{S}}{\delta g_{\mu\nu}(x_1)}\,\frac{\delta\mathcal{S}}{\delta g_{\rho\sigma}(x_2)}\,\frac{\delta\mathcal{S}}{\delta g_{\alpha\beta}(x_3)}\right\rangle^S = C^S_{TTT} V^{\mu\nu}_{T\phi\phi}(i\,\partial^{12}, -i\,\partial^{31})\, V^{\rho\sigma}_{T\phi\phi}(i\,\partial^{23}, -i\,\partial^{12})\, V^{\alpha\beta}_{T\phi\phi}(i\,\partial^{31}, -i\,\partial^{23}) \\ \times \frac{1}{(x_{12}^2)^{d/2-1}\,(x_{23}^2)^{d/2-1}\,(x_{31}^2)^{d/2-1}}\,, \qquad (4.4)$$

$$\left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)} \frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)} \right\rangle^F =$$

$$C^F_{TTT}(-1)\Big(\mathrm{Tr}\big[V^{\mu\nu}_{T\bar\psi\psi}(i\partial^{12}, -i\partial^{31}) i\gamma\cdot\partial^{12} V^{\rho\sigma}_{T\bar\psi\psi}(i\partial^{23}, -i\partial^{12}) i\gamma\cdot\partial^{23} V^{\alpha\beta}_{T\bar\psi\psi}(i\partial^{31}, -i\partial^{23}) i\gamma\cdot\partial^{31}\big]$$

$$+\mathrm{Tr}\big[V^{\mu\nu}_{T\bar\psi\psi}(i\,\partial^{31}, -i\,\partial^{12})\, i\,\gamma\cdot\,\partial^{31}\, V^{\alpha\beta}_{T\bar\psi\psi}(i\,\partial^{23}, -i\,\partial^{31})\, i\,\gamma\cdot\,\partial^{12}\, V^{\rho\sigma}_{T\bar\psi\psi}(i\,\partial^{12}, -i\,\partial^{23})\, i\,\gamma\cdot\partial^{12}\big]\Big)$$

$$\times \frac{1}{(x^2_{12})^{d/2-1}\,(x^2_{23})^{d/2-1}\,(x^2_{31})^{d/2-1}}\,, \tag{4.5}$$

$$\left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)} \frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)} \frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)} \right\rangle^V =$$

$$C^V_{TTT}\,(-1)^3\,\tilde V^{\mu\nu\gamma\delta}_{TAA}(i\,\partial^{12}, -i\,\partial^{31})\,\tilde V^{\rho\sigma\zeta\xi}_{TAA}(i\,\partial^{23}, -i\,\partial^{12})\,\tilde V^{\alpha\beta\chi\omega}_{TAA}(i\,\partial^{31}, -i\,\partial^{23}) \times \frac{\delta_{\gamma\xi}\,\delta_{\delta\chi}\delta_{\zeta\omega}}{x^2_{12}\,x^2_{23}\,x^2_{31}}\,. \tag{4.6}$$

Notice that this last term enters only for $d = 4$. Here and in the following, the dependences of the vertices on the coordinates are obtained by replacing the momenta of (5) with appropriate derivatives respect to the external position variables. For instance

$$V^{\mu\nu}_{T\phi\phi}(p, q) \to V^{\mu\nu}_{T\phi\phi}(\hat p, \hat q) = V^{\mu\nu}_{T\phi\phi}(i\,\partial^{12}, -i\,\partial^{23}) \tag{4.7}$$

with

$$\hat p \to i\,\partial^{12} \qquad\qquad \hat q \to -i\,\partial^{23} \tag{4.8}$$

Explicitly

$$V^{\mu\nu}_{T\phi\phi}(i\,\partial^{12}, -i\,\partial^{23}) = = \frac{1}{2}\,(i\,\partial_{12\,\alpha})\,(-i\,\partial_{23\,\beta})\,C^{\mu\nu\alpha\beta}$$
$$+\chi\Big(\delta^{\mu\nu}\,(i\,\partial_{12} - i\,\partial_{23})^2 - (i\,\partial^{\mu}_{12} - i\,\partial^{\mu}_{23})\,(i\,\partial^{\nu}_{12} - i\,\partial^{\nu}_{23})\Big)\,. \tag{4.9}$$

The replacements of p, q and l, by the operatorial expressions $\hat p, \hat q$ and $\hat l$ in 2.3–2.5 are specific for each vertex. In appendix E we provide some more details on this procedure. Notice that we have chosen the coupling parameter for the scalar field in d dimensions at the corresponding conformal value $\chi = (d-2)/4(d-1)$.

Expanding the derivatives contained in each vertex, the expression given in (4.1) is recovered by setting

$$C^S_{TTT} = -\frac{8}{S^3_d\,(d-2)^3}\,, \quad C^F_{TTT} = \frac{2^{d/2+1}}{S^3_d\,(d-2)^3}\,, \quad C^V_{TTT} = \frac{1}{S^3_4}\,. \tag{4.10}$$

We compute next the contributions with the topology of 2-point functions, which are needed to account for the behavior of the vertex in the short distance limit. In coordinate space we can write them in a manifestly integrable form by pulling out derivatives in the same way as for the triangle diagram. We replace the momenta with derivatives with respect to the corresponding coordinates acting on propagators, obtaining very compact expressions for the vertex. We offer a few more details on this computation in appendix E, quoting here the result. In the scalar case we have

$$\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\mu\nu}(x_1)\delta g_{\alpha\beta}(x_3)} \frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)} \right\rangle^S = \frac{C^S_Q}{2}\,V^{\rho\sigma}_{T\phi\phi}(i\,\partial^{23}, -i\,\partial^{12})\,V^{\mu\nu\alpha\beta}_{TT\phi\phi}(i\,\partial^{12}, -i\,\partial^{23}, i\,\partial^{23} - i\,\partial^{31})$$
$$\times \frac{\delta^{(d)}(x_{31})}{(x^2_{12})^{d/2-1}(x^2_{23})^{d/2-1}}$$

$$\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\mu\nu}(x_1)\delta g_{\rho\sigma}(x_2)}\,\frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)}\right\rangle^S = \frac{C_P^S}{2}\, V^{\alpha\beta}_{T\phi\phi}(i\,\partial^{31}, -i\,\partial^{23})\, V^{\mu\nu\alpha\beta}_{TT\phi\phi}(i\,\partial^{23}, -i\,\partial^{31}, -i\,\partial^{23} + i\,\partial^{12})$$
$$\times \frac{\delta^{(d)}(x_{12})}{(x_{23}^2)^{d/2-1}(x_{31}^2)^{d/2-1}}$$
$$\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\alpha\beta}(x_3)\delta g_{\rho\sigma}(x_2)}\,\frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)}\right\rangle^S = \frac{C_K^S}{2}\, V^{\mu\nu}_{T\phi\phi}(i\,\partial^{12}, -i\,\partial^{31})\, V^{\alpha\beta\rho\sigma}_{TT\phi\phi}(i\,\partial^{31}, -i\,\partial^{12}, i\,\partial^{12} - i\,\partial^{23})$$
$$\times \frac{\delta^{(d)}(x_{23})}{(x_{12}^2)^{d/2-1}(x_{31}^2)^{d/2-1}}. \tag{4.11}$$

Notice that in the three contributions above, the p, q, and l dependence of the vertices correspond to mappings into $\hat{p}, \hat{q}$ and $\hat{l}$ which are specific for each T-bubble. Similarly, in the fermion sector we obtain

$$\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\mu\nu}(x_1)\delta g_{\alpha\beta}(x_3)}\,\frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)}\right\rangle^F =$$
$$-C_Q^F\,\delta^{(d)}(x_{31})\,\mathrm{tr}\left[V^{\mu\nu\alpha\beta}_{TT\bar\psi\psi}(i\,\partial^{12}, -i\,\partial^{23})\, i\,\gamma\cdot\partial^{12} V^{\rho\sigma}_{T\bar\psi\psi}(i\,\partial^{23}, -i\,\partial^{12})\, i\,\gamma\cdot\partial^{23}\right]$$
$$\times \frac{1}{(x_{23}^2)^{d/2-1}(x_{12}^2)^{d/2-1}}, \tag{4.12}$$

and similar expressions for the $k-$ and p-bubles. Finally, for the spin-1 vector field we obtain

$$\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\mu\nu}(x_1)\delta g_{\alpha\beta}(x_3)}\,\frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)}\right\rangle^V =$$
$$\frac{C_Q^V}{2}\,\delta^{(d)}(x_{31})\,\tilde V^{\mu\nu\rho\alpha\beta\chi}_{TTAA}(i\,\partial^{12}, -i\,\partial^{23})\,\tilde V^{\rho\sigma\tau\omega}_{TAA}(i\,\partial^{23}, -i\,\partial^{12})\,\frac{\delta_{\zeta\tau}\,\delta_{\chi\omega}}{x_{12}^2\,x_{23}^2}, \tag{4.13}$$

and similarly for the other bubble-type contributions.

Notice that this expression is affected by terms proportional to derivatives of δ functions. We refer to appendix E for more details on the specific structures of these terms in momentum space, where we illustrate this point in a simple case. The complete structure of the TTT vertex in position space is obtained by combining the triangle and the "K","P" and "Q"-bubble topologies in the form

$$\left\langle T^{\mu\nu}(x_1)\,T^{\rho\sigma}(x_2)\,T^{\alpha\beta}(x_3)\right\rangle = \sum_{I=S,F,V} 8\left[-\left\langle \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)}\,\frac{\delta \mathcal{S}}{\delta g_{\sigma\rho}(x_3)}\,\frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_2)}\right\rangle^I\right.$$
$$+\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\mu\nu}(x_1)\,\delta g_{\alpha\beta}(x_3)}\,\frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}(x_2)}\right\rangle^I + \left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\mu\nu}(x_1)\,\delta g_{\rho\sigma}(x_2)}\,\frac{\delta \mathcal{S}}{\delta g_{\alpha\beta}(x_3)}\right\rangle^I$$
$$\left.+\left\langle \frac{\delta^2 \mathcal{S}}{\delta g_{\alpha\beta}(x_3)\,\delta g_{\rho\sigma}(x_2)}\,\frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x_1)}\right\rangle^I\right]. \tag{4.14}$$

This expression is in agreement with the form of the energy-momentum tensor three point function given in [27]. The integrability of this result is manifest, due to the $(d/2-1)$ exponent of each propagator in position space, which corresponds, generically, to a $1/l^2$ behavior in momentum space. The vector terms, which exist in $d = 4$ are, obviously, Fourier integrable.

5 Moving to momentum space using Lagrangian realizations

At this point we use again the free field theory representation of these correlators to study their expression in momentum space. This will allow us to perform a direct comparison between position space and momentum space approaches for correlators affected by the trace anomaly. We start by investigating the perturbative structure of these theories and derive the corresponding vertices.

The actions for the scalar and the fermion field are respectively

$$\mathcal{S}_{\text{scalar}} = \frac{1}{2}\int d^4x\,\sqrt{g}\left[g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi - \chi\,R\,\phi^2\right], \tag{5.1}$$

$$\mathcal{S}_{\text{fermion}} = \frac{1}{2}\int d^4x\,V\,V_\alpha{}^\rho\left[\bar{\psi}\,\gamma^\alpha\,(\mathcal{D}_\rho\,\psi) - (\mathcal{D}_\rho\,\bar{\psi})\,\gamma^\alpha\,\psi\right], \tag{5.2}$$

where χ is the parameter corresponding to the "improvement term", that we have chosen to be $1/6$ in the diagrammatic calculation so to deal with the classically conformal invariant theory. $V_\alpha{}^\rho$ is the vielbein and V $(=\sqrt{g})$ its determinant, needed in such a way to embed the fermion in the curved background, with its covariant derivative $\mathcal{D}_\mu$ as

$$\mathcal{D}_\mu = \partial_\mu + \Gamma_\mu = \partial_\mu + \frac{1}{2}\,\Sigma^{\alpha\beta}\,V_\alpha{}^\sigma\,\nabla_\mu\,V_{\beta\sigma}\,. \tag{5.3}$$

The $\Sigma^{\alpha\beta}$ are the generators of the Lorentz group in the case of a spin 1/2-field.

The action $\mathcal{S}$ for the photon field is given by

$$\mathcal{S}_{\text{photon}} = \mathcal{S}_M + \mathcal{S}_{gf} + \mathcal{S}_{gh}\,, \tag{5.4}$$

where the three contributions are the Maxwell action, the gauge fixing contribution and the ghost action

$$\mathcal{S}_M = \frac{1}{4}\int d^4x\,\sqrt{g}\,F^{\alpha\beta}F_{\alpha\beta}\,, \tag{5.5}$$

$$\mathcal{S}_{gf} = \frac{1}{2\xi}\int d^4x\,\sqrt{g}\,\left(\nabla_\alpha A^\alpha\right)^2 \tag{5.6}$$

$$\mathcal{S}_{gh} = -\int d^4x\,\sqrt{g}\,\partial^\alpha\bar{c}\,\partial_\alpha c\,. \tag{5.7}$$

We will be using Euclidean conventions for the generating functional, given by

$$\mathcal{W} = \frac{1}{\mathcal{N}}\int \mathcal{D}A_\mu\,\mathcal{D}\bar{c}\,\mathcal{D}c\,e^{-\mathcal{S}_E[A_\mu,\bar{c},c]}\,. \tag{5.8}$$

We will omit the "E" subscript from now on, as already done in (2.6), to keep our notation easy.

The energy-momentum tensor is defined in (2.4), which becomes, in the fermionic case,

$$T^{\mu\nu} = -\frac{1}{V}\,V^{\alpha\mu}\,\frac{\delta\mathcal{S}}{\delta V^\alpha{}_\nu}\,. \tag{5.9}$$

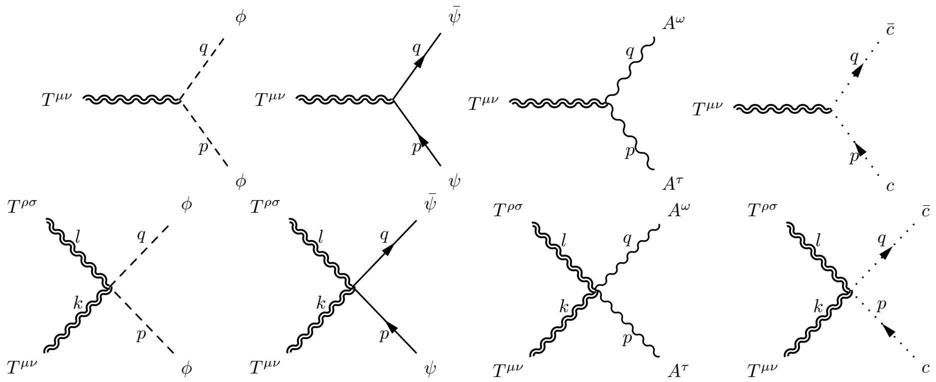

Figure 3. List of the vertices used in the Lagrangian mapping of the conformal correlators

This tensor is not symmetric in general, but its antisymmetric parts do not contribute to our calculations, so that, for our purposes, we can adopt the symmetric definition

$$T^{\mu\nu} \stackrel{def}{\equiv} -\frac{1}{2\,V}\left(V^{\alpha\mu}\,\frac{\delta}{\delta V^{\alpha}{}_{\nu}} + V^{\alpha\nu}\,\frac{\delta}{\delta V^{\alpha}{}_{\mu}}\right)\mathcal{S} \tag{5.10}$$

as well. The energy-momentum tensors for the scalar and the fermion are

$$T^{\mu\nu}_{\text{scalar}} = \nabla^\mu\phi\,\nabla^\nu\phi - \frac{1}{2}\,g^{\mu\nu}\,g^{\alpha\beta}\,\nabla_\alpha\phi\,\nabla_\beta\phi + \chi\left[g^{\mu\nu}\Box - \nabla^\mu\,\nabla^\nu + \frac{1}{2}\,g^{\mu\nu}\,R - R^{\mu\nu}\right]\phi^2 \tag{5.11}$$

$$T^{\mu\nu}_{\text{ferm}} = \frac{1}{4}\left[g^{\mu\rho}\,V_\alpha{}^\nu + g^{\nu\rho}\,V_\alpha{}^\mu - 2\,g^{\mu\nu}\,V_\alpha{}^\rho\right]\left[\bar\psi\,\gamma^\alpha\,\left(\mathcal{D}_\rho\,\psi\right) - \left(\mathcal{D}_\rho\,\bar\psi\right)\,\gamma^\alpha\,\psi\right], \tag{5.12}$$

while the energy-momentum tensor for the photon field is given by the sum of three terms

$$T^{\mu\nu}_{QED} = T^{\mu\nu}_{M} + T^{\mu\nu}_{gf} + T^{\mu\nu}_{gh}\,, \tag{5.13}$$

with

$$T^{\mu\nu}_{M} = F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\,, \tag{5.14}$$

$$T^{\mu\nu}_{gf} = -\frac{1}{\xi}\left\{A^\mu\nabla^\nu(\nabla_\rho A^\rho) + A^\nu\nabla^\mu(\nabla_\rho A^\rho) - g^{\mu\nu}[A^\rho\nabla_\rho(\nabla_\sigma A^\sigma) + \frac{1}{2}(\nabla_\rho A^\rho)^2]\right\}\,, \tag{5.15}$$

$$T^{\mu\nu}_{gh} = g^{\mu\nu}\partial^\rho\bar{c}\,\partial_\rho c - \partial^\mu\bar{c}\,\partial^\nu c - \partial^\nu\bar{c}\,\partial^\mu c\,. \tag{5.16}$$

The computation of the vertices can be done by taking (at most) two functional derivatives of the action with respect to the metric, since the vacuum expectation values of the third order derivatives correspond to massless tadpoles, which are zero in dimensional regularization. Given the complexity of the result and to avoid any error, we have checked that all the expressions obtained for the 1-loop vertices satisfy the corresponding Ward identities derived in the previous sections. They are given in figure 3 and their explicit expressions have been collected in appendix D.

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5.1 The interpretation of the counterterms: the TT case

In this section we begin a discussion of the structure of anomalous correlators in momentum space, starting, for simplicity, from the TT case in the conformal limit. In the non-conformal case this correlator has been investigated in [5, 6] in the worldline approach.

This is a warm-up case before the more involved analysis of the 3-point functions that we will discuss afterwards. As we are going to see, the interpretation of the anomaly and of its origin, in the process of renormalization, can be different in position and in momentum space. In fact, the anomaly can be attributed either to the specific structure of the counterterm in dimensional regularization, which violates conformal invariance in d dimensions, while being traceless in $d = 4$ or, alternatively, to the renormalized amplitude in d=4. In this second case the anomaly emerges as a feature of the $d = 4$ renormalized amplitude and, specifically, of its 4-dimensional trace.

In the TT case conformal symmetry fixes this correlator up to constant, and one can proceed with the Fourier transform without resorting to a specific free field theory realization. Using the inversion matrix in Euclidean space, we define the conformal energy-momentum tensor two-point function as

$$\left\langle T^{\mu\nu}(x)\, T^{\alpha\beta}(y)\right\rangle = \frac{C_T}{s^{2d}}\, \mathcal{I}^{\mu\nu,\alpha\beta}(s)\,, \tag{5.17}$$

where $\mathcal{I}^{\mu\nu,\alpha\beta}(s)$ was defined in (4.2) and (4.3).

In order to move in the framework of differential regularization, we pull out some derivatives and rewrite our correlator as

$$\left\langle T^{\mu\nu}(x)\, T^{\alpha\beta}(0)\right\rangle = \frac{C_T}{4(d-2)^2 d(d+1)}\, \hat{\Delta}^{(d)\,\mu\nu\alpha\beta}\frac{1}{x^{2d-4}}\,, \tag{5.18}$$

where

$$\hat{\Delta}^{(d)\,\mu\nu\alpha\beta} = \frac{1}{2}\left(\hat{\Theta}^{\mu\alpha}\hat{\Theta}^{\nu\beta} + \hat{\Theta}^{\mu\beta}\hat{\Theta}^{\nu\alpha}\right) - \frac{1}{d-1}\hat{\Theta}^{\mu\nu}\hat{\Theta}^{\alpha\beta}\,, \quad \text{with} \quad \hat{\Theta}^{\mu\nu} = \partial^\mu\partial^\nu - \delta^{\mu\nu}\,\Box \tag{5.19}$$

$$\partial_\mu\, \hat{\Delta}^{(d)\,\mu\nu\alpha\beta} = 0\,, \quad \delta_{\mu\nu}\, \hat{\Delta}^{(d)\,\mu\nu\alpha\beta} = 0\,. \tag{5.20}$$

For reasons that will be discussed in section 8, this form of the TT correlator is Fourier-integrable, although it is characterized by a UV divergence as $x \to y$. To move to momentum space we can split the $1/(x^2)^{d-2}$ term into the product of two $1/(x^2)^{d/2-1}$ factors and apply straightforwardly the fundamental transform (c.f. eq. (8.1)), obtaining

$$\begin{aligned}
\left\langle T^{\mu\nu}\, T^{\alpha\beta}\right\rangle(p) &\equiv \int d^dx\, \left\langle T^{\mu\nu}(x) T^{\alpha\beta}(0)\right\rangle e^{-i\,p\cdot x} \\
&= \frac{C_T}{4(d-2)^2 d(d+1)} \int d^dx\, e^{-i\,p\cdot x}\, \hat{\Delta}^{(d)\,\mu\nu\alpha\beta}\, \frac{1}{(x^2)^{d/2-1}}\,\frac{1}{(x^2)^{d/2-1}} \\
&= \frac{(2\pi)^d\, C(d/2-1)^2\, C_T}{4(d-2)^2 d(d+1)}\, \Delta^{(d)\,\mu\nu\alpha\beta}(p) \int d^dl\, \frac{1}{l^2(l+p)^2}\,.
\end{aligned} \tag{5.21}$$

We have also defined

$$\Theta^{\mu\nu}(p) = \delta^{\mu\nu}\, p^2 - p^\mu\, p^\nu \tag{5.22}$$

$$\Delta^{(d)\,\mu\nu\alpha\beta}(p) = \frac{1}{2}\left(\Theta^{\mu\alpha}(p)\,\Theta^{\nu\beta}(p) + \Theta^{\mu\beta}(p)\,\Theta^{\nu\alpha}(p)\right) - \frac{1}{d-1}\,\Theta^{\mu\nu}(p)\,\Theta^{\alpha\beta}(p) \tag{5.23}$$

as the momentum space counterparts of the two operators previously introduced. In our notations $\Delta^{(4)\,\mu\nu\alpha\beta}$ is obtained from the expression above by setting $d = 4$. The tensor indices, however, are still running from 0 to $d-1$.

Notice that in d dimensions the TT correlator is anomaly-free (i.e. traceless)

$$\delta_{\mu\nu}\left\langle T^{\mu\nu}\, T^{\alpha\beta}\right\rangle(p) = \delta_{\alpha\beta}\left\langle T^{\mu\nu}\, T^{\alpha\beta}\right\rangle(p) = 0\,. \tag{5.24}$$

As we move to $d = 4$ the correlator in momentum space has a UV singularity, coming from the 2-point integral

$$\begin{aligned}\mathcal{B}_0(p^2) &= \frac{1}{\pi^2}\int d^dl\,\frac{1}{l^2\,(l+p)^2} = \frac{\left[\Gamma(1-\epsilon/2)\right]^2\Gamma(\epsilon/2)}{\Gamma(2-\epsilon)}\,\frac{1}{(\pi\, p^2)^{\epsilon/2}}\\ &= \frac{2}{\bar\epsilon} + 2 + \ln\left(\frac{\mu^2}{p^2}\right) + O(\epsilon)\,,\end{aligned} \tag{5.25}$$

where $\epsilon = 4 - d$ and we have introduced the quantity $\frac{2}{\bar\epsilon} = \frac{2}{\epsilon} - \gamma - \ln\pi$, typical of the modified minimal subtraction ($\overline{MS}$) scheme. If we work in position space, renormalization is enforced by adding a local (i.e. $\sim\delta(x-y)$) counterterm of the form $c_1/\bar\epsilon\,\hat\Delta^{(4)\mu\nu\alpha\beta}\,\delta(x-y)$. The regulated correlator in $d = 4$ is then defined as

$$\left\langle T^{\mu\nu}(x)\, T^{\alpha\beta}(0)\right\rangle = \frac{C_T}{4(d-2)^2 d(d+1)}\,\hat\Delta^{(d)\,\mu\nu\alpha\beta}\frac{1}{x^{2d-4}} + \frac{c_1}{\bar\epsilon}\,\hat\Delta^{(4)\,\mu\nu\alpha\beta}\,\delta^d(x-y)\,. \tag{5.26}$$

Notice that the counterterm is traceless for $d = 4$ (i.e. contracting the indices with a 4-dimensional metric) but not in general dimensions. Therefore, if we split the d-dimensional metric ($\delta^{(d)}_{\mu\nu}$) as a direct sum ($\oplus$) of a 4-dimensional ($\delta_{\mu\nu} \equiv \delta^{(4)}_{\mu\nu}$) and of a $(d-4)$-dimensional metrics acting on the subspaces M_4 and M_{d-4} (i.e. $M_d = M_4 \oplus M_{d-4}$) we obtain

$$\delta^{(d)}_{\mu\nu}\,\hat\Delta^{(4)\mu\nu\alpha\beta} = \delta^{(4)}_{\mu\nu}\,\hat\Delta^{(4)\mu\nu\alpha\beta} + \delta^{(d-4)}_{\mu\nu}\,\hat\Delta^{(4)\mu\nu\alpha\beta} = \delta^{(d-4)}_{\mu\nu}\,\hat\Delta^{(4)\mu\nu\alpha\beta} \tag{5.27}$$

and using the relation

$$\delta^{(4)}_{\mu\nu}\,\hat\Delta^{(4)\mu\nu\alpha\beta} = 0 \tag{5.28}$$

we find that the d-dimensional trace of $\hat\Delta^{(4)}$ is $O(\epsilon)$

$$\delta^{(d)}_{\mu\nu}\,\hat\Delta^{(4)\,\mu\nu\alpha\beta} = -\frac{\epsilon}{3}\,\Box\,\hat\Theta^{\alpha\beta}\,. \tag{5.29}$$

If we now use the relation $\delta_{\mu\nu}\hat\Delta^{(d)\,\mu\nu\alpha\beta} = 0$, it is clear that the trace of renormalized TT correlator gives the correct anomaly. In particular, the trace operation cancels the $1/\epsilon$ pole of the counterterm

$$\begin{aligned}\delta^{(d)}_{\mu\nu}\left\langle T^{\mu\nu}(x)\, T^{\alpha\beta}(0)\right\rangle &= c_1\,\frac{1}{\bar\epsilon}\,\delta^{(d-4)}_{\mu\nu}\,\hat\Delta^{(4)\,\mu\nu\alpha\beta}\,\delta^d(x-y)\\ &= \frac{c_1}{3}\left[-1 + \frac{\epsilon}{2}(\gamma + \ln\pi)\right]\Box\hat\Theta^{\alpha\beta}\,\delta^d(x-y)\end{aligned} \tag{5.30}$$

which is finite as $\epsilon \to 0$ and reproduces the expected anomaly. The selection of the counterterm is in agreement with the anomalous Ward identity of the 2-point function in momentum space. This can be checked directly from eq. (2.1), by computing its first functional derivative around flat space, which leaves $\Box R$ as the only contribution to the TT anomaly

$$\delta^{(4)}_{\mu\nu} \left\langle T^{\mu\nu}\, T^{\alpha\beta} \right\rangle (p) = 2\, \beta_c \left[\Box R\right]^{\alpha\beta}(p) = 2\, \beta_c\, p^2\, \Theta^{\alpha\beta}(p)\,. \tag{5.31}$$

Below we will be omitting the subscript (4) when referring to a 4-dimensional kronecker $\delta_{\mu\nu}$, unless it is strictly necessary for clarity.

In the expression above, we have introduced the notation $\left[\Box R\right]^{\alpha\beta}(p)$ to indicate the Fourier-transformed functional derivative of the box ($\Box$) of the scalar curvature evaluated in the limit of flat spacetime. The last two equations allow us to fix the final structure of the fully renormalized correlator in the form

$$\begin{aligned} \left\langle T^{\mu\nu}\, T^{\alpha\beta} \right\rangle_{\text{ren}} (p) &= \left\langle T^{\mu\nu}\, T^{\alpha\beta} \right\rangle_{\text{bare}} (p) + 6\, \frac{\beta_c}{\bar\epsilon}\, \Delta^{(4)\,\mu\nu\alpha\beta}(p) \\ &= \left\langle T^{\mu\nu}\, T^{\alpha\beta} \right\rangle_{\text{bare}} (p) - 4\, \frac{\beta_a}{\bar\epsilon}\, \Delta^{(4)\,\mu\nu\alpha\beta}(p)\,, \end{aligned} \tag{5.32}$$

where we have used in the last step eq. (2.3).

In position space, as clear from (5.30), the anomaly can be attributed to the counterterm. This approach allows to write down the solution of the Ward identities as an anomaly free solution (for $x \neq y$) superimposed to the inhomogenous terms, exactly as stated in eq. (5.26). This procedure is general, and can be applied to any correlator.

It is instructive, for comparison, to comment on the same approach in dimensional regularization working in momentum space. One can start from a field theory realization of the same (unrenormalized) correlator obtaining

$$\begin{aligned} \left\langle T^{\mu\nu}\, T^{\alpha\beta} \right\rangle (p) &= \left\{ \frac{1}{2} \left[\Theta^{\mu\alpha}(p)\, \Theta^{\nu\beta}(p) + \Theta^{\mu\beta}(p)\, \Theta^{\nu\alpha}(p) \right] - \frac{1}{3}\, \Theta^{\mu\nu}(p)\, \Theta^{\alpha\beta}(p) \right\} C_1(p) \\ &\quad + \frac{1}{3}\, \Theta^{\mu\nu}(p)\, \Theta^{\alpha\beta}(p)\, C_2(p) \\ &\equiv \Delta^{(4)\,\mu\nu\alpha\beta}(p)\, C_1(p) + \frac{1}{3} \Theta^{\mu\nu}(p)\, \Theta^{\alpha\beta}(p)\, C_2(p)\,, \end{aligned} \tag{5.33}$$

where the form factors are given, in the cases of a conformally coupled scalar, a Dirac fermion and a photon, by

$$C_1(p)\Big|_{\text{conf.scalar}} = \frac{16 + 15\, \mathcal{B}_0(p^2)}{14400\, \pi^2} \qquad C_2(p)\Big|_{\text{conf.scalar}} = -\frac{1}{1440\, \pi^2}\,, \tag{5.34}$$

$$C_1(p)\Big|_{\text{Dir.fermion}} = \frac{2 + 5\, \mathcal{B}_0(p^2)}{800\, \pi^2}\,, \qquad C_2(p)\Big|_{\text{Dir.fermion}} = -\frac{1}{240\, \pi^2}\,, \tag{5.35}$$

$$C_1(p)\Big|_{\text{photon}} = \frac{-11 + 10\, \mathcal{B}_0(p^2)}{800\, \pi^2}\,, \qquad C_2(p)\Big|_{\text{photon}} = -\frac{1}{120\, \pi^2}\,. \tag{5.36}$$

Notice that the singularity of eq. (5.33) is contained in the expressions of $C_1(p)$ due to the presence of the scalar 2-point function $\mathcal{B}_0$ which needs to be renormalized. The constant terms in these coefficients are due to the mass-independent renormalization of the correlator, here performed in dimensional regularization, which, for each separate case, conformal scalar, fermion and photon, can be absorbed into a redefined renormalization scale μ. The two structures in the last line of (5.33) separately respect the energy-momentum conservation Ward identity for the 2-point function 2.38, but only the first one, $\Delta^{(4)\,\alpha\beta\rho\sigma}(p)$, is traceless in $d = 4$, while tracing the second we obtain the anomalous relation

$$\delta_{\mu\nu} \frac{1}{3}\, \Theta^{\mu\nu}(p)\, \Theta^{\alpha\beta}(p) = p^2\, \Theta^{\alpha\beta}(p)\,. \tag{5.37}$$

The singular contribution in eq. (5.33) can be eliminated by the ordinary renormalization procedure, leaving a result that is finite and whose trace can be taken *directly in 4 dimensions*. In this approach the anomaly can be attributed to the regularization procedure and not directly to the counterterm, which is traceless (compare (5.27) for d = 4), while it is the finite part of the correlator, going like $C_2(p)$, to be anomalous.

The complete TT correlation function and its positive spectral functions were calculated in both the tensor and scalar sectors for a scalar field of arbitrary mass and curvature coupling ξ in 4-dimensions in [1]. In the case of general mass and ξ, conformal invariance does not hold and the second tensor structure in (5.33) is always present. By taking $\xi = 1/6$ and the limit of zero mass, one can also see from the spectral function approach in [1] how the trace anomaly appears.

As in the case of the chiral anomaly, a dispersive analysis shows that the spectral density of an anomalous correlator is affected, under certain circumstances, by typical contributions which amount to *anomaly poles*. Anomaly poles emerge from a collinear configuration of a certain amplitude interpreted as a real space-time (on-shell) process. Similar poles have been found in the TT case in 2-dimensions [7]. In higher dimensions because of the kinematics explained in [11] one must go at least to triangle amplitudes at least as complicated as TVV or TTT in order to find these pole terms.

We have stressed this point to emphasize that the approach followed in position space, which consists in the addition of a contact counterterm to regulate the anomaly, is not in contradiction with the ordinary diagrammatic picture. It simply doesn't give the complete kinematical understanding of the origin of the anomaly, which the spectral function dispersive approach attributes to the existence of a collinear region in the (anomalous) diagrams of the perturbative expansion. In the following, we will try to match these two quite different descriptions by discussing more complex correlators.

6 The counterterm for the TVV in position and in momentum space

We now turn to the question of the renormalization of TVV correlator in $d = 4$ dimensions. This can be performed either 1) by solving the renormalized Ward identities in position space or 2) by a perturbative computation in momentum space of all the diagrams in dimensional regularization. The two methods are obviously quite different and the goal of this section is to test their correspondence, given the results of [27].

As already emphasized in section (5.1), the renormalized 3-point functions have to satisfy the requirement of general covariance as well as renormalized anomalous Ward identities. The solution of these identities can be directly found by rewriting them in momentum space. For the $\langle TVV\rangle$ case, the requirement of general covariance is also supplemented with gauge current conservation. If we denote our counterterm by $D_{\mu\nu\alpha\beta}(p,q)$, the algebraic conditions satisfied by the counterterm are given by

$$\begin{aligned}(p+q)^\mu\, D_{\mu\nu\alpha\beta}(p,q) &= q_\nu\,\Theta_{\alpha\beta}(p) - \delta_{\nu\beta}\, q^\mu\,\Theta_{\mu\alpha}(p) + p_\nu\,\Theta_{\alpha\beta}(q) - \delta_{\nu\alpha}\, p^\mu\,\Theta_{\mu\beta}(q)\,,\\ p^\alpha\, D_{\mu\nu\alpha\beta}(p,q) &= q^\beta\, D_{\mu\nu\alpha\beta}(p,q) = 0\,,\end{aligned} \tag{6.1}$$

with $\Theta_{\alpha\beta}(p)$ being the counterterm for the vector-vector 2-point function. In fact, the equations above are just the divergent parts of the general covariance and gauge invariance Ward identities for our three point function,

$$\begin{aligned}(p+q)^\mu\,\left\langle T_{\mu\nu}\,{V^a}_\alpha\,{V^b}_\beta\right\rangle(p,q) &= q_\nu\,\left\langle {V^a}_\alpha\,{V^b}_\beta\right\rangle(p) - \delta_{\nu\beta}\, q^\mu\,\left\langle {V^a}_\mu\,{V^b}_\alpha\right\rangle(p) + \\ &\qquad p_\nu\,\left\langle {V^a}_\alpha\,{V^b}_\beta\right\rangle(q) - \delta_{\nu\alpha}\, p^\mu\,\left\langle {V^a}_\mu\,{V^b}_\beta\right\rangle(q)\,,\\ p^\alpha\,\left\langle T_{\mu\nu}\,{V^a}_\alpha\,{V^b}_\beta\right\rangle(p,q) &= q^\beta\,\left\langle T_{\mu\nu}\,{V^a}_\alpha\,{V^b}_\beta\right\rangle(p,q) = 0\,.\end{aligned} \tag{6.2}$$

To see how they arise, we introduce the counterterms for the two correlators at hand, modulo two constants

$$\begin{aligned}\left\langle {V^a}_\alpha\,{V^b}_\beta\right\rangle_{\rm ren}(p) &= \left\langle {V^a}_\alpha\,{V^b}_\beta\right\rangle_{\rm bare}(p) + \frac{1}{\epsilon}\, C_{VV}\,\Theta_{\alpha\beta}(p)\,,\\ \left\langle T_{\mu\nu}\,{V^a}_\alpha\,{V^b}_\beta\right\rangle_{\rm ren}(p,q) &= \left\langle T_{\mu\nu}\,{V^a}_\alpha\,{V^b}_\beta\right\rangle_{\rm bare}(p,q) + \frac{1}{\epsilon}\, C_{TVV}\, D_{\mu\nu\alpha\beta}(p,q)\,.\end{aligned} \tag{6.3}$$

Replacing them in (6.2) and equating the coefficients of the $1/\epsilon$ terms we immediately obtain (6.1) and the condition $C_{VV} = C_{TVV}$. These constraints are sufficient to state that the counterterm is

$$\begin{aligned}D_{\mu\nu\alpha\beta}(p,q) &= \delta_{\alpha\beta}\,(p_\mu\, q_\nu + q_\mu\, p_\nu) + p\cdot q\,(\delta_{\mu\beta}\,\delta_{\nu\alpha} + \delta_{\mu\alpha}\,\delta_{\nu\beta})\\ &\quad -(\delta_{\beta\nu}\, p_\mu + \delta_{\beta\mu}\, p_\nu)\, q_\alpha - (\delta_{\mu\alpha}\, q_\nu + \delta_{\alpha\nu}\, q_\mu)\, p_\beta - \delta_{\mu\nu}\,(p\cdot q\,\delta_{\alpha\beta} - q_\alpha\, p_\beta)\,.\end{aligned} \tag{6.4}$$

A consistency condition on this tensor, which is easily seen to be satisfied, is that the trace anomaly constraint in d dimensions,

$$\delta^{\mu\nu}\, D_{\mu\nu\alpha\beta}(p,q) = (4-d)\,(p\cdot q\,\delta_{\alpha\beta} - q_\alpha\, p_\beta) \equiv \epsilon\,(p\cdot q\,\delta_{\alpha\beta} - q_\alpha\, p_\beta) \tag{6.5}$$

reproduces the anomaly.

It is instructive to see how the same operation can be performed diagrammatically. For this purpose we just recall that the general form of the TVV amplitude can be expanded in a basis of 13 tensor structures $t_i^{\mu\nu\alpha\beta}(p,q)$ defined in [21]

$$\Gamma_{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{13} F_i(k^2;p^2,q^2)\; t^i_{\mu\nu\alpha\beta}(p,q)\,, \tag{6.6}$$

where we have defined the tensors

$$u^{\alpha\beta}(p,q) \equiv (p\cdot q)\,\delta^{\alpha\beta} - q^\alpha\, p^\beta\,, \tag{6.7}$$

$$w^{\alpha\beta}(p,q) \equiv p^2\, q^2\,\delta^{\alpha\beta} + (p\cdot q)\, p^\alpha\, q^\beta - q^2\, p^\alpha\, p^\beta - p^2\, q^\alpha\, q^\beta\,, \tag{6.8}$$

which are Bose symmetric,

$$u^{\alpha\beta}(p,q) = u^{\beta\alpha}(q,p)\,, \tag{6.9}$$
$$w^{\alpha\beta}(p,q) = w^{\beta\alpha}(q,p)\,. \tag{6.10}$$

Gauge invariance is respected due to the conditions

$$p_\alpha\, u^{\alpha\beta}(p,q) = q_\beta\, u^{\alpha\beta}(p,q) = 0\,, \tag{6.11}$$
$$p_\alpha\, w^{\alpha\beta}(p,q) = q_\beta\, w^{\alpha\beta}(p,q) = 0\,. \tag{6.12}$$

A complete perturbative analysis shows that the only tensor structure which is affected by the renormalization procedure is t_{13}, which coincides with the $D_{\mu\nu\alpha\beta}$ counterterm introduced above. As discussed in [21] for QED and in [3, 4] for QED and QCD by direct computations, renormalization of the TVV vertex affects only this tensor structure. Given the complexity of the computations and the wide difference between the general CFT approach and the ordinary diagrammatic one, this agreement is obviously nontrivial. As in the TT case, the anomaly is generated by the $(d-4)$-dimensional part of the trace, which simplifies with the $1/(d-4)$ factor in front of the counterterm. In particular, all our previous comments concerning the renomalization of the TT case remain valid also here, since in our approach the anomaly is computed after subtracting the infinities, by taking the 4-dimensional trace of the renormalized TVV vertex. In particular, one can check that of the 13 structures t_i only t_1 has a non-vanishing trace, while the remaining ones are traceless. As discussed in [21] for the fermion case, t_1 carries all the information about the anomaly and its corresponding form factor (F_1) contains an anomaly pole. The extraction of this additional information about the TVV correlator indeed requires a complete analysis of the same in momentum space.

6.1 TVV on-shell in $d = 4$ and the anomaly poles

As we have mentioned, the complete TVV correlator can be obtained in any dimension as a superposition of a scalar and of a fermion sectors. Obviously, this result holds for any CFT, and the explicit evaluation that we provide is completely general. In the off-shell case the fermion loop has been analyzed in [3, 21]. Explicit resuls for this sector can be found in [3]. In this section we extend the computation to the scalar sector, focusing on the on-shell case for the two external vectors, since the expressions in the general case are far lengthier.

In the on-shell case the 13 structures t^i simplify drastically. We use three structures A^1, A^2 and D, with D being the counterterm discussed above, to describe the parameterization of this vertex. In terms of the momenta of the two outgoing gauge bosons (p,q), with $p^2 = q^2 = 0$ and $p \cdot q = k^2/2$ we have

$$\Gamma^{ab}_{\mu\nu\alpha\beta}(p,q)^{f/s} = F^{ab}_1(p\cdot q)^{f/s} A^1_{\mu\nu\alpha\beta}(p,q) + F^{ab}_2(p\cdot q)^{f/s} A^2_{\mu\nu\alpha\beta}(p,q) + F^{ab}_3(p\cdot q)^{f/s} D_{\mu\nu\alpha\beta}(p,q) \tag{6.13}$$

with

$$A^1_{\mu\nu\alpha\beta} = (2p\cdot q\,\delta^{\mu\nu} - k^\mu k^\nu)\, u^{\alpha\beta}(p,q)\,, \tag{6.14}$$
$$A^2_{\mu\nu\alpha\beta} = -2\, u^{\alpha\beta}(p,q)\,(2p\cdot q\,\delta^{\mu\nu} + 2(p^\mu\, p^\nu + q^\mu\, q^\nu) - 4\,(p^\mu\, q^\nu + q^\mu\, p^\nu))\,, \tag{6.15}$$

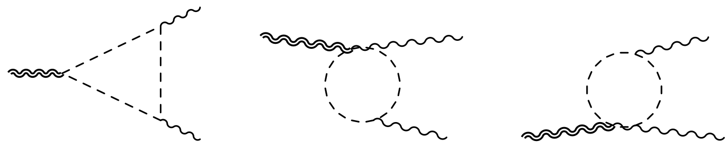

Figure 4. The fermion/scalar sectors in the TVV vertex.

with form factors given by

$$
\begin{aligned}
F_1^{ab}(p\cdot q)^f &= \delta^{ab}\,\frac{1}{72\,\pi^2\,p\cdot q}\,, && (6.16)\\
F_2^{ab}(p\cdot q)^f &= \delta^{ab}\,\frac{1}{576\pi^2\,p\cdot q}\,, && (6.17)\\
F_3^{ab}(p\cdot q)^f &= -\delta^{ab}\,\frac{1}{288\pi^2}\Big[12\mathcal{B}_0(2p\cdot q,0,0)+11\Big]\,, && (6.18)
\end{aligned}
$$

for the fermion sector and

$$
\begin{aligned}
F_1^{ab}(p\cdot q)^s &= \delta^{ab}\,\frac{1}{144\pi^2\,p\cdot q}\,, && (6.19)\\
F_2^{ab}(p\cdot q)^s &= -\delta^{ab}\,\frac{1}{576\pi^2\,p\cdot q}\,, && (6.20)\\
F_3^{ab}(p\cdot q)^s &= -\delta^{ab}\,\frac{1}{576\pi^2}\Big[6\mathcal{B}_0(2p\cdot q,0,0)+7\Big]\,, && (6.21)
\end{aligned}
$$

for the scalar sector. Notice that both the scalar (s) and the fermion (f) sectors have anomaly poles. The anomaly is attributed to the tensor structure A_1 which has a nonzero trace. As we have clarified above, the anomaly is not attributed to D (i.e. t_{13}), which is the counterterm found in position space, but to the tensor structure A_1, after renormalization. The remaining structures A_2 and D are, in fact, traceless in 4-dimensions. This structure coincides with the form factor t_1 of [21], which has a nonzero trace. As remarked before, the dynamical origin of the trace anomaly has necessarily to be found in momentum space.

6.2 TVV in d dimension

These results can be generalized, with some extra effort, to d dimensions. By our inverse mapping procedure the result of the computation in this case remains valid for any conformal theory, since the two sectors, scalar and fermion, are sufficient to describe the general solution of the Ward identities. The result can be given in a form which is quite similar to those in (6.13). We obtain

$$
\begin{aligned}
\Gamma^{ab}_{\mu\nu\alpha\beta}(p,q)_f &= f_1^{ab}(p\cdot q)C^f_{\mu\nu\alpha\beta}(p,q)+f_2^{ab}(p\cdot q)D_{\mu\nu\alpha\beta}(p,q)\,,\\
\Gamma^{ab}_{\mu\nu\alpha\beta}(p,q)_s &= s_1^{ab}(p\cdot q)C^s_{\mu\nu\alpha\beta}(p,q)+s_2^{ab}(p\cdot q)D_{\mu\nu\alpha\beta}(p,q)\,. && (6.22)
\end{aligned}
$$

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The form factors are found to be

$$
\begin{aligned}
f_1^{ab}(p\cdot q) &= \frac{1\,\delta^{ab}}{(2\pi)^d p\cdot q}\frac{d-4}{d(d-1)(d-2)}\pi^2\mathcal{B}_0(2p\cdot q,0,0)\\
f_2^{ab}(p\cdot q) &= -\frac{2\,\delta^{ab}}{(2\pi)^d}\frac{d(d-3)+4}{d(d-1)(d-2)}\pi^2\mathcal{B}_0(2p\cdot q,0,0)\,,\\
s_1^{ab}(p\cdot q) &= \frac{4\delta^{ab}}{(2\pi)^d}\frac{d-4}{d(d-1)(d-2)p\cdot q}\pi^2\mathcal{B}_0(2p\cdot q,0,0)\,,\\
s_2^{ab}(p\cdot q) &= -\frac{2\,\delta^{ab}}{(2\pi)^d}\frac{1}{d(d-1)}\pi^2\mathcal{B}_0(2p\cdot q,0,0)\,,
\end{aligned}
\tag{6.23}
$$

where the tensors in the basis are given by

$$
\begin{aligned}
C^f_{\mu\nu\alpha\beta}(p,q) &= (p\cdot q\,\delta_{\alpha\beta}-q_\alpha\,p_\beta)\,(d(p_\mu\,p_\nu+q_\mu\,q_\nu)+(d-4)(p_\mu\,q_\nu+q_\mu\,p_\nu)\\
&\quad -2(d-2)p\cdot q\,\delta_{\mu\nu})\ ,\\
C^s_{\mu\nu\alpha\beta}(p,q) &= (p\cdot q\,\delta_{\alpha\beta}-q_\alpha\,p_\beta)\,(p_\mu\,q_\nu+q_\mu\,p_\nu-p\cdot q\,\delta_{\mu\nu})\ ,\\
D_{\mu\nu\alpha\beta}(p,q) &= \delta_{\alpha\beta}\,(p_\mu\,q_\nu+q_\mu\,p_\nu)+p\cdot q\,(\delta_{\mu\beta}\,\delta_{\nu\alpha}+\delta_{\mu\alpha}\,\delta_{\nu\beta})\\
&\quad -(\delta_{\beta\nu}\,p_\mu+\delta_{\beta\mu}\,p_\nu)\,q_\alpha-(\delta_{\mu\alpha}\,q_\nu+\delta_{\alpha\nu}\,q_\mu)\,p_\beta-\delta_{\mu\nu}\,(p\cdot q\,\delta_{\alpha\beta}-q_\alpha\,p_\beta)\,.
\end{aligned}
\tag{6.24}
$$

Notice that in this case all the structures (C, D) are traceless since there is no anomaly. As a final observation, we remark that in the on-shell case, the only topology that survives in the expansion of this correlator corresponds to a master integral of type $\mathcal{B}_0$ which corresponds to a massless 2-point function. The other master integral which also heavily appears in the perturbative expansion, $\mathcal{C}_0$, which corresponds to the scalar triangle diagram, drops out in the on-shell limit.

6.3 Renormalization of the TTT

In this section we address the problem of the renormalization of the 3-graviton vertex and compare the standard Lagrangian approach with the deductive method of [27], which is developed for the analysis in d dimensions. Since our interest, for this vertex, is sharply focused on the $d=4$ case, we need to clarify a few points. Notice that one of the two counterterms that appear at Lagrangian level, G, is a total divergence in 4 but not in d dimensions. In particular, G generates a counterterm which is effectively a projector on the extra $(d-4)$-dimensional space and as such, gives a contribution which needs to be included in order to perform a correct renormalization of the vertex. This has been verified by an explicit computation in dimensional regularization.

We recall that in perturbation theory the one loop counterterm Lagrangian is

$$
S_{\text{counter}} = -\frac{1}{\epsilon}\sum_{I=f,s,V} n_I\int d^dx\sqrt{g}\Big(\beta_a(I)F+\beta_b(I)G\Big)\,. \tag{6.25}
$$

We have used the 4-dimensional realization of F

$$
F = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-2\,R^{\alpha\beta}R_{\alpha\beta}+\frac{1}{3}\,R^2 \tag{6.26}
$$

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which is obtained from (A.6) with $d \to 4$. G, obviously does not contribute to every correlator. For instance, in the case of the TT, the counterterm is obtained by functional differentiation twice of S_{counter}, but one can easily check (see eq. (B.12)) that the second variation of G vanishes in the flat limit. Hence, the only counterterm is given by

$$D_F^{\alpha\beta\rho\sigma}(x_1, x_2) = 4\,\frac{\delta^2}{\delta g_{\alpha\beta}(x_1)\delta g_{\rho\sigma}(x_2)} \int d^d w\, \sqrt{g}\, F\,. \tag{6.27}$$

Its form in momentum space is given by

$$D_F^{\alpha\beta\rho\sigma}(p) = 4\,\Delta^{(4)\,\alpha\beta\rho\sigma}(p)\,, \tag{6.28}$$

and we recover the renormalized 2-point function in (5.32) just with its inclusion, i.e.

$$\left\langle T^{\alpha\beta}\, T^{\rho\sigma} \right\rangle_{\text{ren}}(p) = \left\langle T^{\alpha\beta}\, T^{\rho\sigma} \right\rangle(p) - \frac{\beta_a}{\bar{\epsilon}}\, D_F^{\alpha\beta\rho\sigma}(p)\,. \tag{6.29}$$

In the case of the 3-graviton vertex the counterterm action (6.25) generates the vertices

$$-\,\frac{1}{\epsilon}\left(\beta_a D_F^{\mu\nu\rho\sigma\alpha\beta}(z, x, y) + \beta_b\, D_G^{\mu\nu\rho\sigma\alpha\beta}(z, x, y)\right), \tag{6.30}$$

where

$$D_F^{\mu\nu\rho\sigma\alpha\beta}(x_1, x_2, x_3) = 8\,\frac{\delta^3}{\delta g_{\mu\nu}(x_1)\delta g_{\rho\sigma}(x_2)\delta g_{\alpha\beta}(x_3)} \int d^d w\, \sqrt{g}\, F\,, \tag{6.31}$$

$$D_G^{\mu\nu\rho\sigma\alpha\beta}(z, x, y) = 8\,\frac{\delta^3}{\delta g_{\mu\nu}(x_1)\delta g_{\rho\sigma}(x_2)\delta g_{\alpha\beta}(x_3)} \int d^d w\, \sqrt{g}\, G\,. \tag{6.32}$$

(6.32) and (6.31) are obtained by functionally deriving three times the general functional

$$\mathcal{I}(a, b, c) \equiv \int d^4x\, \sqrt{g}\,\left(a\, R^{abcd} R_{abcd} + b\, R^{ab} R_{ab} + c\, R^2\right), \tag{6.33}$$

with respect to the metric for appropriate a, b and c, i.e.

$$a = 1\,, \qquad b = -2\,, \qquad c = \frac{1}{3}\,,$$
$$a = 1\,, \qquad b = -4\,, \qquad c = 1\,.$$

Some of the computations are, for convenience, reproduced in appendix B.

It is known that $D_G^{\mu\nu\alpha\beta\rho\sigma}(p, q)$ is found to vanish identically in four dimensions. In fact, its explicit form is

$$D_G^{\mu\nu\alpha\beta\rho\sigma}(p, q) = -240\big(E^{\mu\sigma\alpha\gamma\kappa,\nu\rho\beta\delta\lambda} + E^{\mu\rho\alpha\gamma\kappa,\nu\sigma\beta\delta\lambda} + \alpha \leftrightarrow \beta\big)\, q_\gamma\, q_\delta\, p_\kappa\, p_\lambda\,, \tag{6.34}$$

where $E^{\mu\sigma\alpha\gamma\kappa,\nu\rho\beta\delta\lambda}$ is a projector onto completely antisymmetric tensors with five indices, so that it would yield zero in four dimensions, reflecting the fact that the integral of the Euler density is a topological invariant in such dimensions. We have explicitly checked by an explicit computation that, given the structure of the counterterm Lagrangian in (6.25), one needs necessarily to include the contribution from the G part of the functional, in

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the form given by D_G, in order to remove all the divergences. This choice brings us to a counterterm contribution which regulates TTT which is slightly different from the approach followed in [27]. The two approaches, in fact, differ by a finite renormalization, since in our case we reproduce the entire anomaly, including the local contribution ($\beta_c \neq 0$). The fully renormalized 3-point correlator in momentum space can be written down as

$$\left\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}\right\rangle_{\rm ren}(p,q) = \left\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}\right\rangle_{\rm bare}(p,q) - \frac{1}{\epsilon}\left(\beta_a\, D_F^{\mu\nu\alpha\beta\rho\sigma}(p,q) + \beta_b\, D_G^{\mu\nu\alpha\beta\rho\sigma}(p,q)\right) \tag{6.35}$$

and the goal is to proceed with an identification both of D_F and D_G from the diagrammatic expansion in momentum space. The cancellation of all of the ultraviolet poles, for suitable expressions of D_F and D_G, has been thoroughly checked from our explicit results. As we have already discussed in the previous cases, after renormalization, we can take the trace of (6.35) (in four dimensions) and obtain the entire trace anomaly.

In parallel, it is instructive to see how one can derive the analogue of (6.35), using our expression of F, which is 4-dimensional, but following the same approach of [27], i.e. by using the Ward identities. In this case we are bound to introduce the generic counterterms to the TTT vertex

$$\left\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}\right\rangle_{\rm ren}(p,q) = \left\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}\right\rangle_{\rm bare}(p,q) + \frac{1}{\epsilon}\left(C_F\, D_F^{\mu\nu\alpha\beta\rho\sigma}(p,q) + C_G\, D_G^{\mu\nu\alpha\beta\rho\sigma}(p,q)\right), \tag{6.36}$$

written in terms of arbitrary coefficients C_F and C_G. Notice that, for convenience, we have formulated (6.36) in momentum space, but the $1/\epsilon$ corrections are supported only at the coincidence point ($x_1 = x_2 = x_3$), for appropriate D_F and D_G, as one could check by performing a transform of this expression.

With the addition of the new contact terms which guarantee the regularization of the correlator, the new renormalized vertex must satisfy (2.41) and two similar identities which follow exchanging indices and momenta properly.

One can check that $D_G^{\mu\nu\alpha\beta\rho\sigma}(p,q)$ is transverse, as (6.34) shows clearly,

$$k_\nu D_G^{\mu\nu\alpha\beta\rho\sigma}(p,q) = 0\,, \quad p_\alpha D_G^{\mu\nu\alpha\beta\rho\sigma}(p,q) = 0 \quad q_\sigma D_G^{\mu\nu\alpha\beta\rho\sigma}(p,q) = 0\,, \tag{6.37}$$

so that by inserting the expressions (6.29) and (6.36) into these Ward identities and taking (6.37) into account, one obtains three conditions on the F-contribution to the counterterm, the first being

$$\begin{aligned} C_F\, k_\nu D_F^{\mu\nu\alpha\beta\rho\sigma}(p,q) = -4\,\beta_a \Big\{ & q^\mu \Delta^{(4)\,\rho\sigma\alpha\beta}(p) + p^\mu \Delta^{(4)\,\alpha\beta\rho\sigma}(q) \\ & - q_\nu \left[\delta^{\mu\rho}\Delta^{(4)\,\nu\sigma\alpha\beta}(p) + \delta^{\mu\sigma}\Delta^{(4)\,\nu\rho\alpha\beta}(p)\right] - p_\nu\left[\delta^{\mu\alpha}\Delta^{(4)\,\nu\beta\rho\sigma}(q) + \delta^{\mu\beta}\Delta^{(4)\,\nu\alpha\rho\sigma}(q)\right]\Big\}\,, \end{aligned} \tag{6.38}$$

and the other two coming from a permutation of the indices and of the momenta. They are seen to be satisfied if $C_F = -\beta_a$.

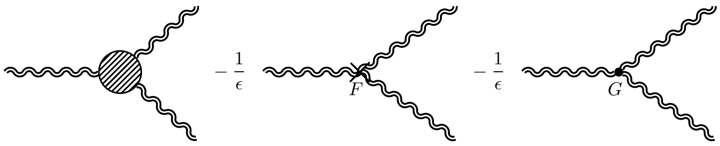

Figure 5. *TTT* and its counterterms generated with the choice of the square of the Weyl (F) tensor in 4 dimensions and the Euler density (G).

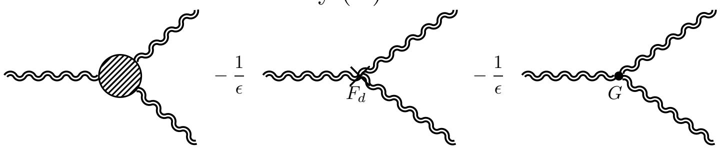

Figure 6. The contributions to the renormalized *TTT* vertex from the square of the Weyl tensor in d-dimensions (F^d) and the Euler density (G).

Exactly the same argument can be applied to the three anomalous trace identities in $d = 4 + \epsilon$ dimensions in order to fix C_G. Notice that, in this approach, the anomaly is reproduced by taking the traces of $D_F^{\mu\nu\alpha\beta}(p,q)$ and $D_G^{\mu\nu\alpha\beta}(p,q)$ in d dimensions, obtaining

$$\begin{aligned} \delta_{\mu\nu} D_F^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= -4\,\epsilon \left(\big[F\big]^{\alpha\beta\rho\sigma}(p,q) - \frac{2}{3}\big[\sqrt{g}\Box\, R\big]^{\alpha\beta\rho\sigma}(p,q) \right) \\ &\qquad -8 \left(\Delta^{(4)\,\alpha\beta\rho\sigma}(p) + \Delta^{(4)\,\alpha\beta\rho\sigma}(q) \right) \\ \delta_{\mu\nu} D_G^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= -4\,\epsilon\, \big[G\big]^{\alpha\beta\rho\sigma}(p,q)\,. \end{aligned} \tag{6.39}$$

According to the previously established notation, $\big[F\big]^{\alpha\beta\rho\sigma}(p,q)$ and $\big[G\big]^{\alpha\beta\rho\sigma}(p,q)$ are the Fourier-transformed second functional derivatives of the squared Weyl tensor and the Euler density respectively. Requiring (2.43) to be satisfied by the renormalized 2 and 3-point correlators we get

$$\begin{aligned} &\delta_{\mu\nu} \left(-\,\beta_a\;\, D_F^{\mu\nu\alpha\beta\rho\sigma}(p,q) + C_G\, D_G^{\mu\nu\alpha\beta\rho\sigma}(p,q) \right) = \\ &\quad 4\,\epsilon \left[\beta_a \left(\big[F\big]^{\alpha\beta\rho\sigma}(p,q) - \frac{2}{3}\big[\sqrt{g}\Box\, R\big]^{\alpha\beta\rho\sigma}(p,q) \right) \right. \\ &\quad \left. +\beta_b \,\big[G\big]^{\alpha\beta\rho\sigma}(p,q) \right] - 8 \left(\Delta^{(4)\,\alpha\beta\rho\sigma}(p) + \Delta^{(4)\,\alpha\beta\rho\sigma}(q) \right), \end{aligned} \tag{6.40}$$

and other two similar equations, obtained by shuffling indices and momenta as for the general covariance Ward identites.

In this way the conditions (6.29), (6.39), (6.39) and (2.3) allow us to obtain the relation $C_G = -\beta_b$, as expected. We have verified by direct computation for scalar, fermion and vector fields that the approach followed in ref. [27] of solving the Ward identities by adding contact terms to the homogenous expression of vertex (obtained for separate points) matches precisely the renormalization procedure above in momentum space.

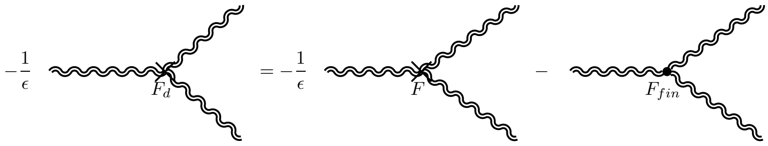

Figure 7. The relation between the counterterm generated by F^d and the same obtained from F. The difference is a finite renormalization (F_{fin}) generated by the $\sqrt{g}R^2$ term in the counterterm Lagrangian, which generates the local contirbution to the trace anomaly.

Notice that in [27] the choice of F is slightly different from ours, since the authors essentially define a counterterm which at a Lagrangian level would be of the form

$$\tilde{\mathcal{S}}_{\text{counter}} = -\frac{1}{\epsilon} \int d^4x \sqrt{g} \left(\beta_a F^d + \beta_b G \right) \tag{6.41}$$

based on the d-dimensional expression of the squared of the Weyl tensor (F^d). Such a choice does not generate a local anomaly contribution proportional to $\square R$ as $d \to 4$. In fact the authors choose to work with $\beta_c = 0$ from the beginning, since the inclusion of the local anomaly contribution amounts just to a finite renormalization with respect to (6.41). Notice that in d dimensions, if we take the trace of the functional derivative in (B.12) for $a = 1$, $b = -4/(d-2)$, $c = 2/((d-1)(d-2))$, which are the d-dimensional coefficients appearing in F^d, one can explicitly check that the contribution proportional to $\square R$ in the anomalous trace cancels. For this purpose we can expand the integrand of (6.41) around $d = 4$ (in $\epsilon = 4 - d$) up to $O(\epsilon)$, obtaining that the counterterm action can be separated in a pole plus a finite part, i.e.

$$\tilde{\mathcal{S}}_{\text{counter}} = \mathcal{S}_{\text{counter}} + \mathcal{S}_{\text{fin. ren.}} = \mathcal{S}_{\text{counter}} + \beta_a \int d^4x \sqrt{g} \left(R^{\alpha\beta} R_{\alpha\beta} - \frac{5}{18} R^2 \right) + O(\epsilon) \,. \tag{6.42}$$

Recalling the definition (2.5) and using (B.12), we see that the contribution of this finite part to the vev of the energy-momentum tensor is

$$g_{\mu\nu} \langle T^{\mu\nu} \rangle_{\text{fin.ren.}} = -\beta_c \square R \,. \tag{6.43}$$

Comparing this with (2.1), we see that this extra contribution will cancel the local anomaly.

So this approach is equivalent, for what concerns the anomaly, to supplying the action of the theory with the finite renormalization usually met in the literature, i.e.

$$\mathcal{S}^{(2)}_{\text{fin. ren.}} \equiv -\frac{\beta_c}{12} \int d^4x \sqrt{g} R^2 \,, \tag{6.44}$$

which is known to cancel the local anomaly, due to the similar relation

$$g_{\mu\nu} \frac{2}{\sqrt{g}} \frac{\delta \mathcal{S}^2_{\text{fin. ren.}}}{\delta g_{\mu\nu}} = -\beta_c \square R \,, \tag{6.45}$$

which holds in $d = 4$ as well.

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6.4 The renormalized on-shell 3-graviton vertex in 4 dimensions

In all of the three cases examined, the vertex $\Gamma^{\mu\nu\alpha\beta\rho\sigma}(p,q)$ can be expanded on a basis made up of thirteen tensors, if we go on shell on the two outgoing gravitons, which amounts to contract the amplitude with polarization tensors which are transverse and traceless

$$e^{s}_{\lambda\kappa}(p)\,, \quad (e^{s})^{\lambda}{}_{\lambda} = 0\,, \quad p^{\lambda}\, e^{s}_{\lambda\kappa} = 0\,, \tag{6.46}$$

where the superscript denotes the helicity state.

It is easy to see that the contraction of the amplitude with the polarization tensors with the properties (6.46) for the two outgoing gravitons is equivalent to the replacements

$$p^2 \to 0\,, \quad q^2 \to 0\,, \quad p^{\alpha} \to 0\,, \quad p^{\beta} \to 0\,, \quad q^{\rho} \to 0\,, \quad q^{\sigma} \to 0\,, \tag{6.47}$$

so that we will give the amplitude in terms of tensors which are non-vanishing after this limit is taken.

The expansion of our Green's function for a theory with n_S scalars, n_F fermions and n_V vector bosons can be written in general as

$$\begin{aligned} \left\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}\right\rangle(p,q)\Big|_{\mathrm{On-Shell}} &= \sum_{n_I = n_S, n_F, n_V} n_I \sum_{i=1}^{13} \Omega^{I}_{i}(s)\, t_i^{\mu\nu\alpha\beta\rho\sigma}(p,q)\,, \\ s &= k^2 = (p+q)^2 = 2\,p\cdot q\,. \end{aligned} \tag{6.48}$$

The form factors for the three theories at hand are listed in table 2, modulo the three overall factors, in the first row. The 13 tensors $t_i^{\mu\nu\alpha\beta\rho\sigma}(p,q)$ are listed below. They are given by

$$\begin{aligned}
t_1^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^{\mu}p^{\nu} + q^{\mu}q^{\nu}\right)p^{\rho}p^{\sigma}q^{\alpha}q^{\beta} \\
t_2^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^{\mu}q^{\nu} + p^{\nu}q^{\mu}\right)p^{\rho}p^{\sigma}q^{\alpha}q^{\beta} \\
t_3^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^{\mu}p^{\nu} + q^{\mu}q^{\nu}\right)\left(p^{\sigma}q^{\beta}\delta^{\alpha\rho} + p^{\sigma}q^{\alpha}\delta^{\beta\rho} + p^{\rho}q^{\beta}\delta^{\alpha\sigma} + p^{\rho}q^{\alpha}\delta^{\beta\sigma}\right) \\
t_4^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= p^{\rho}p^{\sigma}\left(q^{\beta}q^{\nu}\delta^{\alpha\mu} + q^{\beta}q^{\mu}\delta^{\alpha\nu} + q^{\alpha}q^{\nu}\delta^{\beta\mu} + q^{\alpha}q^{\mu}\delta^{\beta\nu}\right) \\
&\quad + q^{\alpha}q^{\beta}\left(p^{\nu}p^{\sigma}\delta^{\mu\rho} + p^{\nu}p^{\rho}\delta^{\mu\sigma} + p^{\mu}p^{\sigma}\delta^{\nu\rho} + p^{\mu}p^{\rho}\delta^{\nu\sigma}\right) \\
t_5^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^{\mu}q^{\nu} + q^{\mu}p^{\nu}\right)\Big(p^{\rho}\left(q^{\alpha}\delta^{\beta\sigma} + q^{\beta}\delta^{\alpha\sigma}\right) + p^{\sigma}\left(q^{\alpha}\delta^{\beta\rho} + q^{\beta}\delta^{\alpha\rho}\right)\Big) \\
t_6^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \delta^{\mu\nu}p^{\rho}p^{\sigma}q^{\alpha}q^{\beta} \\
t_7^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= p^{\rho}p^{\sigma}\left(\delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha}\right) + q^{\alpha}q^{\beta}\left(\delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}\right) \\
&\quad - \frac{1}{2}\Big(p^{\mu}p^{\rho}\left(\delta^{\alpha\sigma}\delta^{\nu\beta} + \delta^{\beta\sigma}\delta^{\nu\alpha}\right) + p^{\nu}p^{\rho}\left(\delta^{\alpha\sigma}\delta^{\mu\beta} + \delta^{\beta\sigma}\delta^{\mu\alpha}\right) \\
&\quad + p^{\mu}p^{\sigma}\left(\delta^{\alpha\rho}\delta^{\nu\beta} + \delta^{\beta\rho}\delta^{\nu\alpha}\right) + p^{\nu}p^{\sigma}\left(\delta^{\alpha\rho}\delta^{\mu\beta} + \delta^{\beta\rho}\delta^{\mu\alpha}\right) \\
&\quad + q^{\mu}q^{\alpha}\left(\delta^{\beta\sigma}\delta^{\nu\rho} + \delta^{\beta\rho}\delta^{\nu\sigma}\right) + q^{\nu}q^{\alpha}\left(\delta^{\beta\sigma}\delta^{\mu\rho} + \delta^{\beta\rho}\delta^{\mu\sigma}\right) \\
&\quad + q^{\mu}q^{\beta}\left(\delta^{\alpha\sigma}\delta^{\nu\rho} + \delta^{\alpha\rho}\delta^{\nu\sigma}\right) + q^{\nu}q^{\beta}\left(\delta^{\alpha\sigma}\delta^{\mu\rho} + \delta^{\alpha\rho}\delta^{\mu\sigma}\right)\Big) \\
t_8^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^{\mu}p^{\nu} + q^{\mu}q^{\nu}\right)\left(\delta^{\alpha\sigma}\delta^{\beta\rho} + \delta^{\alpha\rho}\delta^{\beta\sigma}\right)
\end{aligned}$$

i	$\Omega_i^S(s)$	$\Omega_i^F(s)$	$\Omega_i^V(s)$
1	$-\frac{1}{720\,\pi^2}\times\frac{1}{2\,s}$	$-\frac{1}{240\,\pi^2}\times\frac{1}{s}$	$\frac{1}{1152\,\pi^2}\times\frac{72}{5\,s}$
2	$-\frac{1}{720\,\pi^2}\times\frac{1}{s}$	$-\frac{1}{240\,\pi^2}\times\frac{1}{3\,s}$	$\frac{1}{1152\,\pi^2}\times\frac{64}{5\,s}$
3	$-\frac{1}{720\,\pi^2}\times\frac{7+30\,\mathcal{B}_0(s)}{120}$	$\frac{1}{240\,\pi^2}\times\frac{13-30\,\mathcal{B}_0(s)}{60}$	$\frac{1}{1152\,\pi^2}\times\frac{82-120\,\mathcal{B}_0(s)}{25}$
4	$-\frac{1}{720\,\pi^2}\times\frac{2+5\,\mathcal{B}_0(s)}{10}$	$\frac{1}{240\,\pi^2}\times\frac{7-70\,\mathcal{B}_0(s)}{120}$	$\frac{1}{1152\,\pi^2}\times\frac{2\,(482+130\,\mathcal{B}_0(s))}{25}$
5	$\frac{1}{720\,\pi^2}\times\frac{1}{6}$	$-\frac{1}{240\,\pi^2}\times\frac{-1+10\,\mathcal{B}_0(s)}{48}$	$-\frac{1}{1152\,\pi^2}\times\frac{79+50\,\mathcal{B}_0(s)}{5}$
6	$\frac{1}{720\,\pi^2}\times\frac{23+20\,\mathcal{B}_0(s)}{20}$	$\frac{1}{240\,\pi^2}\times\frac{33+70\,\mathcal{B}_0(s)}{60}$	$-\frac{1}{1152\,\pi^2}\times\frac{104\,(22+5\,\mathcal{B}_0(s))}{25}$
7	$-\frac{1}{720\,\pi^2}\times\frac{s\,(16+15\,\mathcal{B}_0(s))}{20}$	$-\frac{1}{240\,\pi^2}\times\frac{3\,s\,(2+5\,\mathcal{B}_0(s))}{10}$	$-\frac{1}{1152\,\pi^2}\times\frac{s\,(-11+10\,\mathcal{B}_0(s))}{80}$
8	$-\frac{1}{720\,\pi^2}\times\frac{s\,(47+30\,\mathcal{B}_0(s))}{80}$	$-\frac{1}{240\,\pi^2}\times\frac{3\,s\,(9+10\,\mathcal{B}_0(s))}{40}$	$\frac{1}{1152\,\pi^2}\times\frac{s\,(2+5\,\mathcal{B}_0(s))}{40}$
9	$\frac{1}{720\,\pi^2}\times\frac{s\,(2+5\,\mathcal{B}_0(s))}{40}$	$-\frac{1}{240\,\pi^2}\times\frac{7s\,(1-10\,\mathcal{B}_0(s))}{480}$	$-\frac{1}{1152\,\pi^2}\times\frac{s\,(487+130\,\mathcal{B}_0(s))}{50}$
10	$\frac{1}{720\,\pi^2}\times\frac{s\,(9+10\,\mathcal{B}_0(s))}{20}$	$\frac{1}{240\,\pi^2}\times\frac{s\,(137+430\,\mathcal{B}_0(s))}{480}$	$-\frac{1}{1152\,\pi^2}\times\frac{s\,(883-230\,\mathcal{B}_0(s))}{50}$
11	$-\frac{1}{720\,\pi^2}\times\frac{s\,(7+5\,\mathcal{B}_0(s))}{20}$	$-\frac{1}{240\,\pi^2}\times\frac{7\,s\,(9+10\,\mathcal{B}_0(s))}{240}$	$\frac{1}{1152\,\pi^2}\times\frac{s\,(467+130\,\mathcal{B}_0(s))}{25}$
12	$-\frac{1}{720\,\pi^2}\times\frac{s\,(121+90\,\mathcal{B}_0(s))}{240}$	$-\frac{1}{240\,\pi^2}\times\frac{s\,(97+130\,\mathcal{B}_0(s))}{240}$	$\frac{1}{1152\,\pi^2}\times\frac{2\,s\,(299+35\,\mathcal{B}_0(s))}{25}$
13	$\frac{1}{720\,\pi^2}\times\frac{5\,s^2\,(3+2\,\mathcal{B}_0(s))}{32}$	$\frac{1}{240\,\pi^2}\times\frac{5\,s^2\,(9+10\,\mathcal{B}_0(s))}{96}$	$-\frac{s^2\,(13-\mathcal{B}_0(s))}{1152\,\pi^2}$

Table 2. Form factors for the vertex $\Gamma^{\mu\nu\alpha\beta\rho\sigma}(p,q)$ in the on-shell limit.

$$
\begin{aligned}
t_9^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= p^\rho\left(q^\mu(\delta^{\alpha\sigma}\delta^{\beta\nu}+\delta^{\alpha\nu}\delta^{\beta\sigma})+q^\nu(\delta^{\alpha\sigma}\delta^{\beta\mu}+\delta^{\alpha\mu}\delta^{\beta\sigma})\right)\\
&\quad +p^\sigma\left(q^\mu(\delta^{\alpha\rho}\delta^{\beta\nu}+\delta^{\alpha\nu}\delta^{\beta\rho})+q^\nu(\delta^{\alpha\rho}\delta^{\beta\mu}+\delta^{\alpha\mu}\delta^{\beta\rho})\right)\\
&\quad +q^\alpha\left(p^\mu(\delta^{\beta\sigma}\delta^{\nu\rho}+\delta^{\beta\rho}\delta^{\nu\sigma})+p^\nu(\delta^{\beta\sigma}\delta^{\mu\rho}+\delta^{\beta\rho}g_{\mu\sigma})\right)\\
&\quad +q^\beta\left(p^\mu(\delta^{\alpha\sigma}\delta^{\nu\rho}+\delta^{\alpha\rho}\delta^{\nu\sigma})+p^\nu(\delta^{\alpha\sigma}\delta^{\mu\rho}+\delta^{\alpha\rho}\delta^{\mu\sigma})\right)\\
t_{10}^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= p^\rho\left(q^\alpha(\delta^{\beta\nu}\delta^{\mu\sigma}+\delta^{\beta\mu}\delta^{\nu\sigma})+q^\beta(\delta^{\alpha\nu}\delta^{\mu\sigma}+\delta^{\alpha\mu}\delta^{\nu\sigma})\right)\\
&\quad +p^\sigma\left(q^\alpha(\delta^{\beta\nu}\delta^{\mu\rho}+\delta^{\beta\mu}\delta^{\nu\rho})+q^\beta(\delta^{\alpha\nu}\delta^{\mu\rho}+\delta^{\alpha\mu}\delta^{\nu\rho})\right)\\
&\quad -p.q\left(\delta^{\alpha\rho}(\delta^{\beta\nu}\delta^{\mu\sigma}+\delta^{\beta\mu}\delta^{\nu\sigma})+\delta^{\alpha\nu}(\delta^{\beta\sigma}\delta^{\mu\rho}+\delta^{\beta\rho}\delta^{\mu\sigma})\right.\\
&\quad \left.+\delta^{\alpha\mu}(\delta^{\beta\sigma}\delta^{\nu\rho}+\delta^{\beta\rho}\delta^{\nu\sigma})+\delta^{\alpha\sigma}(\delta^{\beta\nu}\delta^{\mu\rho}+\delta^{\beta\mu}\delta^{\nu\rho})\right)\\
t_{11}^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^\nu q^\mu+p^\mu q^\nu\right)\left(\delta^{\alpha\sigma}\delta^{\beta\rho}+\delta^{\alpha\rho}\delta^{\beta\sigma}\right)\\
t_{12}^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \delta^{\mu\nu}\left(p^\rho\left(q^\beta\delta^{\alpha\sigma}+q^\alpha\delta^{\beta\sigma}\right)+p^\sigma\left(q^\beta\delta^{\alpha\rho}+q^\alpha\delta^{\beta\rho}\right)\right)\\
t_{13}^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \delta^{\mu\nu}\left(\delta^{\alpha\sigma}\delta^{\beta\rho}+\delta^{\alpha\rho}\delta^{\beta\sigma}\right).
\end{aligned}
$$

The correlator is affected by ultraviolet divergences coming from the two-point integrals $\mathcal{B}_0(s)$ (see eq. (5.25)). This is true in the off-shell case too, as all the other contributions to the scalar coefficients of its tensor expansion are made up of the three invariants p^2, q^2 and $p \cdot q$ plus the scalar 3-point integral

$$\mathcal{C}_0(s, s_1, s_2) = \frac{1}{\pi^2} \int d^d l \, \frac{1}{l^2 \, (l+p_1)^2 \, (l+p_2)^2} \, , \qquad s = (p_1+p_2)^2 \, , s_i = p_i^2 \, , \qquad i = 1, 2 \, , \tag{6.49}$$

which is finite for $d = 4$. In the $\overline{MS}$ scheme the renormalized two-point integral is defined as

$$\mathcal{B}_0^{\overline{MS}}(p^2) = 2 + \ln\left(\frac{\mu^2}{p^2}\right) \, , \tag{6.50}$$

which simply replaces the unrenormalized expression $B_0(p^2)$ (5.25) given in table 2, after using the renormalization procedure discussed above. We have checked that by taking the trace of these 13 tensors one reproduces the Weyl, Euler and local contributions to the trace anomaly satisfied by the vertex which in this on-shell case are given by

$$\begin{aligned}
\delta_{\mu\nu} \left\langle T^{\mu\nu} T^{\alpha\beta} T^{\mu\nu} \right\rangle (p,q) \Big|_{\text{On-Shell}} &= 4 \Big\{ \beta_a \left([F]^{\alpha\beta\rho\sigma}(p,q) - \frac{2}{3} [\sqrt{g} \Box \, R]^{\alpha\beta\rho\sigma}(p,q) \right) \\
&\quad + \beta_b \, [G]^{\alpha\beta\rho\sigma}(p,q) \Big\} \Big|_{\text{On-Shell}} \qquad (6.51) \\
\delta_{\alpha\beta} \left\langle T^{\mu\nu} T^{\alpha\beta} T^{\mu\nu} \right\rangle (p,q) \Big|_{\text{On-Shell}} &= 4 \Big\{ \beta_a \left([F]^{\mu\nu\rho\sigma}(-k,q) - \frac{2}{3} [\sqrt{g} \Box \, R]^{\mu\nu\rho\sigma}(-k,q) \right) \\
&\quad + \beta_b \, [G]^{\mu\nu\rho\sigma}(-k,q) - \frac{1}{2} \left\langle T^{\mu\nu} T^{\rho\sigma} \right\rangle (k) \Big\} \Big|_{\text{On-Shell}} \qquad (6.52) \\
\delta_{\rho\sigma} \left\langle T^{\mu\nu} T^{\alpha\beta} T^{\mu\nu} \right\rangle (p,q) \Big|_{\text{On-Shell}} &= 4 \Big\{ \beta_a \left([F]^{\mu\nu\alpha\beta}(-k,p) - \frac{2}{3} [\sqrt{g} \Box \, R]^{\mu\nu\alpha\beta}(-k,p) \right) \\
&\quad + \beta_b \, [G]^{\mu\nu\alpha\beta}(-k,p) - \frac{1}{2} \left\langle T^{\mu\nu} T^{\alpha\beta} \right\rangle (k) \Big\} \Big|_{\text{On-Shell}} \qquad (6.53)
\end{aligned}$$

with

$$\begin{aligned}
[F]^{\alpha\beta\rho\sigma}(p,q) \Big|_{\text{On-Shell}} &= 2 p^\rho p^\sigma q^\alpha q^\beta - p \cdot q \left(p^\sigma q^\beta \delta^{\alpha\rho} - p^\rho q^\beta \delta^{\alpha\sigma} - p^\sigma q^\alpha \delta^{\beta\rho} - p^\rho q^\alpha \delta^{\beta\sigma} \right) \\
&\quad + (p \cdot q)^2 \left(\delta^{\alpha\sigma} \delta^{\beta\rho} + \delta^{\alpha\rho} \delta^{\beta\sigma} \right) \qquad (6.54) \\
[G]^{\alpha\beta\rho\sigma}(p,q) \Big|_{\text{On-Shell}} &= 2 p^\rho p^\sigma q^\alpha q^\beta - p \cdot q \left(p^\sigma q^\beta \delta^{\alpha\rho} - p^\rho q^\beta \delta^{\alpha\sigma} - p^\sigma q^\alpha \delta^{\beta\rho} - p^\rho q^\alpha \delta^{\beta\sigma} \right) \\
&\quad + (p \cdot q)^2 \left(\delta^{\alpha\sigma} \delta^{\beta\rho} + \delta^{\alpha\rho} \delta^{\beta\sigma} \right) \qquad (6.55) \\
[\sqrt{g} \Box R]^{\alpha\beta\rho\sigma}(p,q) \Big|_{\text{On-Shell}} &= \frac{1}{2} p \cdot q \left(p^\sigma q^\beta \delta^{\alpha\rho} + p^\rho q^\beta \delta^{\alpha\sigma} + p^\sigma q^\alpha \delta^{\beta\rho} + p^\rho q^\alpha \delta^{\beta\sigma} \right) \\
&\quad - \frac{3}{2} (p \cdot q)^2 \left(g^{\alpha\sigma} \delta^{\beta\rho} - \delta^{\alpha\rho} \delta^{\beta\sigma} \right). \qquad (6.56)
\end{aligned}$$

The on-shell limits of the two point functions appearing in the r.h.s. of (6.52) and (6.53) are obtained from (5.33) replacing $p \to k$ and using (6.47) in (5.22).

7 The $T\phi^2\phi^2$ and VVV correlators in momentum space

A similar analysis allows to obtain the expression in momentum space of the $T\phi^2\phi^2$, discussed before in position space. We give the complete d-dimensional off-shell expression. It can be decomposed into four independent tensor structures

$$
\begin{aligned}
\Gamma^{T\phi^2\phi^2}_{\mu\nu}(p,q) &= F_1(p,q)\left(p_\mu p_\nu - \frac{p^2}{d}\delta_{\mu\nu}\right) + F_1(q,p)\left(q_\mu q_\nu - \frac{q^2}{d}\delta_{\mu\nu}\right) \\
&\quad + F_2(p,q)\left(p_\mu q_\nu + p_\nu q_\mu - \frac{2\,p\cdot q}{d}\delta_{\mu\nu}\right) + F_3(p,q)\frac{1}{d}\delta_{\mu\nu}
\end{aligned}
\tag{7.1}
$$

where the first three tensors are traceless while the last one has a non-vanishing trace.

The three form factors are given by

$$
\begin{aligned}
F_1(p,q) &= \frac{1}{(2\pi)^d}\,\frac{\pi^2}{2(d-2)(p\cdot q^2 - p^2q^2)^2}\Big\{(d-1)p\cdot q(p\cdot q+q^2)(p+q)^2\,\mathcal{B}_0((p+q)^2) \\
&\quad -\mathcal{B}_0(p^2)\Big[p^2((d-1)p\cdot q^2 + 2\,p\cdot q\,q^2 + q^2) + (d-2)p\cdot q^2(2\,p\cdot q+q^2)\Big] \\
&\quad -\mathcal{B}_0(q^2)\Big[q^2\,p\cdot q((3d-5)p\cdot q+p^2) - q^4((d-3)p^2-(d-1)p\cdot q) + (d-2)p\cdot q^3\Big] \\
&\quad +(p\cdot q+q^2)(p+q)^2((d-2)p\cdot q^2+p^2q^2)\mathcal{C}_0(p^2,(p+q)^2,q^2)\Big\}, \\
F_2(p,q) &= \frac{1}{(2\pi)^d}\,\frac{\pi^2}{2(d-2)(p\cdot q^2 - p^2q^2)^2}\Big\{\mathcal{B}_0(p^2)\Big[(d-1)p^2\,p\cdot q^2 + (d-2)p\cdot q^3 + p^2q^2\,p\cdot q\Big] \\
&\quad +\mathcal{B}_0(q^2)\Big[(d-1)q^2\,p\cdot q^2 + (d-2)p\cdot q^3 + p^2q^2\,p\cdot q\Big] \\
&\quad -p\cdot q\,\mathcal{B}_0((p+q)^2)\Big[(d-1)p\cdot q(p^2+q^2) + (d-2)p^2q^2 + d\,p\cdot q^2\Big] \\
&\quad -\frac{(d-2)p\cdot q^2+p^2q^2}{d-1}\mathcal{C}_0(p^2,(p+q)^2,q^2)\Big[(d-1)p\cdot q(p^2+q^2) + (d-2)p^2q^2 + d\,p\cdot q^2\Big]\Big\}, \\
F_3(p,q) &= \frac{1}{(2\pi)^d}\pi^2\left(\mathcal{B}_0(p^2)+\mathcal{B}_0(q^2)\right).
\end{aligned}
\tag{7.2}
$$

Finally, we present here the expression of the conformal contributions to the VVV with two external legs on mass-shell. This limit is achieved contracting with the two polarization vectors $(e_\alpha(p)\,, e_\beta(q))$ and sending the invariants p^2, q^2 to zero. The fermion sector, for instance gives

$$
\begin{aligned}
\Gamma^{VVV_{\text{ferm}}}_{\alpha\beta\lambda}(p,q) &= \frac{1}{(2\pi)^d}\,\frac{f^{abc}}{(d-2)(d-1)}\Big\{d(d-3)\delta_{\alpha\beta}(p-q)_\lambda + 2(d-2)^2\left(\delta_{\beta\lambda}q_\alpha - \delta_{\alpha\lambda}p_\beta\right) \\
&\quad -\frac{d-4}{p\cdot q}(p-q)_\lambda p_\beta q_\alpha\Big\}\pi^2\,\mathcal{B}_0(2p\cdot q,0,0),
\end{aligned}
\tag{7.3}
$$

while the scalar sector gives

$$
\begin{aligned}
\Gamma^{VVV_{\text{scalar}}}_{\alpha\beta\lambda}(p,q) &= -\frac{1}{(2\pi)^d}\,\frac{f^{abc}}{(d-2)(d-1)}\Big\{\delta_{\alpha\beta}(p-q)_\lambda + (d-2)\left(\delta_{\beta\lambda}q_\alpha - \delta_{\alpha\lambda}p_\beta\right) \\
&\quad +\frac{d-4}{2\,p\cdot q}(p-q)_\lambda p_\beta q_\alpha\Big\}\pi^2\,\mathcal{B}_0(2p\cdot q,0,0).
\end{aligned}
\tag{7.4}
$$

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8 Handling any massless correlator: a direct approach in d dimensions

In the previous sections we have tried to compare perturbative results in free field theory with general ones coming from the requirement of conformal symmetry imposed on certain correlators. We have also seen that in this case, working backward from the explicit field theory representation of the lowest order realization of these correlators, one can match the general solutions given by conformal field theory. This is the case of the VVV, TVV and TOO correlators in general dimensions, while for the TTT the 4-dimensional solution of the Ward identities is completely matched by a combination of scalar, vector and fermion sectors. As we consider the same 3-graviton vertex in d-dimensions, the vector contribution is not conformally invariant, and therefore the combination of the scalar and the fermion sectors does not match the most general d-dimensional solution. This raises the issue if there is, in general, a free field theory that can reproduce a given CFT correlator, and there is no simple answer. The goal of CFT, in fact, is to bootstrap certain correlation functions bypassing, if necessary, a Lagrangian formulation.

In fact, one of the main features of the standard CFT approach in the identification of the correlators of a given conformal field theory is to work in position space with no reference to a Lagrangian. The finiteness of the Fourier transform is the necessary requirement in order to proceed with the identification, if this turns out to exist, of the corresponding field theory, since this can always be defined in momentum space.

Checking the finiteness (in momentum space) of a general solution given in position space are not obvious steps, since a correlator in position space such as the TTT, once expanded, contains several hundreds of terms, most of them characterized by a divergent Fourier expression in momentum space. For this reason we are going here to illustrate a very general algorithm that allows to compute correlators of such a complexity using a direct approach. Our analysis will be formulated in general but illustrated with few examples only up to correlators of rank-4. We will be choosing, as a test of our approach, some of the correlators defined in the previous sections and for obvious reasons. These, in fact, as we have seen, can be deduced from a Lagrangian formulation and therefore their expressions in momentum space are well defined. Obviously, we need some intermediate regularization of the integrands (in position space) of these correlators in order to proceed with the definition of the Fourier transform of each individual term. This is obtained by introducting a power-like regulator (ω) which is the analogous of the ϵ regulator of ordinary dimensional regularization but, for the rest, completely unrelated to it.

The algorithm implements a sequence of integration by parts before proceeding with the identification of the ω-regulated transforms. As a consistency condition, the correlators that we investigate have finite Fourier expressions, as expected, and we check the direct cancellation of all their Fourier singularities, which appear as poles (double and single) in $1/\omega$.

The finite parts of the procedure, which correspond to the Fourier space integrands, manifest specific logarithmic terms. These, in general, are a new feature of the momentum space form of a given (position space) CFT correlator. They are expected to appear once we rewrite any CFT correlation function from position to momentum space. Obviously, these log terms, in some cases, can be rewritten as ordinary (non-logarithmic) integrals, in other cases not, and we can think of the log-integrals, in all these second cases, as of new irreducible contributions.

In the correlators that we investigate, obviously, we know beforehand that they have to be matched by free field theory. In this case, a brute force application of the algorithm would produce log-integrals which are, therefore, reducible to ordinary (non logarithmic) ones. When the ω singularities cancel, which indicates that it is possible to recollect the terms in position space (and using integration by parts) in such a way that the Fourier expression is manifestly finite, the logarithmic terms are absent. The use of the previous (Fourier integrable) vertices allows to test this approach showing its consistency.

The steps. As we have already mentioned in the previous sections, given any correlator, we can formulate a general procedure which allows us to transform its expression to momentum space, with the following steps:

1) expansion of the correlator into its single tensor components;

2) rewriting of each component in terms of some "R-substitutions", that we will define below;

3) application of the dimensional shift $d \to d - 2\omega$ which can be performed generically in the expression resulting from point 2); and

4) implementation of the transform. The transform is implemented by eq. (8.1) for each single difference $x_{ij} = x_i - x_j$. For correlators of higher orders, say of rank n ($n > 3$), the transform is used n times.

As we are going to describe below, this method and the regularization imposed by the dimensional shift allows to test quite straighforwardly the integrability of any correlator, a point already emphasized in [27] where this regularization has been first introduced. The transform can be applied in several independent ways. These features share some similarities with the so called "method of uniqueness" (see for instance [22]) used for massless integrals in momentum or in configuration space.

8.1 Pulling out derivatives

One of the main steps that we will follow in the computation of the transform of the x-space expression of the correlators consists in the rewriting of a given x-space tensor in terms of derivatives of other terms. We call this rule a "derivative relation." It allows one to reduce the degree of singularity of a given tensor structure, when the variables are coincident, in the spirit of differential regularization. Differently from the standard approach given by differential regularization, which is 4-dimensional, we will be working in d dimensions. We will be using the term "integrable" to refer to expressions for which the Fourier transform exists and that are well defined in d-dimensions, although they may be singular in d=4.

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Derivative relations, combined with the basic transform

$$\begin{aligned}\frac{1}{(x^2)^\alpha} &= \frac{1}{4^\alpha \pi^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \int d^d l \, \frac{e^{il\cdot x}}{(l^2)^{d/2-\alpha}} \equiv C(\alpha) \int d^d l \, \frac{e^{il\cdot x}}{(l^2)^{d/2-\alpha}} \\ C(\alpha) &= \frac{1}{4^\alpha \, \pi^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)}\end{aligned} \tag{8.1}$$

allow one to perform a direct mapping of these correlators to momentum space. We proceed with a few examples to show how the lowering of the singularity takes place.

We start from tensors of rank-1. At this rank we use the relation

$$\begin{aligned}\frac{x^\mu}{(x^2)^\alpha} &= -\frac{1}{2(\alpha-1)} \partial_\mu \frac{1}{(x^2)^{\alpha-1}} \\ &= -\frac{i}{2^{2\alpha-1}\pi^{d/2}} \frac{\Gamma(d/2+1-\alpha)}{\Gamma(\alpha)} \int d^d l \, e^{il\cdot x} \frac{l_\mu}{(l^2)^{d/2-\alpha+1}}\end{aligned} \tag{8.2}$$

to extract the derivative, where in the last step we have used (8.1). Notice that by using (8.1) with $\alpha = d/2 - 1$ one can immediately obtain the equation

$$\Box \frac{1}{(x^2)^{d/2-1}} = -\frac{4\,\pi^{d/2}}{\Gamma(d/2-1)} \, \delta^{(d)}(x) \tag{8.3}$$

which otherwise needs Gauss' theorem to be derived.

Scalar 2-point functions describing loops in x-space are next in difficulty. As an illustration, consider the generalized 2-point function

$$\frac{1}{[(x-y)^2]^\alpha [(x-y)^2]^\beta} \, . \tag{8.4}$$

Using (8.1) separately for the $1/[(x-y)^2]^\alpha$ and the $1/[(x-y)^2]^\beta$ factors, the Fourier transform ($\mathcal{FT}$) of this expression is found to be

$$\begin{aligned}\mathcal{FT}\left[\frac{1}{[(x-y)^2]^\alpha [(x-y)^2]^\beta}\right] &\equiv \int d^d x \, d^d y \, \frac{e^{-i(p\cdot x + q\cdot y)}}{[(x-y)^2]^\alpha [(x-y)^2]^\beta} \\ &= (2\pi)^{2d} \, C(\alpha) \, C(\beta) \int d^d l \, \frac{1}{[l^2]^\alpha [(l+p)^2]^\beta} \, .\end{aligned} \tag{8.5}$$

Uniqueness allows to reformulate the transform by combining the powers of the propagators into a single factor

$$\mathcal{FT}\left[\frac{1}{[(x-y)^2]^{\alpha+\beta}}\right] = (2\pi)^{2d} \, \frac{C(\alpha+\beta)}{(p^2)^{d/2-\alpha-\beta}}, \tag{8.6}$$

giving, for consistency, a functional relation for the integral in (8.5)

$$\begin{aligned}\int d^d l \, \frac{1}{[l^2]^\alpha [(l+p)^2]^\beta} &= \frac{C(\alpha+\beta)}{C(\alpha)\, C(\beta)} \frac{1}{(p^2)^{d/2-\alpha-\beta}} \\ &= \pi^{d/2} \, \frac{\Gamma(d/2-\alpha)\Gamma(d/2-\beta)\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)} \, \frac{1}{(p^2)^{\alpha+\beta-d/2}} \, .\end{aligned} \tag{8.7}$$

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In the TT and TVV cases, x-space expressions such as $x^{\mu_1}\dots x^{\mu_n}/(x^2)^\alpha$ up to rank-4 are common, and the use of derivative relations — before proceeding with their final transform to momentum space — can be done in several ways. Also in this case, as for the scalar functions, uniqueness shows that the result does not depend on the way we combine the factors at the denominators with the corresponding numerators.

To deal with tensor expressions in position space we introduce some notation. We denote by

$$
R^n{}_{\mu_1,\dots,\mu_n}(x,\alpha) \equiv \frac{x_{\mu_1},\dots x_{\mu_n}}{(x^2)^\alpha}\,, \tag{8.8}
$$

the ratio between a generic tensor monomial in the vector x and a power of x^2. We do so to denote in a compact way the tensor structures that appear in the expansion of any correlator. We call these expressions "R-terms".

After some differential and algebraic manipulation we can easily derive the first four R-terms as

$$
\begin{aligned}
R^1{}_\mu(x,\alpha) &= -\frac{1}{2\,(\alpha-1)}\,\partial_\mu\,\frac{1}{(x^2)^{\alpha-1}}\,,\\
R^2{}_{\mu\nu}(x,\alpha) &= \frac{1}{4\,(\alpha-2)\,(\alpha-1)}\,\partial_\mu\,\partial_\nu\,\frac{1}{(x^2)^{\alpha-2}} + \frac{\delta_{\mu\nu}}{2\,(\alpha-1)}\,\frac{1}{(x^2)^{\alpha-1}}\,,\\
R^3{}_{\mu\nu\rho}(x,\alpha) &= -\frac{1}{8(\alpha-3)(\alpha-2)(\alpha-1)}\,\partial_\mu\,\partial_\nu\,\partial_\rho\frac{1}{(x^2)^{\alpha-3}}\\
&\quad+\frac{1}{2(\alpha-1)}\left[\delta_{\mu\nu}R^1{}_\rho+\delta_{\mu\rho}R^1{}_\nu+\delta_{\nu\rho}R^1{}_\mu\right](x,\alpha-1)\,,\\
R^4{}_{\mu\nu\rho\sigma}(x,\alpha) &= \frac{1}{16(\alpha-4)(\alpha-3)(\alpha-2)(\alpha-1)}\,\partial_\mu\,\partial_\nu\,\partial_\rho\,\partial_\sigma\,\frac{1}{(x^2)^{\alpha-4}}\\
&\quad+\frac{1}{2(\alpha-1)}\left[\delta_{\mu\nu}R^2{}_{\rho\sigma}+\delta_{\rho\sigma}R^2{}_{\mu\nu}+\delta_{\mu\rho}R^2{}_{\nu\sigma}+\delta_{\nu\sigma}R^2{}_{\mu\rho}\right.\\
&\quad\left.+\delta_{\mu\sigma}R^2{}_{\nu\rho}+\delta_{\nu\rho}R^2{}_{\mu\sigma}\right](x,\alpha-1)\\
&\quad-\frac{1}{4(\alpha-2)(\alpha-1)}\left(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}\right)\frac{1}{(x^2)^{\alpha-2}}\,.
\end{aligned} \tag{8.9}
$$

The use of R-terms allows to extract immediately the leading singularities of the correlators, as we show below. One can use several different forms of R-substitutions for a given tensor component and the procedure is in fact not unique. For example, a second rank tensor can be rewritten in R-form in several ways

$$
\begin{aligned}
\frac{(x-y)_\mu(x-y)_\nu}{[(x-y)^2]^{d+1}} &= R^2{}_{\mu\nu}(x-y,d+1)\\
&= R^1{}_\mu(x-y,d/2+1)\,R^1{}_\nu(x-y,d/2)\\
&= \frac{1}{(x-y)^2}\,R^1{}_\mu(x-y,d/2)\,R^1{}_\nu(x-y,d/2)\,.
\end{aligned} \tag{8.10}
$$

The derivative relations in the three cases shown above are obviously different, but the transform is unique. One can also artificially rewrite the numerators at will by introducing trivial identities in position space, without affecting the final expression of the mapping.

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We will be using this method in order to extract some of the logarithmic integrals generated by this procedure. Obviously, this is possible only if we guarantee an intermediate regularization. We implement it by a dimensional shift of the exponents of the propagators. The regulator will allow to smooth out the singularity of the correlators around the value $\alpha = d/2$, which is the critical value beyond which a function such as $1/[x^2]^\alpha$ is not integrable.

The structure of the singularities in x-space of the corresponding scalars and tensor correlators can be identified using the basic transform. For instance, using (8.1) for $\alpha = d/2$ one encounters a pole in the expression of the transform. For this reason we regulate dimensionally in x-space such a singularity by shifting $d \to d - 2\omega$. At the same time we compensate with a regularization scale μ to preserve the dimension of the redefined correlator. A similar approach has been discussed in [17], in an attempt to relate differential and dimensional regularization. In our case as in [27], however, ω is an independent regulator which serves to test integrability in momentum space, and for this reason is combined with a fundamental transform which is given by

$$\frac{\mu^{2\omega}}{[x^2]^{d/2-\omega}} = \frac{\mu^{2\omega}}{4^{d/2-\omega}\pi^{d/2}} \frac{\Gamma(\omega)}{\Gamma(d/2-\omega)} \int d^d l \, \frac{e^{il\cdot x}}{[l^2]^\omega} \tag{8.11}$$

that we can expand around $\omega \sim 0$ to obtain

$$\begin{aligned} \frac{\mu^{2\omega}}{[x^2]^{d/2-\omega}} &= \frac{\pi^{d/2}}{\Gamma(d/2)} \delta^{(d)}(x) \left[\frac{1}{\omega} - \gamma + \log 4 + \psi(d/2) \right] \\ &\quad - \frac{1}{(4\pi)^{d/2}\,\Gamma(d/2)} \int d^d l \, e^{il\cdot x} \log\left(\frac{l^2}{\mu^2}\right) + O(\omega)\,. \end{aligned} \tag{8.12}$$

The subtraction of this pole in d dimensions is obviously related to the need of redefining correlators which are not integrable, in analogy with the approach followed in differential regularization. The most popular example is $1/[x^2]^2$, which has no transform for $d = 4$, but is rewritten in the derivative form as [20]

$$\frac{1}{x^4} = \Box\, G(x^2), \tag{8.13}$$

where $G(x^2)$ is defined by

$$G(x^2) = \frac{\log x^2 M^2}{x^2} + c\,, \tag{8.14}$$

with c being a constant. This second approach can be easily generalized to d dimensions. One can use derivative relations such as

$$\frac{1}{[x^2]^\alpha} = \frac{1}{2(\alpha-1)(2\alpha-d)} \Box \frac{1}{[x^2]^{\alpha-1}} \tag{8.15}$$

which is correct as far as $\alpha \neq d/2$. For $\alpha = d/2$ this relation misses the singularity at $x = 0$, which is apparent from (8.3). For this reason, as far as $\alpha = d/2 - \omega$ eq. (8.15) remains valid and it can be used together with (8.3) and an expansion in ω to give

$$\begin{aligned}\frac{\mu^{2\omega}}{[x^2]^{d/2-\omega}} &= -\frac{1}{2\,\omega}\frac{\mu^{2\omega}}{d-2-2\omega}\,\Box\frac{1}{[x^2]^{d/2-1-\omega}}\\ &= \frac{1}{4-2\,d}\left(\frac{1}{\omega}+\frac{2}{d-2}\right)\Box\frac{1}{[x^2]^{d/2-1}} - \frac{1}{2(d-2)}\Box\frac{\log(\mu^2x^2)}{[x^2]^{d/2-1}}\\ &= \frac{\pi^{d/2}}{\Gamma(d/2)}\left(\frac{1}{\omega}+\frac{2}{d-2}\right)\delta^{(d)}(x) - \frac{1}{2(d-2)}\Box\frac{\log(\mu^2x^2)}{(x^2)^{d/2-1}}. \end{aligned} \tag{8.16}$$

The d-dimensional version of differential regularization (DfR) can be obtained by requiring the subtraction of all the terms in (8.16) which are proportional to $\delta^d(x)$, giving

$$\left.\frac{1}{[x^2]^{d/2}}\right|_{DfR} \equiv -\frac{1}{2(d-2)}\Box\frac{\log(\mu^2x^2)}{(x^2)^{d/2-1}}. \tag{8.17}$$

This procedure clearly agrees with the traditional version of differential regularization in $d=4$ [20]

$$\frac{1}{x^4} \equiv -\frac{1}{4}\Box\frac{\log(x^2\mu^2)}{x^2}. \tag{8.18}$$

Notice that this analysis shows that, according to (8.16), the logarithmic integral in (8.12) is given by

$$\begin{aligned}\int d^dl e^{il\cdot x}\log\left(\frac{l^2}{\mu^2}\right) &= (2\pi)^d\left[-\gamma+\log 4+\psi(d/2)-\frac{2}{d-2}\right]\delta^{(d)}(x)+\frac{(4\pi)^{d/2}}{2(d-2)}\Gamma(d/2)\,\Box\,\frac{\log(\mu^2x^2)}{[x^2]^{d/2-1}}\\ &= \frac{(4\pi)^{d/2}}{2(d-2)}\Gamma(d/2)\,\Box\,\frac{\log(\bar\mu^2x^2)}{[x^2]^{d/2-1}}\,, \end{aligned} \tag{8.19}$$

having redefined the regularization scale properly

$$\log\bar\mu^2 = \log\mu^2+\gamma-\log 4-\psi(d/2)+\frac{2}{d-2}\,. \tag{8.20}$$

Notice that a regulated (but singular) correlator can be mapped in several ways into momentum space, with identical results. For instance, we can take $1/[x^2]^{d/2}$ and use on it eq. (8.1) once

$$\begin{aligned}\int d^dx\, e^{ik\cdot x}\frac{1}{[x^2]^{d/2}} &\to \int d^dx\, e^{ik\cdot x}\frac{\mu^{2\omega}}{[x^2]^{d/2-\omega}} = \frac{1}{4^{d/2-\omega}\,\pi^{d/2}}\frac{\Gamma(\omega)}{\Gamma(d/2-\omega)}\int d^dx\, d^dl\, e^{i(k+l)\cdot x}\frac{\mu^{2\omega}}{[l^2]^{\omega}}\\ &= 4^{\omega}\,\pi^{d/2}\frac{\Gamma(\omega)}{\Gamma(d/2-\omega)}\frac{\mu^{2\omega}}{[k^2]^{\omega}}\,, \end{aligned} \tag{8.21}$$

twice

$$\begin{aligned}\int d^dx\,\frac{\mu^{2\omega}}{x^2[x^2]^{d/2-1-\omega}} &= \frac{1}{4^{d/2-\omega}\pi^d}\frac{\Gamma(d/2-1)\,\Gamma(1+\omega)}{\Gamma(d/2-1-\omega)}\int d^dx\, d^dl_1\, d^dl_2\, e^{i(k+l_1+l_2)\cdot x}\frac{\mu^{2\omega}}{[l_1^2]^{d/2-1}[l_2^2]^{1+\omega}}\\ &= 4^{\omega}\,\pi^{d/2}\frac{\Gamma(\omega)}{\Gamma(d/2-\omega)}\frac{\mu^{2\omega}}{[k^2]^{\omega}}\,, \end{aligned} \tag{8.22}$$

(where in the last step (8.7) was used) or any number of times, obtaining the same transform.

As one can easily work out, the use of the dimensional regulator generates, after a Laurent expansion in ω, some logarithmic integrals in momentum space. As we shall show, if the $1/\omega$ poles cancel, then these integrals can be avoided, in the sense that it will be possible to rewrite the correlator in such a way that they are absent. This means that in this case one has to go back and try to rewrite the correlator in such a way that it takes a finite form already in position space. In this case the mapping of the correlators to momentum space is similar to the usual Feynman expansion typical of perturbation theory. The condition of Fourier transformability is in fact necessary in order to have, eventually, a Lagrangian description of the correlator. On the other hand, if the same poles do not cancel, then the logarithms are a significant aspect of the correlator which, for sure, can't be reproduced by a local field theory Lagrangian in any simple way, in particular not by a free field theory. We have left to appendix F a few more examples on the correct handling of these distributional identities.

8.2 Regularization of tensors

The regularization of other tensor contributions using this extension of differential regularization can be handled in a similar and straightforward way. The use of the derivative relations on the R-terms, that map the tensor structures into derivative of less singular terms, combined at the last stage with the basic transform, allows to get full control of any correlator and guarantee their consistent mappings into momentum space. We provide a few examples to illustrate the procedure.

Consider for instance the tensor structure

$$t_\mu = \frac{(x-y)_\mu}{[(x-y)^2]^{d/2+1}}\,, \tag{8.23}$$

whose R-form is, trivially,

$$t_\mu = {R^1}_\mu\left(x-y, \frac{d}{2}+1\right) = -\frac{1}{d}\,\partial_\mu \frac{1}{[(x-y)^2]^{d/2}}\,, \tag{8.24}$$

where the derivative is intended with respect to $x-y$. Now we send $d \to d-2\omega$ in the exponent of the denominator, since $d/2$ is a critical value for the integrability of the exponent, introducing the proper mass scale. This allows us to use the basic transform (8.1), getting

$$t_\mu(\omega) = -\frac{i\,\mu^{2\omega}}{(d-2\omega)\,4^{d/2-\omega}\,\pi^{d/2}}\,\frac{\Gamma(\omega)}{\Gamma(d/2-\omega)}\int d^dl\,\frac{l_\mu}{[l^2]^\omega}\,e^{il\cdot(x-y)}\,. \tag{8.25}$$

We can expand in ω obtaining

$$\begin{aligned} t_\mu(\omega) &= \frac{i}{d\,2^d\pi^{d/2}\,\Gamma(d/2)}\bigg[-\left(\frac{1}{\omega}+\frac{2}{d}-\gamma+\log 4+\psi(d/2)\right)\int d^dl\,e^{il\cdot(x-y)}\,l_\mu \\ &\quad +\int d^dl\,e^{il\cdot(x-y)}\,l_\mu\,\log\left(\frac{l^2}{\mu^2}\right)\bigg]+O(\omega) \\ &= \frac{\pi^{d/2}}{d\,\Gamma(d/2)}\,\partial_\mu\bigg[-\left(\frac{1}{\omega}+\frac{4(d-1)}{d(d-2)}\right)\delta^{(d)}(x-y)+\frac{\Gamma(d/2)}{2(d-2)\pi^{d/2}}\,\Box\frac{\log(\bar\mu^2(x-y)^2)}{[(x-y)^2]^{d/2-1}}\bigg]\,, \end{aligned} \tag{8.26}$$

where in the last step we have used (8.19). Notice that the strength of the singularity has increased from $\delta(x)/\omega$ to $\partial_\mu \delta(x)/\omega$, due to the higher power $(d/2)$ of the denominator in position space. It is clear that for finite correlators these singular contributions must cancel. In general, the introduction of the regulator ω allows to perform algorithmically all the computations of any lengthy expression leaving its implementation to a symbolic manipulation program. Obviously, for finite correlators this approach might look redundant, but it can be extremely useful in order to check the cancellation of all the multiple and single pole singularities in a very efficient way. We will present more examples of this approach in the next sections.

A more involved example is given by

$$t_{\mu\nu} = \frac{(x-y)_\mu (x-y)_\nu}{[(x-y)^2]^{d/2+1}} \tag{8.27}$$

to which corresponds the regulated expression

$$t_{\mu\nu}(\omega) = \frac{\mu^{2\omega}(x-y)_\mu (x-y)_\nu}{[(x-y)^2]^{d/2+1-\omega}} \tag{8.28}$$

and a minimal R-form which is given by

$$t_{\mu\nu}(\omega) = \mu^{2\omega}\, R^2{}_{\mu\nu}\left(x-y, \frac{d}{2}+1-\omega\right). \tag{8.29}$$

Using the list of replacements given in (8.9), the derivative form of $t_{\mu\nu}$ is given by

$$t_{\mu\nu}(\omega) = \frac{\mu^{2\omega}}{(d-2-2\omega)\,(d-2\omega)}\,\partial_\mu\,\partial_\nu \frac{1}{[(x-y)^2]^{d/2-\omega-1}} + \frac{\delta_{\mu\nu}}{d+2-2\omega}\,\frac{\mu^{2\omega}}{[(x-y)^2]^{d/2-\omega}} \tag{8.30}$$

whose singularities are all contained in the second term, whose Fourier transform is given by

$$\mathcal{FT}\left[\frac{\delta_{\mu\nu}}{d+2-2\omega}\,\frac{\mu^{2\omega}}{[(x-y)^2]^{d/2-\omega}}\right] = \frac{1}{\omega}\,\frac{\delta_{\mu\nu}}{2^d\,\pi^{d/2}\,(d+2)\,\Gamma(d/2)} + O(\omega^0) \tag{8.31}$$

where we have omitted the regular terms. The procedure therefore allows to identify quite straightforwardly the leading singularities of any tensor in x-space, giving, in this specific case

$$\frac{(x-y)_\mu (x-y)_\nu}{[(x-y)^2]^{d/2+1-\omega}} \sim \frac{1}{\omega}\,\frac{\delta_{\mu\nu}}{2^d\,\pi^{d/2}\,(d+2)\,\Gamma(d/2)}\,. \tag{8.32}$$

We can repeat the procedure for correlators of higher rank. The singularities, after performing all the substitutions, are proportional to the non-derivative terms isolated by the repeated replacement of eq. (8.9).

8.3 Regularization of 3-point functions

In the case of 3-point functions the analysis of the corresponding singularities can be extracted quite simply. Let's consider, for instance, the identity

$$\begin{aligned}\mathcal{FT}\left[\frac{1}{[(x-y)^2]^{\alpha_1}[(z-x)^2]^{\alpha_2}[(y-z)^2]^{\alpha_3}}\right] &\equiv \int d^dx\, d^dy\, d^dz\, \frac{e^{-i(k\cdot z+p\cdot x+q\cdot y)}}{[(x-y)^2]^{\alpha_1}[(z-x)^2]^{\alpha_2}[(y-z)^2]^{\alpha_3}} \\ &= (2\pi)^{3d}\prod_{i=1}^{3}\left(\frac{\Gamma(d/2-\alpha_i)}{4^{\alpha_i}\pi^{d/2}\Gamma(\alpha_i)}\right)\delta^{(d)}(k+p+q)\int\frac{d^dl}{[l^2]^{d/2-\alpha_1}[(l+p)^2]^{d/2-\alpha_2}[(l-q)^2]^{d/2-\alpha_3}}\,,\end{aligned} \tag{8.33}$$

obtained using the fundamental transform (8.1), where all the physical momenta (k, p, q) are treated as incoming. The convention for matching the momenta in (8.1) with the couples of coordinate is

$$l_1 \leftrightarrow x - y\,, \qquad l_2 \leftrightarrow z - x\,, \qquad l_3 \leftrightarrow y - z\,, \tag{8.34}$$

and the shift $l \to l - q$ (which is always possible in a regularized expression) has been performed at the end.

It is clear that the prefactor on the r.h.s. of this relation has poles for $\alpha_i = d/2 + n$, with $n \geq 0$. At the same time the loop integral is asymptotically divergent if $d = \sum_i \alpha_i$, where it develops a logarithmic singularity. In dimensional regularization such a singularity corresponds to a single pole in $\epsilon = d - \sum_i \alpha_i$. One can be more specific by discussing further examples of typical 3-point functions.

For instance, consider the tensor structure

$$\mathcal{Q}^1{}_{\alpha\beta\mu\nu} = \frac{(y-z)_\alpha\,(y-z)_\beta\,(y-z)_\mu\,(y-z)_\nu}{[(x-y)^2]^{d/2+1}\,[(z-x)^2]^{d/2-1}\,[(y-z)^2]^{d/2+1}}\,, \tag{8.35}$$

which appears in the TVV correlator and can be reduced to its R-form in several ways. We use a minimal substitution and have

$$\mathcal{Q}^1{}_{\alpha\beta\mu\nu} = \frac{1}{[(x-y)^2]^{d/2+1}}\,\frac{1}{[(z-x)^2]^{d/2-1}}\,R^4{}_{\alpha\beta\mu\nu}\left(y-z, \frac{d}{2}+1\right) \tag{8.36}$$

and application of the derivative reductions in (8.9) gives

$$\begin{aligned}
\mathcal{Q}^1{}_{\alpha\beta\mu\nu} &= \frac{1}{(d-6)\,(d-4)\,(d-2)\,d}\,\frac{1}{[(x-y)^2]^{d/2+1}}\,\frac{1}{[(x-z)^2]^{d/2-1}} \\
&\times\bigg\{\partial_\alpha\,\partial_\beta\,\partial_\mu\,\partial_\nu\,\frac{1}{[(y-z)^2]^{d/2-3}} + (d-6)\,(d-4)\,\frac{\delta_{\mu\nu}\,\delta_{\alpha\beta} + \delta_{\mu\alpha}\,\delta_{\nu\beta} + \delta_{\mu\beta}\,\delta_{\nu\alpha}}{[(y-z)^2]^{d/2-1}} \\
&+(d-6)\,(\delta_{\mu\nu}\,\partial_\alpha\,\partial_\beta + \delta_{\alpha\beta}\,\partial_\mu\,\partial_\nu + \delta_{\mu\alpha}\,\partial_\nu\,\partial_\beta + \delta_{\nu\beta}\,\partial_\mu\,\partial_\alpha + \delta_{\nu\alpha}\,\partial_\mu\,\partial_\beta + \delta_{\mu\beta}\,\partial_\nu\,\partial_\alpha)\,\frac{1}{[(y-z)^2]^{d/2-2}}\bigg\}.
\end{aligned} \tag{8.37}$$

Before moving to momentum space, a quick glance at this equation shows that its transform does not exist. This appears obvious from the presence of the overall factor $1/([(x-y)^2]^{d/2+1})$ which needs regularization. The mapping can be performed using the rules defined above, which give, for instance, for the coefficient of $\delta_{\mu\nu}\,\delta_{\alpha\beta} + \delta_{\mu\alpha}\,\delta_{\nu\beta} + \delta_{\mu\beta}\,\delta_{\nu\alpha}$,

$$\begin{aligned}
&\mathcal{FT}\left[\frac{1}{d(d-2)}\,\frac{\mu^{2\omega}}{[(x-y)^2]^{d/2+1-\omega}[(z-x)^2]^{d/2-1}[(y-z)^2]^{d/2-1}}\right] \\
&= \frac{(2\pi)^{3d}\delta^{(d)}(k+p+q)}{d(d-2)}\,\frac{4^{1+\omega}}{(4\pi)^{3d/2}}\,\frac{\Gamma(\omega-1)}{\Gamma(d/2-1)^2\Gamma(d/2-1-\omega)}\int d^d l\,\frac{\mu^{2\omega}}{(l^2)^{\omega-1}(l+p)^2(l-q)^2} \\
&= \frac{\delta^d(k+p+q)}{d(d-2)}\,\frac{4\pi^{3d/2}}{\Gamma(d/2-1)^3}\left[-\frac{1}{\omega}\int d^d l\,\frac{l^2}{(l+p)^2(l-q)^2} + \int d^d l\,\frac{l^2\log\left(l^2/\bar{\mu}^2\right)}{(l+p)^2(l-q)^2}\right] + O(\omega).
\end{aligned} \tag{8.38}$$

In a similar way the Fourier transform of the first term is

$$\mathcal{FT}\Big[\frac{1}{(d-6)\,(d-4)\,(d-2)\,d}\,\frac{\mu^{2\omega}}{[(x-y)^2]^{d/2+1-\omega}}\,\frac{1}{[(z-x)^2]^{d/2-1}}\,\partial_\mu\,\partial_\nu\,\partial_\alpha\,\partial_\beta\,\frac{1}{[(y-z)^2]^{d/2-3}}\Big]$$
$$= \frac{(2\pi)^{3d}\,\delta^{(d)(k+p+q)}}{(d-6)\,(d-4)\,(d-2)\,d}\,\frac{4^{3+\omega}}{(4\pi)^{3d/2}}\,\frac{2\,\Gamma(\omega-1)}{\Gamma(d/2-3)\,\Gamma(d/2-1)\,\Gamma(d/2+1-\omega)}$$
$$\times\int d^d l\,\frac{(l-q)_\alpha\,(l-q)_\beta\,(l-q)_\mu\,(l-q)_\nu}{(l^2)^{\omega-1}\,(l+p)^2\,[(l-q)]^3}$$
$$= \frac{\delta^{(d)(k+p+q)}}{d\,(d-2)}\,\frac{32\,\pi^{3d/2}}{\Gamma(d/2-1)^3}\left[-\frac{1}{\omega}\int d^d l\,\frac{l^2\,(l-q)_\alpha\,(l-q)_\beta\,(l-q)_\mu\,(l-q)_\nu}{(l+p)^2[(l-q)^2]^3}\right]$$
$$+\int d^d l\,\frac{\log\left(l^2/\bar{\mu}^2\right)\,(l-q)_\alpha\,(l-q)_\beta\,(l-q)_\mu\,(l-q)_\nu}{(l+p)^2[(l-q)^2]^3}+O(\omega)\,, \tag{8.39}$$

illustrating quite clearly how the general procedure can be implemented.

At this point we pause for some comments. The regularization can be performed by sending $d\to d-2\omega$ — with no distinction among the various terms — or, alternatively, one can regulate only the non integrable terms. The two approaches, in a generic computation, will differ only at $O(\omega)$ and as such they are equivalent. One can obviously check this by an explicit computation.

Another important point concerns the possibility of performing an explicit computation of the logarithmic integrals. They are indeed calculable in terms of generalized hypergeometric functions (for general ω), but the small ω expansion of these functions is rather difficult to re-express as a combination of ordinary functions and polylogs. This is due to the need of performing a double expansion (in ϵ and in ω) if we move to $d=4$ and insist, as we should, on the use of dimensional regularization in the computation of the momentum integrals. This difficulty is attributed to the absence of simple expansions of hypergeometric functions (ordinary and generalized) about non integer (real) values of their indices. However, if the $1/\omega$ terms for a combination of terms similar to those shown above cancel, there are some steps which can be taken in order to simplify this final part of the computation.

8.4 Application to the VVV case

To illustrate the way to proceed in general, we reconsider the VVV case, that we know to be integrable, but treated this time with the general algorithm. We expand the correlator and perform the R-substitutions (8.9). The direct algorithm gives an expression which is not immediately recognized as being integrable

$$f^{abc}\left\{\frac{(a-2\,b)}{(d-2)^3}\times\left[\partial^{31}_\mu\,\frac{1}{(x_{31}^2)^{d/2-1}}\,\partial^{12}_\nu\,\frac{1}{(x_{12}^2)^{d/2-1}}\,\partial^{23}_\rho\,\frac{1}{(x_{23}^2)^{d/2-1}}\right.\right.$$
$$\left.+\,\partial^{12}_\mu\,\frac{1}{(x_{12}^2)^{d/2-1}}\,\partial^{23}_\nu\,\frac{1}{(x_{23}^2)^{d/2-1}}\,\partial^{31}_\rho\,\frac{1}{(x_{31}^2)^{d/2-1}}\right]$$
$$+\frac{a}{d(d-2)^2}\times\left[\frac{1}{(x_{12}^2)^{d/2-1}}\left(\partial^{31}_\mu\frac{1}{(x_{31}^2)^{d/2-1}}\partial^{23}_\nu\partial^{23}_\rho\frac{1}{(x_{23}^2)^{d/2-1}}+\partial^{23}_\nu\frac{1}{(x_{23}^2)^{d/2-1}}\partial^{31}_\mu\partial^{31}_\rho\frac{1}{(x_{31}^2)^{d/2-1}}\right)\right.$$
$$+\,\frac{1}{(x_{23}^2)^{d/2-1}}\left(\partial^{31}_\rho\frac{1}{(x_{31}^2)^{d/2-1}}\partial^{12}_\mu\partial^{12}_\nu\frac{1}{(x_{12}^2)^{d/2-1}}+\partial^{12}_\nu\frac{1}{(x_{12}^2)^{d/2-1}}\partial^{31}_\mu\partial^{31}_\rho\frac{1}{(x_{31}^2)^{d/2-1}}\right)$$

$$+\frac{1}{(x_{31}^2)^{d/2-1}}\left(\partial^{23}_\rho\frac{1}{(x_{23}^2)^{d/2-1}}\partial^{12}_\mu\partial^{12}_\nu\frac{1}{(x_{12}^2)^{d/2-1}}+\partial^{12}_\mu\frac{1}{(x_{12}^2)^{d/2-1}}\partial^{23}_\nu\partial^{23}_\rho\frac{1}{(x_{23}^2)^{d/2-1}}\right)\Bigg]$$

$$-\frac{1}{d-2}\left(b-\frac{a}{d+2}\right)\times\Bigg[\frac{1}{(x_{31}^2)^{d/2-1}}\left(\frac{\delta_{\mu\nu}}{(x_{12}^2)^{d/2}}\partial^{23}_\rho\frac{1}{(x_{23}^2)^{d/2-1}}+\frac{\delta_{\nu\rho}}{(x_{23}^2)^{d/2}}\partial^{12}_\mu\frac{1}{(x_{12}^2)^{d/2-1}}\right)$$

$$+\frac{1}{(x_{23}^2)^{d/2-1}}\left(\frac{\delta_{\mu\nu}}{(x_{12}^2)^{d/2}}\partial^{31}_\rho\frac{1}{(x_{31}^2)^{d/2-1}}+\frac{\delta_{\mu\rho}}{(x_{31}^2)^{d/2}}\partial^{12}_\nu\frac{1}{(x_{12}^2)^{d/2-1}}\right)$$

$$+\frac{1}{(x_{12}^2)^{d/2-1}}\left(\frac{\delta_{\mu\rho}}{(x_{31}^2)^{d/2}}\partial^{23}_\nu\frac{1}{(x_{23}^2)^{d/2-1}}+\frac{\delta_{\nu\rho}}{(x_{23}^2)^{d/2}}\partial^{31}_\mu\frac{1}{(x_{31}^2)^{d/2-1}}\right)\Bigg]\Bigg\}. \tag{8.40}$$

The apparent non-integrability is due to terms of the form $1/(x_{ij}^2)^{d/2}$ in the last addend. For this reason, ignoring any further information, to test the approach we proceed with a regularization of the non-integrable terms. The expression in momentum space is obtained by sending $d \to d-2\omega$ in all the terms of the form $1/(x_{ij}^2)^{d/2}$. Expanding in ω the result, one can show that, as expected, the $1/\omega$ terms cancel, proving its integrability. We fill in few more details to clarify this point. A typical not manifestly integrable term in VVV is

$$\frac{1}{(x_{31}^2)^{d/2-1}}\frac{1}{(x_{12}^2)^{d/2}}\partial^{23}_\rho\frac{1}{(x_{23}^2)^{d/2-1}}+\frac{1}{(x_{23}^2)^{d/2-1}}\frac{1}{(x_{12}^2)^{d/2}}\partial^{31}_\rho\frac{1}{(x_{31}^2)^{d/2-1}} \tag{8.41}$$

which in momentum space after ω regularization gives (omitting an irrelevant constant)

$$\mu^{2\omega}\,\Gamma(\omega)\int d^dl\,\frac{2l^\rho-q^\rho}{(l^2)(l-q)^2[(l+p)^2]^\omega}. \tag{8.42}$$

Expanding in ω, the residue at the pole is given by the integral

$$\int d^dl\frac{2l^\rho-q^\rho}{l^2(l-q)^2} \tag{8.43}$$

which vanishes in dimensional regularization. The finite term is logarithmic and it is given by

$$\int d^dl\frac{\log\left((l+p)^2/\mu^2\right)\,(2l^\rho-q^\rho)}{l^2(l-q)^2}. \tag{8.44}$$

The scale dependence also disappears, since the $\log\mu^2$ term is also multiplied by the same vanishing integral. Obviously, the nontrivial part of the computation is in the appearance of a finite logarithmic integral which, due to the finiteness of the correlator, has to be re-expressed in terms of other non-logarithmic contributions, i.e. of ordinary Feynman integrals. There is no simple way to relate one single integral to an ordinary non-logarithmic contribution unless one performs the entire computation and expresses the result in terms of special polylogarithmic functions, using consistency. For correlators which are integrable, however, it is possible to relate two log integrals to regular Feynman integrals. Single log integrals, at least in this case, can also be evaluated explicitly, as we illustrate in an appendix.

By applying the algorithm we get

$$\left\langle V^a{}_\mu\, V^b{}_\nu\, V^c{}_\rho\right\rangle(p,q)=(2\pi)^{3d}\,\delta^{(d)}(k+p+q)\,i\,f^{abc}$$
$$\times\Bigg\{C(d/2-1)^3\left[\frac{a\,(6-4d)+2\,b\,d}{d(d-2)^3}\Big(2\,J_{\mu\nu\rho}(p,-q)+(p+q)_\mu\,J_{\nu\rho}(p,-q)+p_\nu\,J_{\mu\rho}(p,-q)\right.$$

$$
\begin{aligned}
& - q_\rho \, J_{\mu\nu}(p,-q) - p_\nu \, q_\mu \, J_\rho(p,-q) - p_\mu \, q_\rho \, J_\nu(p,-q) \Big) \Big] \\
& + \frac{a}{d(d-2)^2} \Big(- 2 \, (p_\mu + q_\mu) \, \big(p_\nu \, J_\rho(p,-q) + q_\rho \, J_\nu(p,-q)\big) \\
& + q_\rho p_\nu \big(2 \, J_\mu(p,-q) + (p-q)_\mu \, J(p,-q)\big) \Big) \\
& - \frac{C(d/2-1)^2}{(4\pi)^{d/2} \, \Gamma(d/2) \, (d-2)} \left(\frac{a}{d+2} - b \right) \Big[\delta_{\mu\nu} \Big(2 \, IL_\rho(p,0,-q) - q_\rho \, IL(p,0,-q) \Big) \\
& + \delta_{\mu\rho} \Big(2 \, IL_\nu(-q,0,p) + p_\nu \, IL(-q,0,p) \Big) \\
& + \delta_{\nu\rho} \Big(2 \, IL_\mu(q,0,k) + k_\mu \, IL(q,0,k) \Big) \Big] \Big\} . \qquad (8.45)
\end{aligned}
$$

The notations introduced for the momentum space integrals here and in the following point are explained in appendix F. One can easily show the scale independence of the result, which is related to the finiteness of the expressions and to the fact that the logarithmic contributions, in this case, are an artifact of the approach. For this reason, when the scale independence of the regulated expressions has been proved, then one can go back and try to rewrite the correlator in such a way that it is manifestly integrable. Obviously this may not be a straightforward thing to do, especially if the expression is given by hundreds of terms in configuration space. If, even after proving the finiteness of the expression, one is unable to rewrite it in an integrable form, one can always continue applying the algorithm that we have presented, generating the logarithmic integrals. Pairs of log integrals can be related to ordinary Feynman integrals by applying appropriate tricks. We have illustrated in an appendix an example where we discuss the computation of the single log-integral appearing in VVV as an example. In the TOO case one encounters both single and double-log integrals. For non-conformal correlators these second type of integrals are, in general, expected and turn out to be a characteristic feature of these correlators in momentum space.

8.5 Direct methods for the TOO case and double logs

A similar analysis can be pursued in the TOO case. Also for this correlator we can apply a direct approach in order to show the way to proceed in the test of its regularity. Using our basic transform (8.33) and introducing the regulator ω to regulate the intermediate singularities we can easily transform it to momentum space

$$
\begin{aligned}
& \mathcal{FT}\Big[\langle T_{\mu\nu}(x_1) \, O(x_2), \; O(x_3) \rangle \Big] \equiv \langle T_{\mu\nu} \, O \, O \rangle \, (p,q) = (2\pi)^{3d} \, \delta^{(d)}(k+p+q) \, a \\
& \times \Big\{ \frac{C(d/2-1)^3}{d\,(d-2)^2} \Big[- 4 \, (d-1) \, J_{\mu\nu}(p,-q) - 2 \, (d-1) \Big((q_\nu - p_\nu) \, J_\mu(p,-q) + (q_\mu - p_\mu) \, J_\nu(p,-q) \Big) \\
& + \Big(d \, (p_\mu q_\nu + p_\nu q_\mu) - (d-2) \, (p_\mu p_\nu + q_\mu q_\nu) \Big) \, J(p,-q) \Big] \\
& + \frac{C(d/2-1)^2 \, C(d/2-\omega)}{d} \, \delta_{\mu\nu} \Big(\int d^d l \, \frac{\mu^{2\omega}}{l^2 [(l+p)^2]^\omega (l-q)^2} + \int d^d l \, \frac{\mu^{2\omega}}{l^2 (l+p)^2 [(l-q)^2]^\omega} \Big) \\
& - \frac{C(d/2-1) \, C(d/2-\omega)^2}{d} \, \delta_{\mu\nu} \int d^d l \, \frac{(\mu^{2\omega})^2}{[l^2]^2 [(l+p)^2]^\omega [(l-q)^2]^\omega} \Big\} . \qquad (8.46)
\end{aligned}
$$

The expression above is affected by double and single poles in ω once we perform an expansion in this parameter, which are expected to vanish in order to guarantee a finite result.

The coefficient of the double pole is easily seen to take the form

$$- \delta_{\mu\nu} \frac{a\,(2\,\pi^2)^d\, d\, C(d/2-1)}{\Gamma(d/2)^2}\, I(0)\,, \tag{8.47}$$

where the integral vanishes in dimensional regularization, being a massless tadpole.

The coefficient of the simple pole is instead given by

$$\delta_{\mu\nu} \frac{4^d\,\pi^{5d/2}\, C(d/2-1)^2}{d\,\Gamma(d/2)} \left\{ \frac{1}{\Gamma(d/2-2)\,\Gamma(d/2)^2} \left[2 \Big(\gamma - \log 4 - \psi(d/2) \Big) I(0) \right.\right.$$
$$\left.\left. + \Big(IL(p,0,0) + IL(-q,0,0) \Big) \right] + \frac{1}{\Gamma(d/2-1)^2\,\Gamma(d/2)} \Big[I(p) + I(q) \Big] \right\}. \tag{8.48}$$

The first term of (8.48) vanishes as in the case of the double pole, while for the remaining contributions we use the relation

$$IL(p,0,0) = \int d^d l\, \frac{\log\left(\frac{(l+p)^2}{\mu^2}\right)}{[l^2]^2} = -\frac{\partial}{\partial\omega} \int d^d l\, \frac{\mu^{2\omega}}{[l^2]^2\,[(l+p)^2]^\omega}\bigg|_{\omega=0}. \tag{8.49}$$

It is easy to see that the contributions in the last line in (8.48) cancel after inserting the explicit value for the 2-point function in (8.7).

The finite part of the expression is found to be, after removing some additional tadpoles,

$$\langle T_{\mu\nu}\, O\, O\rangle\,(p,q) = (2\pi)^{3d}\,\delta^{(d)}(k+p+q)\, a$$
$$\times \left\{ \frac{C(d/2-1)^3}{d\,(d-2)^2} \left[-4\,(d-1)\, J_{\mu\nu}(p,-q) - 2\,(d-1) \Big((q_\nu - p_\nu)\, J_\mu(p,-q) + (q_\mu - p_\mu)\, J_\nu(p,-q) \Big) \right.\right.$$
$$\left. + \Big(d\,(p_\mu q_\nu + p_\nu q_\mu) - (d-2)\,(p_\mu p_\nu + q_\mu q_\nu) \Big)\, J(p,-q) \right]$$
$$-\delta_{\mu\nu} \left[\frac{C(d/2-1)^2}{d\,\pi^{d/2}\,2^d\,\Gamma(d/2)} \Big((\gamma - \log 4 - \psi(d/2))\, \big(I(p) + I(-q)\big) + \big(IL(p,0,-q) + IL(-q,0,p)\big) \Big) \right.$$
$$+ \frac{C(d/2-1)}{3\,d\,2^{2d+1}\,\pi^d\,\Gamma(d/2)^2} \Big(12\,(\gamma - \log 4 - \psi(d/2))\, \big(IL(p,0,0) + IL(-q,0,0)\big)$$
$$\left.\left. + 3\,\big(ILL(p,p,0,0) + 2\,ILL(p,-q,0) + ILL(-q,-q,0,0)\big) \Big) \right] \right\} \tag{8.50}$$

where now also double logarithmic integrals have appeared. Using the relations (8.7) and (8.49), the terms proportional to $(\gamma - \log 4 - \psi(d/2))$, which are just a remain of the regularization procedure, cancel out, leaving us with the simplified result

$$\langle T_{\mu\nu}\, O\, O\rangle\,(p,q) = (2\pi)^{3d}\,\delta^{(d)}(k+p+q)\, a$$
$$\times \left\{ \frac{C(d/2-1)^3}{d\,(d-2)^2} \left[-4\,(d-1)\, J_{\mu\nu}(p,-q) - 2\,(d-1) \Big((q_\nu - p_\nu)\, J_\mu(p,-q) + (q_\mu - p_\mu)\, J_\nu(p,-q) \Big) \right.\right.$$
$$\left. + \Big(d\,(p_\mu q_\nu + p_\nu q_\mu) - (d-2)\,(p_\mu p_\nu + q_\mu q_\nu) \Big)\, J(p,-q) \right]$$
$$-\delta_{\mu\nu} \frac{C(d/2-1)}{d\,(4\pi)^d\,\Gamma(d/2)^2} \left[(4\pi)^{d/2}\,\Gamma(d/2)\, C(d/2-1) \Big(\big(IL(p,0,-q) + IL(-q,0,p)\big) \Big) \right.$$
$$\left.\left. + (d-4)\,\big(ILL(p,p,0) + 2\,ILL(p,-q,0,0) + ILL(-q,-q,0,0)\big) \right] \right\}. \tag{8.51}$$

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It is slightly lengthy but quite straightforward to show that (8.51) can be re-expressed in terms of ordinary Feynman integrals. This can be obtained by reducing all the tensor integrals (logarithmic and non-logarithmic) to scalar forms. After the reduction, one can check directly that specific combinations of logarithmic integrals can be expressed in terms of ordinary master integrals. In this case these relations hold since the integrands of the logarithmic expansion (linear combinations thereof) are equivalent to non-logarithmic ones, given the finiteness of the correlators. Obviously for a correlator which is not integrable such a correspondence does not exist and the logarithmic integrals cannot be avoided. This would be another signal, obviously, that the theory does not have a realization in terms of a local Lagrangian, since a Lagrangian field theory has a diagrammatic description only in terms of ordinary Feynman integrals.

We conclude this section with few more remarks concerning the treatment of correlators with more general scaling dimensions (2Δ). For instance one could consider correlators of the generic form

$$\langle O_i(x_i)O_j(x_j)O_k(x_k)\rangle = \frac{\lambda_{ijk}}{((x_i-x_j)^2)^{\Delta_i+\Delta_j-\Delta_k}((x_j-x_k)^2)^{\Delta_j+\Delta_k-\Delta_i}((x_k-x_i)^2)^{\Delta_k+\Delta_i-\Delta_j}}. \tag{8.52}$$

In this case their expression in momentum space can be found by applying Mellin-Barnes methods. They can be reconducted to integrals in momentum space of the form

$$J(\nu_1,\nu_2,\nu_3) = \int \frac{d^d l}{(l^2)^{\nu_1}((l-k)^2)^{\nu_2}((l+p)^2)^{\nu_3}} \tag{8.53}$$

$$\nu_1 = d/2-\Delta_i-\Delta_j+\Delta_k \qquad \nu_2 = d/2-\Delta_j-\Delta_k+\Delta_i \qquad \nu_3 = d/2-\Delta_k-\Delta_i+\Delta_j \tag{8.54}$$

which can be expressed [13] in terms of generalized hypergeometric functions $F_4[a,b,c,d;x,y]$ of two variables (x,y), the two ratios of the 3 external momenta. The computation of these integrals with arbitrary exponents at the denominators is by now standard lore in perturbation theory, with recursion relations which allow to relate shifts in the exponents in a systematic way. The problem is more involved for correlators which require an intermediate regularization in order to be transformed to momentum space. In this case one can show, in general, that the pole structure (in $1/\omega$) of these can be worked out closely, but the finite $O(1)$ contributions involve derivatives of generalized hypergeometric functions respect to their indices a,b,c,d. The latter can be re-expressed in terms of poly-logarithmic functions, which are typical and common in ordinary perturbation theory, only in some cases. The possibility to achieve this is essentially related to finding simple expansions of the hypergeometric functions around non integer (and not just rational) indicial points. For integrable correlators the analysis of Mellin-Barnes methods remains, however, a significant option, which will probably deserve a closer look.

9 Perspectives: the integrated anomaly and the nonlocal action

Before coming to our conclusions, we offer here a brief discussion of the possible extensions of our analysis in the context of the emergence of massless degrees of freedom in the

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computation of correlators of the form TVV and TTT, as predicted by Riegert's Lagrangian solution [30] of the anomaly equation. We recall that an action that formally solves the anomaly equation takes the form

$$S_{\text{anom}}[g,A] = \frac{1}{8}\int d^4x\sqrt{g}\int d^4x'\sqrt{-g'}\left(G-\frac{2}{3}\Box R\right)_x G_4(x,x')\left[2b\,F+b'\left(G-\frac{2}{3}\Box R\right)+2c\,F_{\mu\nu}F^{\mu\nu}\right]_{x'} \tag{9.1}$$

where b, b' and c are parameters. For the case of a single fermion in an abelian gauge theory they are given by $b = 1/320\,\pi^2$, $b' = -11/5760\,\pi^2$, and $c = -e^2/24\,\pi^2$. F is the square of the Weyl tensor and G is the Euler density. The notation $G_4(x,x')$ denotes the Green's function of the differential operator defined by

$$\Delta_4 \equiv \nabla_\mu\left(\nabla^\mu\nabla^\nu + 2R^{\mu\nu} - \frac{2}{3}Rg^{\mu\nu}\right)\nabla_\nu = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}(\nabla^\mu R)\nabla_\mu - \frac{2}{3}\Box R. \tag{9.2}$$

As shown in [21, 26] performing repeated variations of the "anomaly induced" action (9.1) with respect to the background metric $g_{\mu\nu}$ and to the A_α gauge field, here taken as a background, one can reproduce the anomalous contribution of correlators with multiple insertions of the EMT or of gauge currents. This action does not reproduce the homogeneous contributions to the anomalous trace Ward identity, which require an independent computation in order to be identified. The action can be reformulated in such a way that its interactions become local [26], by introducing two auxiliary scalar fields. After some manipulations, one can show that the quartic pole reduces to a single pole and the anomaly induced action near a flat background takes the simpler form

$$S_{\text{anom}}[g,A] \to -\frac{c}{6}\int d^4x\sqrt{g}\int d^4x'\sqrt{-g'}\,R_x\,\Box^{-1}_{x,x'}\,[F_{\alpha\beta}F^{\alpha\beta}]_{x'}. \tag{9.3}$$

Notice that this action is valid to first order in metric variations around flat space. Its local expression is given by

$$S_{\text{anom}}[g,A;\varphi,\psi'] = \int d^4x\,\sqrt{g}\left[-\psi'\Box\,\varphi - \frac{R}{3}\,\psi' + \frac{c}{2}F_{\alpha\beta}F^{\alpha\beta}\varphi\right], \tag{9.4}$$

with ψ' and φ defined as in [21]. R, in the equations above, is the linearized version of the Ricci scalar

$$R \equiv \partial^x_\mu\,\partial^x_\nu\,h^{\mu\nu} - \Box\,h, \qquad h = \eta_{\mu\nu}\,h^{\mu\nu}. \tag{9.5}$$

eq. (9.4) shows the appearance of coupled massless degrees of freedom whose interpretation has been offered in [21] using the approach of dispersion relations and to which we refer for further details. This analysis, so far, has been limited to the TVV correlator and can be obviously extended, with some effort, to the case of the TTT vertex whose explicit computation has been discussed in this paper. In particular this analysis could test directly if the pole structure present in the expression of the TTT vertex will match the prediction of the same vertex once this is computed using (9.1) by functional differentiation respect to the metric. This point is technically very involved since it requires a comparison between the result of a direct computation in perturbative field theory of the TTT, as done in this work, with the same correlator computed from Riegert' s variational solution. We hope to come back to discuss this point in a related work.

10 Conclusions

In this work we have tried to close the gap between two analysis of several CFT correlators, such as the TVV and TTT vertices, characterized by the presence of one, two and three gravitons on the external lines. We have tried to map position space and momentum space approaches, showing their interrelation. We have used free field theory realizations of the general solutions of these correlators in order to establish their expression in momentum space. These expressions, obviously, remain valid for any CFT. We have also drawn a parallel between the approach to renormalization typical of standard perturbation theory and the same approach based on the solution of the anomalous Ward identities, as discussed in [18, 27]. As a nontrivial test of the equivalence of both methods in 4 dimensions, we have verified that the counterterms predicted by the general analysis in position space coincide with those obtained from momentum space in the Lagrangian predictions derived from one-loop free field theory calculations.

In our approach, based on dimensional regularization, the anomaly is generated by tracing in 4 dimensions the renormalized vertex, and in some cases, such as in the TVV vertex, it can be thought as due to a single specific tensor structure. This is characterized by the appearance of an anomaly pole. In the TTT case, the explicit expression of this vertex that we have presented is the starting point for further analysis. For instance it is a necessary intermediate step in demonstrating the correspondence between general CFT calculations in d-dimensional Euclidean position space, perturbative calculations by Feynman diagrams in momentum space, and the anomaly effective action of [24–26, 30]. This will remove a possible objection to the anomaly effective action raised in [18]) by the consistent inclusion of all the terms required by conformal invariance, including the non-anomalous ones for which the anomaly effective action is mute. The origin of an effective massless degree of freedom (an effective "dilaton-like" field) coupled to gravity in the Standard Model will then be made fully explicit. As we have mentioned, this point has already been proven in the TVV case [3, 4, 21] and is expected on general grounds of anomalous Ward identities and the associated non-trivial cohomology of Weyl transformations [24].

We have also discussed a general algorithm that should prove useful to regulate and map correlators from position space to momentum space, and we have illustrated how to perform such a mapping in a systematic way with a number of examples. The method can be applied in the analysis of more complex correlators, for which a manifest proof of finiteness may not be available. The power of the approach has been shown by re-analysing finite conformal correlators investigated in the first part, offering a complete test of its consistency.

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A The computation of TTT

A.1 Definitions and conventions

The covariant derivatives of a contravariant vector A^μ and of a covariant one B_μ are respectively

$$\nabla_\nu A^\mu \equiv \partial_\nu A^\mu + \Gamma^\mu_{\nu\rho} A^\rho \,, \tag{A.1}$$

$$\nabla_\nu B_\mu \equiv \partial_\nu B_\mu - \Gamma^\rho_{\nu\mu} B_\rho \,, \tag{A.2}$$

with the Christoffel symbols defined as

$$\Gamma^\alpha_{\beta\gamma}(z) = \frac{1}{2} g^{\alpha\kappa}(z) \left[-\partial_\kappa g_{\beta\gamma}(z) + \partial_\beta g_{\kappa\gamma}(z) + \partial_\gamma g_{\kappa\beta}(z) \right] \,. \tag{A.3}$$

Our definition of the Riemann tensor is

$$R^\lambda{}_{\mu\kappa\nu} = \partial_\nu \Gamma^\lambda_{\mu\kappa} - \partial_\kappa \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\kappa} - \Gamma^\lambda_{\kappa\eta}\Gamma^\eta_{\mu\nu}. \tag{A.4}$$

The Ricci tensor is defined by the contraction $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and the scalar curvature by $R = g^{\mu\nu} R_{\mu\nu}$.

The traceless part of the Riemann tensor in d dimension is the Weyl tensor,

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} &- \frac{2}{d-2}(g_{\alpha\gamma}\, R_{\delta\beta} + g_{\alpha\delta}\, R_{\gamma\beta} - g_{\beta\gamma}\, R_{\delta\alpha} - g_{\beta\delta}\, R_{\gamma\alpha}) \\ &+ \frac{2}{(d-1)(d-2)}\, (g_{\alpha\gamma}\, g_{\delta\beta} + g_{\alpha\delta}\, g_{\gamma\beta}) R \,, \end{aligned} \tag{A.5}$$

and its square, F^d, whose $d = 4$ realization, called simply F, appears in the trace anomaly equation (2.1), is

$$F^d \equiv C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - \frac{4}{d-2} R^{\alpha\beta} R_{\alpha\beta} + \frac{2}{(d-2)(d-1)} R^2 \,. \tag{A.6}$$

The Euler density is instead

$$G = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4\, R^{\alpha\beta} R_{\alpha\beta} + R^2 \,. \tag{A.7}$$

The functional variations with respect to the metric tensor are computed using the relations

$$\begin{aligned} \delta\sqrt{g} &= -\frac{1}{2}\sqrt{g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta} \qquad & \delta\sqrt{g} &= \frac{1}{2}\sqrt{g}\, g^{\alpha\beta}\, \delta g_{\alpha\beta} \\ \delta g_{\mu\nu} &= -g_{\mu\alpha} g_{\nu\beta}\, \delta g^{\alpha\beta} \qquad & \delta g^{\mu\nu} &= -g^{\mu\alpha} g^{\nu\beta}\, \delta g_{\alpha\beta} \,. \end{aligned} \tag{A.8}$$

The structure $s^{\alpha\beta\gamma\delta}$ has been repeatedly used throughout the calculations: it comes from

$$-\left.\frac{\delta g^{\alpha\beta}(z)}{\delta g_{\gamma\delta}(x)}\right|_{g_{\mu\nu}=\delta_{\mu\nu}} = \frac{1}{2}\left[\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right]\, \delta^{(4)}(z-x) = s^{\mu\nu\alpha\beta}\, \delta^{(4)}(z-x) \,. \tag{A.9}$$

B Functional derivation of invariant integrals

In this appendix we briefly show how to evaluate the functional variation of the invariant integral $\mathcal{I}(a,b,c)$

$$\mathcal{I}(a,b,c) \equiv \int d^dx \sqrt{g}\, K \equiv \int d^dx \sqrt{g} \left(a\, R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + b\, R^{\alpha\beta} R_{\alpha\beta} + c\, R^2\right), \tag{B.1}$$

needed to compute the counterterms found in section 6.3.

Our index conventions for the Riemann and Ricci tensors are those in (A.4). We have

$$\begin{aligned}
\delta(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}) &= \delta(g_{\alpha\sigma} g^{\beta\eta} g^{\gamma\zeta} g^{\delta\rho} R^{\alpha}{}_{\beta\gamma\delta} R^{\sigma}{}_{\eta\zeta\rho}) \\
&= \delta(g_{\alpha\sigma} g^{\beta\eta} g^{\gamma\zeta} g^{\delta\rho}) R^{\alpha}{}_{\beta\gamma\delta} R^{\sigma}{}_{\eta\zeta\rho} + g_{\alpha\sigma} g^{\beta\eta} g^{\gamma\zeta} g^{\delta\rho} \delta(R^{\alpha}{}_{\beta\gamma\delta} R^{\sigma}{}_{\eta\zeta\rho}) \\
&= \delta(g_{\alpha\sigma} g^{\beta\eta} g^{\gamma\zeta} g^{\delta\rho}) R^{\alpha}{}_{\beta\gamma\delta} R^{\sigma}{}_{\eta\zeta\rho} + 2\, \delta(R^{\alpha}{}_{\beta\gamma\delta}) R_{\alpha}{}^{\beta\gamma\delta}\,.
\end{aligned} \tag{B.2}$$

Using (A.8) and (A.9) and the product rule for derivatives one easily finds out that the variation can be written at first as

$$\begin{aligned}
\delta\mathcal{I}(a,b,c) = \int d^dx \sqrt{g} \Bigg\{ &\left[\frac{1}{2} g^{\mu\nu} K - 2a\, R^{\mu\alpha\beta\gamma} R^{\nu}{}_{\alpha\beta\gamma} - 2b\, R^{\mu\alpha} R^{\nu}{}_{\alpha} - 2c\, R R^{\mu\nu}\right] \delta g_{\mu\nu} \\
&+ 2a\, R_{\alpha}{}^{\beta\gamma\delta} \delta R^{\alpha}{}_{\beta\gamma\delta} + 2b\, R^{\alpha\beta} \delta R_{\alpha\beta} + 2c\, R g^{\alpha\beta} \delta R_{\alpha\beta} \Bigg\}\,.
\end{aligned} \tag{B.3}$$

Exploiting the Palatini identities,

$$\delta R^{\alpha}{}_{\beta\gamma\delta} = (\delta\Gamma^{\alpha}_{\beta\gamma})_{;\delta} - (\delta\Gamma^{a}_{\beta\delta})_{;\gamma} \quad \Rightarrow \quad \delta R_{\beta\delta} = (\delta\Gamma^{\lambda}_{\beta\lambda})_{;\delta} - (\delta\Gamma^{\lambda}_{\beta\delta})_{;\lambda}\,, \tag{B.4}$$

and the Bianchi identities we get

$$\begin{aligned}
R_{\alpha\beta\gamma\delta;\eta} + R_{\alpha\beta\eta\gamma;\delta} + R_{\alpha\beta\delta\eta;\gamma} = 0 \quad &\Rightarrow \quad R_{\beta\delta;\eta} - R_{\beta\eta;\delta} + R^{\gamma}{}_{\beta\delta\eta;\gamma} = 0 \\
\Rightarrow \quad R_{;\delta} = 2\, R^{\alpha}{}_{\delta;\alpha} \quad &\Leftrightarrow \quad \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right)_{;\beta} = 0\,.
\end{aligned} \tag{B.5}$$

After an integration by parts and a reshuffling of indices we get

$$\begin{aligned}
\delta\mathcal{I}(a,b,c) = \int d^dx \sqrt{g} \Bigg\{ &\left[\frac{1}{2} g^{\mu\nu} K - 2\big(a\, R^{\mu\alpha\beta\gamma} R^{\nu}{}_{\alpha\beta\gamma} + b\, R^{\mu\alpha} R^{\nu}{}_{\alpha} + c\, R R^{\mu\nu}\big)\right] \delta g_{\mu\nu} \\
+ \Big[4a\, g_{\beta\delta} g^{\gamma\eta} (\delta\Gamma^{\delta}_{\alpha\gamma})_{;\eta} &- (4a+2b)\, (\delta\Gamma^{\gamma}_{\alpha\beta})_{;\gamma} + (4c+2b)\, (\delta\Gamma^{\lambda}_{\alpha\lambda})_{;\beta} - 4c g^{\eta\delta} g_{\gamma\alpha} (\delta\Gamma^{\gamma}_{\eta\delta})_{;\beta}\Big] R^{\alpha\beta} \Bigg\}.
\end{aligned} \tag{B.6}$$

The variations of the Christoffel symbols and of their covariant derivatives in terms of covariant derivatives of the metric tensors variations are

$$\begin{aligned}
\delta\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2}\, g^{\alpha\delta} \big[- (\delta g_{\beta\gamma})_{;\delta} + (\delta g_{\beta\delta})_{;\gamma} + (\delta g_{\gamma\delta})_{;\beta} \big]\,, \\
(\delta\Gamma^{\alpha}_{\beta\gamma})_{;\delta} &= \frac{1}{2}\, g^{\alpha\eta} \big[- (\delta g_{\beta\gamma})_{;\eta;\delta} + (\delta g_{\beta\eta})_{;\gamma;\delta} + (\delta g_{\gamma\eta})_{;\beta;\delta} \big]\,.
\end{aligned} \tag{B.7}$$

Now we use them to rewrite (B.6) as

$$\begin{aligned}\delta\mathcal{I}(a,b,c) = \int d^dx\,\sqrt{g}\Big\{ &\Big[\frac{1}{2}g^{\mu\nu}K - 2\big(a\,R^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma} + b\,R^{\mu\alpha}R^{\nu}{}_{\alpha} + c\,RR^{\mu\nu}\big)\Big]\delta g_{\mu\nu} \\ &+\Big[2a\big[-(\delta g_{\alpha\delta})_{;\beta;\gamma} + (\delta g_{\alpha\beta})_{;\gamma;\delta} + (\delta g_{\beta\delta})_{;\alpha;\gamma}\big] \\ &-(2a{+}b)\big[-(\delta g_{\alpha\beta})_{;\delta;\gamma}{+}(\delta g_{\alpha\delta})_{;\beta;\gamma}{+}(\delta g_{\beta\delta})_{;\alpha;\gamma}\big]{+}(2c{+}b)(\delta g_{\gamma\delta})_{;\alpha;\beta} \\ &-2c\big[-(\delta g_{\gamma\delta})_{;\alpha;\beta} + (\delta g_{\alpha\delta})_{;\gamma;\beta} + (\delta g_{\alpha\gamma})_{;\delta;\beta}\big]\Big]g^{\gamma\delta}\,R^{\alpha\beta}\Big\}\,. \end{aligned} \tag{B.8}$$

The presence of the factor $g^{cd}R^{ab}$ imposes two symmetry constraints on the terms in the last contribution in square brackets. By adding and subtracting $-(4a+2b)\,(\delta g_{ac})_{;d;b}$ we obtain the expression

$$\begin{aligned}\delta\mathcal{I}(a,b,c) = \int d^dx\,\sqrt{g}\,\Big\{ &\Big[\frac{1}{2}g^{\mu\nu}K - 2\big(a\,R^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma} + b\,R^{\mu\alpha}R^{\nu}{}_{\alpha} + c\,RR^{\mu\nu}\big)\Big]\delta g_{\mu\nu} \\ &+\Big[(4a+2b)\,\big[(\delta g_{\alpha\gamma})_{;\beta;\delta} - (\delta g_{\alpha\gamma})_{;\delta;\beta}\big] + (4a+b)(\delta g_{\alpha\beta})_{;\gamma;\delta} \\ &+(4c+b)(\delta g_{\gamma\delta})_{;\alpha;\beta} - (4a+2b+4c)(\delta g_{\alpha\gamma})_{;\delta;\beta}\Big]g^{\gamma\delta}R^{\alpha\beta}\Big\}\,. \end{aligned} \tag{B.9}$$

The commutation of covariant derivatives allows us to write

$$\begin{aligned} g^{\gamma\delta}\big[(\delta g_{\alpha\gamma})_{;\beta;\delta} - (\delta g_{\alpha\gamma})_{;\delta;\beta}\big]R^{\alpha\beta} &= g^{\gamma\delta}\big[-\delta g_{\alpha\sigma}R^{\sigma}{}_{\gamma\delta\beta} - \delta g_{\gamma\sigma}R^{\sigma}{}_{\alpha\beta\delta}\big]R^{\alpha\beta} \\ &= g^{\gamma\delta}\big[-s^{\mu\nu}_{\alpha\sigma}R^{\sigma}{}_{\gamma\beta\delta} - s^{\mu\nu}_{c\sigma}R^{\sigma}{}_{\alpha\beta\delta}\big]R^{\alpha\beta}\,\delta g_{\mu\nu} \\ &= (-R^{\mu\alpha}R^{\nu}{}_{\alpha} + R^{\mu\alpha\nu\beta}R_{\alpha\beta})\delta g_{\mu\nu}\,. \end{aligned} \tag{B.10}$$

Inserting this back into (B.9) we get

$$\begin{aligned}&\delta\mathcal{I}(a,b,c) = \\ &\int d^dx\sqrt{g}\Big\{\Big[\frac{1}{2}g^{\mu\nu}K - 2aR^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma} + 4aR^{\mu\alpha}R^{\nu}{}_{\alpha} - (4a+2b)R^{\mu\alpha\nu\beta}R_{\alpha\beta} - 2cRR^{\mu\nu}\Big]\delta g_{\mu\nu} \\ &+\Big[(4a+b)(\delta g_{\alpha\beta})_{;\gamma;\delta} + (4c+b)(\delta g_{\gamma\delta})_{;\alpha;\beta} - (4a+2b+4c)(\delta g_{\alpha\gamma})_{;\delta;\beta}\Big]g^{\gamma\delta}R^{\alpha\beta}\Big\}. \end{aligned} \tag{B.11}$$

If the coefficients are $a = c = 1$ and $b = -4$, i.e. if the integrand is the Euler density, the last three terms are zero.

All that is left to do is a double integration by parts for each one of the last three terms, to factor out $\delta g_{\mu\nu}$. This is easily performed and the final result can be written as

$$\begin{aligned}\frac{\delta}{\delta g_{\mu\nu}}\mathcal{I}(a,b,c) &= \frac{\delta}{\delta g_{\mu\nu}}\int d^dx\sqrt{g}\big(aR^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + bR^{\alpha\beta}R_{\alpha\beta} + cR^2\big) \\ &= \sqrt{g}\Big\{\frac{1}{2}g^{\mu\nu}K - 2aR^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma} + 4aR^{\mu\alpha}R^{\nu}{}_{\alpha} - (4a{+}2b)R^{\mu\alpha\nu\beta}R_{\alpha\beta} - 2cRR^{\mu\nu} \\ &\quad + (4a{+}b)\Box R^{\mu\nu} + (4c{+}b)g^{\mu\nu}R^{\alpha\beta}{}_{;\alpha;\beta} - (4a{+}2b{+}4c)R^{\nu\beta}{}_{;\beta}{}^{;\mu}\Big\}. \end{aligned} \tag{B.12}$$

C List of functional derivatives

We list here the contributions to the trace anomalies for three point function coming from the elementary quadratic objects. They are given by

$$
\begin{aligned}
\big[\Box\, R\big]^{\alpha\beta\rho\sigma}(p,q) &= \big[g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\lambda_{\mu\nu}\partial_\lambda)R\big]^{\alpha\beta\rho\sigma}(p,q) \\
&= i^2\,(p+q)^2\,\big[R\big]^{\alpha\beta\rho\sigma}(p,q) - \big\{i^2\,q^\alpha q^\beta - \delta^{\mu\nu}\,\big[\Gamma^\lambda_{\mu\nu}\big]^{\alpha\beta}(p)\,i\,q_\lambda\big\}R^{\rho\sigma}(q) \\
&\quad -\big\{i^2\,p^\rho p^\sigma - \delta^{\mu\nu}\,\big[\Gamma^\lambda_{\mu\nu}\big]^{\rho\sigma}(q)\,i\,p_\lambda\big\}R^{\alpha\beta}(p) \\
&= (p+q)^2\bigg\{-\frac{1}{2}\delta^{\alpha\beta}\big(p^\rho q^\sigma + p^\sigma q^\rho + 2\,p^\rho p^\sigma\big) - \frac{1}{2}\delta^{\rho\sigma}\big(q^\alpha p^\beta + q^\beta q^\alpha + 2\,q^\alpha q^\beta\big) \\
&\quad +\frac{1}{2}p\cdot q\,\delta^{\alpha\beta}\delta^{\rho\sigma} + \frac{1}{4}\big(p^\rho q^\beta\delta^{\alpha\sigma} + p^\rho q^\alpha\delta^{\beta\sigma} + p^\sigma q^\beta\delta^{\alpha\rho} + p^\sigma q^\alpha\delta^{\beta\rho}\big) \\
&\quad +\frac{1}{2}\Big[\big(q^\rho p^\beta\delta^{\alpha\sigma} + q^\rho p^\alpha\delta^{\beta\sigma} + q^\sigma p^\beta\delta^{\alpha\rho} + q^\sigma p^\alpha\delta^{\beta\rho}\big) \\
&\quad +\delta^{\alpha\rho}\big(p^\beta p^\sigma + q^\beta q^\sigma\big) + \delta^{\alpha\sigma}\big(p^\beta p^\rho + q^\beta q^\rho\big) + \delta^{\beta\rho}\big(p^\alpha p^\sigma + q^\alpha q^\sigma\big) \\
&\quad +\delta^{\beta\sigma}\big(p^\alpha p^\rho + q^\alpha q^\rho\big) - \big(\delta^{\alpha\sigma}\delta^{\beta\rho} + \delta^{\alpha\rho}\delta^{\beta\sigma}\big)\big(p^2 + q^2 + \frac{3}{2}p\cdot q\big)\Big]\bigg\} \\
&\quad +\frac{1}{2}\big(p^2\delta^{\alpha\beta} - p^\alpha p^\beta\big)\big(p\cdot q\,\delta^{\rho\sigma} - (p^\rho q^\sigma + p^\sigma q^\rho) - 2\,p^\rho p^\sigma\big) \\
&\quad +\frac{1}{2}\big(q^2\delta^{\rho\sigma} - q^\sigma q^\rho\big)\big(p\cdot q\,\delta^{\alpha\beta} - (p^\alpha q^\beta + p^\beta q^\alpha) - 2\,q^\alpha q^\beta\big)\,,
\end{aligned}
\tag{C.1}
$$

with

$$
\begin{aligned}
\big[\Gamma^\lambda_{\alpha\beta}\big]^{\rho\sigma}(l) &= \frac{1}{2}\delta^{\lambda\kappa}\,i\,\Big[s^{\rho\sigma}_{\alpha\kappa}\,l_\beta + s^{\rho\sigma}_{\beta\kappa}\,l_\alpha - s^{\rho\sigma}_{\alpha\beta}\,l_\kappa\Big]\,, \\
\big[R_{\alpha\beta}\big]^{\rho\sigma}(l) &= -i\,l_\alpha\,[\Gamma^\lambda_{\lambda\beta}]^{\rho\sigma}(l) + i\,l_\lambda\,[\Gamma^\lambda_{\alpha\beta}]^{\rho\sigma}(l)\,.
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\big[R_{\lambda\mu\kappa\nu}R^{\lambda\mu\kappa\nu}\big]^{\alpha\beta\rho\sigma}(p,q) &= 2\,\big[R_{\lambda\mu\kappa\nu}\big]^{\alpha\beta}(p)\big[R^{\lambda\mu\kappa\nu}\big]^{\rho\sigma}(q) \\
&= p\cdot q\,\big[p\cdot q\big(\delta^{\alpha\rho}\delta^{\beta\sigma} + \delta^{\alpha\sigma}\delta^{\beta\rho}\big) - \big(\delta^{\alpha\rho}p^\sigma q^\beta + \delta^{\alpha\sigma}p^\rho q^\beta \\
&\quad +\delta^{\beta\rho}p^\sigma q^\alpha + \delta^{\beta\sigma}p^\rho q^\alpha\big)\big] + 2\,p^\rho p^\sigma q^\alpha q^\beta\,, \\
\big[R_{\mu\nu}R^{\mu\nu}\big]^{\alpha\beta\rho\sigma}(p,q) &= 2\,\big[R_{\mu\nu}\big]^{\alpha\beta}(p)\big[R^{\mu\nu}\big]^{\rho\sigma}(q) \\
&= \frac{1}{4}p\cdot q\big(\delta^{\alpha\rho}p^\beta q^\sigma + \delta^{\alpha\sigma}p^\beta q^\rho + \delta^{\beta\rho}p^\alpha q^\sigma + \delta^{\beta\sigma}p^\alpha q^\rho\big) \\
&\quad +\frac{1}{2}(p\cdot q)^2\delta^{\alpha\beta}\delta^{\rho\sigma} + \frac{1}{4}p^2q^2\big(\delta^{\alpha\rho}\delta^{\beta\sigma} + \delta^{\alpha\sigma}\delta^{\beta\rho}\big) \\
&\quad -\Big[\frac{1}{4}p^2\big(q^\alpha q^\rho\delta^{\beta\sigma} + q^\alpha q^\sigma\delta^{\beta\rho} + q^\beta q^\rho\delta^{\alpha\sigma} + q^\beta q^\sigma\delta^{\alpha\rho}\big) \\
&\quad +\frac{1}{2}\delta^{\alpha\beta}\big(p\cdot q\,(p^\rho q^\sigma + p^\sigma q^\rho) - q^2p^\rho p^\sigma\big) + (\alpha,\beta,p)\leftrightarrow(\rho,\sigma,q)\Big]\,, \\
\big[R^2\big]^{\alpha\beta\rho\sigma}(p,q) &= 2\,\delta^{\mu\nu}\big[R_{\mu\nu}\big]^{\alpha\beta}(p)\delta^{\tau\omega}\big[R_{\tau\omega}\big]^{\rho\sigma}(q) \\
&= 2\big(p^\alpha p^\beta q^\rho q^\sigma - p^2q^\rho q^\sigma\delta^{\alpha\beta} - q^2p^\alpha p^\beta\delta^{\rho\sigma} + p^2\,q^2\delta^{\alpha\beta}\delta^{\rho\sigma}\big)\,,
\end{aligned}
\tag{C.3}
$$

The dependence on the momenta is obviously determined by (2.25).

D Vertices

We have shown in figure 3 a list of all the vertices which are needed for the momentum space computation of the various correlators in d dimensions. We list them below: notice that they are computed differentiating the first and second functional derivatives of the action, because this allows to keep multi-graviton correlators symmetric (see 2.22).

$$
\begin{aligned}
V^{\mu\nu}_{T\phi\phi}(p,q) &= \frac{1}{2}p_\alpha q_\beta C^{\mu\nu\alpha\beta} + \chi\Big(\delta^{\mu\nu}\,(p+q)^2 - (p^\mu+q^\mu)\,(p^\nu+q^\nu)\Big),\\
V^{\mu\nu}_{T\bar\psi\psi}(p,q) &= \frac{1}{8}A^{\mu\nu\alpha\lambda}\gamma_\alpha\,(p_\lambda - q_\lambda)\,,\\
V^{\mu\nu\tau\omega}_{TAA}(p,q) &= \frac{1}{2}\Big[p\cdot q C^{\mu\nu\tau\omega} + D^{\mu\nu\tau\omega}(p,q) + \frac{1}{\xi}E^{\mu\nu\tau\omega}(p,q)\Big] = \Big(\tilde V_{TAA} + \frac{1}{\xi}\bar V_{TAA}\Big)^{\mu\nu\tau\omega}(p,q),\\
V^{\mu\nu}_{T\bar c c}(p,q) &= -V^{\mu\nu}_{T\phi\phi}(p,q)\Big|_{\chi=0},
\end{aligned}
$$

for the graviton (T)- to two scalars (ϕ), fermions, photons and ghost pairs. Quadrilinear interactions involving 2 gravitons are far more involved and are given by the expressions

$$
\begin{aligned}
V^{\mu\nu\rho\sigma}_{TT\phi\phi}(p,q,l) &= \frac{1}{2}p\cdot q s^{\mu\nu\rho\sigma} - \frac{1}{4}G^{\mu\nu\rho\sigma}(p,q) + \frac{1}{4}\delta^{\rho\sigma}p_\alpha q_\beta C^{\mu\nu\alpha\beta}\\
&\quad + \chi\bigg\{\bigg[\Big(\delta^{\mu\lambda}\delta^{\alpha\kappa}\delta^{\nu\beta} + \delta^{\mu\alpha}\delta^{\nu\kappa}\delta^{\beta\lambda} - \delta^{\mu\kappa}\delta^{\nu\lambda}\delta^{\alpha\beta} - \delta^{\mu\nu}\delta^{\alpha\lambda}\delta^{\beta\kappa}\Big)s^{\rho\sigma}_{\lambda\kappa}\\
&\quad + \frac{1}{2}\delta^{\rho\sigma}\Big(\delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\mu\nu}\delta^{\alpha\beta}\Big)\bigg]\,(p_\alpha q_\beta + p_\beta q_\alpha + p_\alpha p_\beta + q_\alpha q_\beta)\\
&\quad + \bigg[\Big(\delta^{\mu\nu}\delta^{\alpha\beta} - \delta^{\mu\alpha}\delta^{\nu\beta}\Big)\,[\Gamma^\lambda_{\alpha\beta}]^{\rho\sigma}(l)i\,(p_\lambda + q_\lambda)\\
&\quad + \Big(\delta^{\mu\alpha}\delta^{\nu\beta} - \frac{1}{2}\delta^{\mu\nu}\delta^{\alpha\beta}\Big)\,[R_{\alpha\beta}]^{\rho\sigma}(l)\bigg]\bigg\},\\
V^{\mu\nu\rho\sigma}_{TT\bar\psi\psi}(p,q) &= \frac{1}{16}\bigg[-4\,s^{\mu\nu\rho\sigma} - 2\,\delta^{\mu\nu}\,s^{\alpha\lambda\rho\sigma} + 2\,\delta^{\alpha\mu}\,s^{\nu\lambda\rho\sigma} + 2\,\delta^{\alpha\nu}\,s^{\mu\lambda\rho\sigma} + \delta^{\mu\lambda}\,s^{\alpha\nu\rho\sigma}\\
&\quad + \delta^{\nu\lambda}\,s^{\alpha\mu\rho\sigma} + \delta^{\rho\sigma}\,A^{\mu\nu\alpha\lambda}\bigg]\,\gamma_\alpha\,(p_\lambda - q_\lambda)\,,\\
V^{\mu\nu\rho\sigma\tau\omega}_{TTAA}(p,q,l) &= \frac{1}{2}\bigg\{\Big[B^{\alpha\mu\rho\sigma\beta\lambda\gamma\nu} + \frac{1}{4}B^{\mu\nu\rho\sigma\alpha\lambda\gamma\beta}\Big]F_{\alpha\beta\gamma\lambda}{}^{\tau\omega}(p,q)\\
&\quad + \frac{1}{\xi}\Big(H^{\mu\nu\rho\sigma\tau\omega}(p,q,l) + I^{\mu\nu\rho\sigma\tau\omega}(p,q,l)\Big)\bigg\}\\
&\quad + \frac{1}{4}\delta^{\rho\sigma}\Big[p\cdot q C^{\mu\nu\tau\omega} + D^{\mu\nu\tau\omega}(p,q) + \frac{1}{\xi}E^{\mu\nu\tau\omega}(p,q)\Big]\\
&= \big(\tilde V_{TTAA}(p,q) + \bar V_{TTAA}(p,q,l)\big)^{\mu\nu\rho\sigma\tau\omega},\\
V^{\mu\nu\rho\sigma}_{TT\bar c c}(p,q,l) &= -V^{\mu\nu\rho\sigma}_{TT\phi\phi}(p,q,l)\Big|_{\chi=0},
\end{aligned}
\tag{D.1}
$$

describing interactions similar to those shown above in the trilinear case, but now with the insertion of one extra graviton. We have simplified the notation by introducing, for convenience, the tensor components

$$
\begin{aligned}
A^{\mu\nu\alpha\lambda} &= 2\,\delta^{\mu\nu}\,\delta^{\alpha\lambda} - \delta^{\alpha\mu}\,\delta^{\lambda\nu} - \delta^{\alpha\nu}\,\delta^{\lambda\mu} \\
B^{\alpha\mu\rho\sigma\beta\lambda\gamma\nu} &= s^{\alpha\mu\rho\sigma}\,\delta^{\beta\lambda}\,\delta^{\gamma\nu} + s^{\beta\lambda\rho\sigma}\,\delta^{\alpha\mu}\,\delta^{\gamma\nu} + s^{\gamma\nu\rho\sigma}\,\delta^{\alpha\mu}\,\delta^{\beta\lambda} \\
C^{\mu\nu\rho\sigma} &= \delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho} - \delta^{\mu\nu}\delta^{\rho\sigma}\,, \\
D^{\mu\nu\rho\sigma}(p,q) &= \delta^{\mu\nu}p^\sigma q^\rho + \delta^{\rho\sigma}\left(p^\mu q^\nu + p^\nu q^\mu\right) - \delta^{\mu\sigma}p^\nu q^\rho - \delta^{\mu\rho}p^\sigma q^\nu - \delta^{\nu\sigma}p^\mu q^\rho - \delta^{\nu\rho}p^\sigma q^\mu \\
E^{\mu\nu\rho\sigma}(p,q) &= \delta^{\mu\nu}\left[p^\rho p^\sigma + q^\rho q^\sigma + p^\rho q^\sigma\right] - \left[\delta^{\nu\sigma}p^\mu p^\rho + \delta^{\nu\rho}q^\mu q^\sigma + \delta^{\mu\sigma}p^\nu p^\rho + \delta^{\mu\rho}q^\nu q^\sigma\right], \\
F^{\mu\nu\rho\sigma\tau\omega}(p,q) &= -\delta^{\tau\rho}\delta^{\omega\mu}p^\sigma q^\nu + \delta^{\tau\rho}\delta^{\omega\nu}p^\sigma q^\mu + \delta^{\tau\sigma}\delta^{\omega\mu}p^\rho q^\nu - \delta^{\tau\sigma}\delta^{\omega\nu}p^\rho q^\mu + (\tau,p)\leftrightarrow(\omega,q) \\
G^{\mu\nu\rho\sigma}(p,q) &= \delta^{\mu\sigma}\left[p^\rho q^\nu + q^\rho p^\nu\right] + \delta^{\nu\sigma}\left[p^\rho q^\mu + q^\rho p^\mu\right] + \delta^{\mu\rho}\left[p^\sigma q^\nu + q^\sigma p^\nu\right] + \delta^{\nu\rho}\left[p^\sigma q^\mu + q^\sigma p^\mu\right] \\
&\quad -\delta^{\mu\nu}\left[p^\rho q^\sigma + q^\rho p^\sigma\right] \\
H^{\mu\nu\rho\sigma\tau\omega}(p,q,l) &= \Bigg[\left(s^{\mu\omega\rho\sigma}\,\delta^{\nu\lambda} + s^{\nu\lambda\rho\sigma}\,\delta^{\mu\omega}\right)p_\lambda\,p^\tau + \delta^{\mu\omega}\left(s^{\lambda\tau\rho\sigma}\,l^\nu + s^{\lambda\tau\rho\sigma}\,p^\nu\right)p_\lambda \\
&\quad +\frac{1}{2}\delta^{\mu\omega}\,(p+l)^\nu\left(-l^\tau\,\delta^{\rho\sigma} + 2\,l_\lambda\,s^{\tau\lambda\rho\sigma}\right) + (\mu\leftrightarrow\nu)\Bigg] + (\tau,p)\leftrightarrow(\omega,q) \\
I^{\mu\nu\rho\sigma\tau\omega}(p,q,l) &= \delta^{\mu\nu}\left\{\frac{1}{2}\,\delta^{\rho\sigma}\,l^\tau\,(p+q+l)^\omega - s^{\lambda\tau\rho\sigma}\left[q^\omega\,p_\lambda + l_\lambda\,(p+q+l)^\omega\right]\right. \\
&\quad \left.-s^{\lambda\omega\rho\sigma}\left[p^\tau\,p_\lambda + q_\lambda\,(q+l)^\tau\right]\right\} - s^{\mu\nu\rho\sigma}\left(p^\omega\,p^\tau + q^\omega\,p^\tau\right) + (\tau,p)\leftrightarrow(\omega,q).
\end{aligned}
\tag{D.2}
$$

We have performed all our computations in the Feynman gauge ($\xi = 1$) The Euclidean propagators of the fields in this case are

$$
\begin{aligned}
\langle\phi\,\phi\rangle\,(p) &= \frac{1}{p^2} \\
\langle\bar\psi\,\psi\rangle\,(p) &= \frac{p\cdot\gamma}{p^2}\,. \\
\langle A^\mu\,A^\nu\rangle\,(p) &= \frac{\delta^{\mu\nu}}{p^2}\,, \\
\langle\bar c\,c\rangle\,(p) &= -\frac{1}{p^2}\,.
\end{aligned}
\tag{D.3}
$$

E Comments on the inverse mapping

In this appendix we offer some calculational details in the derivation of the expression of the TTT correlator in position space. The remarks apply as well to any other correlator.

For example eq. (4.4) refers to the contribution coming from the triangle diagram shown in figure 2. We assign the loop momentum l to flow from the upper external point (x_3) to the lower one (x_2) on the right, the other two flows being determined by momentum conservation. We denote the third external point as x_1. For the scalar case, for instance, the complete one-loop triangle diagram is

$$
\int \frac{d^d l}{(2\pi)^d}\,\frac{V^{\mu\nu}_{T\phi\phi}(l-q,-l-p)V^{\rho\sigma}_{T\phi\phi}(l,-l+q)V^{\alpha\beta}_{T\phi\phi}(l+p,-l)}{l^2\,(l-q)^2\,(l+p)^2}
\tag{E.1}
$$

The vertices are defined in eq. (D.1). The first argument in each vertex denotes the momentum of the incoming particle, the second argument is the momentum of the outgoing one. A typical term appearing in the loop integral is then

$$I \equiv \int \frac{d^d l}{(2\pi)^d} \frac{l^\mu\, l^\nu\, (l+p)^\rho\, (l+p)^\sigma (l-q)^\alpha\, (l-q)^\beta}{l^2\, (l-q)^2\, (l+p)^2}\,. \tag{E.2}$$

From (8.1) the propagators in configuration space are

$$\frac{1}{l^2\,(l-q)^2\,(l+p)^2} = C(1)^3 \int d^d x_{12}\, d^d x_{23}\, d^d x_{31}\, \frac{e^{i\,[l\cdot x_{23}+(l-q)\cdot x_{12}+(l+p)\cdot x_{31}]}}{(x_{12}^2)^{d/2-1}\,(x_{23}^2)^{d/2-1}\,(x_{31}^2)^{d/2-1}}\,, \tag{E.3}$$

where $C(\alpha)$ has been defined in (8.1). It is straightforward to see that (E.2) is given by

$$\begin{aligned}&\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu\, l^\nu\, (l+p)^\rho\, (l+p)^\sigma (l-q)^\alpha\, (l-q)^\beta}{l^2\, (l-q)^2\, (l+p)^2} = \\ &\quad C(1)^3 \int \frac{d^d l}{(2\pi)^d}\, d^d x_{12}\, d^d x_{23}\, d^d x_{31}\, \frac{(-i)^6\, \partial^\mu_{23}\, \partial^\nu_{23}\, \partial^\rho_{31}\, \partial^\sigma_{31}\, \partial^\alpha_{12}\, \partial^\beta_{12}\, e^{i\,[l\cdot x_{23}+(l-q)\cdot x_{12}+(l+p)\cdot x_{31}]}}{(x_{12}^2)^{d/2-1}\,(x_{23}^2)^{d/2-1}\,(x_{31}^2)^{d/2-1}}\,.\end{aligned} \tag{E.4}$$

We can now integrate by parts moving the derivatives onto the propagators, getting

$$\begin{aligned}I = C(1)^3 &\int \frac{d^d l}{(2\pi)^d}\, d^d x_{12}\, d^d x_{23}\, d^d x_{31}\, e^{i\,[l\cdot x_{23}+(l-q)\cdot x_{12}+(l+p)\cdot x_{31}]} \\ &\times i^6\, \partial^\mu_{23}\, \partial^\nu_{23}\, \partial^\rho_{31}\, \partial^\sigma_{31}\, \partial^\alpha_{12}\, \partial^\beta_{12}\, \frac{1}{(x_{12}^2)^{d/2-1}\,(x_{23}^2)^{d/2-1}\,(x_{31}^2)^{d/2-1}}\,.\end{aligned} \tag{E.5}$$

The second line is immediately identified with the coordinate space Green's function.

This can be done for each term of (E.1), justifying the rule quoted in section 4.2, that we have used for all the inverse mappings of the paper. According to this the correlators in coordinate space can be obtained replacing the momenta in the vertices with "i" times the respective derivative which then act directly on the propagators after a partial integration.

The same arguments could be applied to the bubbles. Nevertheless, we have seen in 4.2 that derivatives of delta functions appear in the scalar case. These are generated by the dependence of the $V^{\mu\nu\rho\sigma}_{TT\phi\phi}(p,q,l)$ from the momentum l of the graviton bringing the pair of indices $\rho\sigma$ (see eq. (D.1)). They are due to coupling of the scalar with derivatives of the metric through the Ricci scalar R in the improvement term (see eq. (5.1)) and state that the graviton feels the metric gradient. We discuss this below, showing how to inverse-map the third bubble in figure 2, getting (4.11).

This bubble can be seen as the $(x_2 \to x_3)$ limit of the triangle and its diagrammatic momentum-space expression at one-loop is

$$\int \frac{d^d l}{(2\pi)^d} \frac{V^{\mu\nu}_{T\phi\phi}(l-q,-l-p)V^{\alpha\beta\rho\sigma}_{TT\phi\phi}(l+p,-l+q,-q)}{(l-q)^2\,(l+p)^2}\,. \tag{E.6}$$

As the two propagators are expressed by

$$\frac{1}{(l+q)^2\,(l+p)^2} = C(1)^2 \int d^d x_{12}\, d^d x_{31}\, \frac{e^{i\,[(l-q)\cdot x_{12}+(l+p)\cdot x_{31}]}}{(x_{12}^2)^{d/2-1}\,(x_{31}^2)^{d/2-1}}\,, \tag{E.7}$$

the dependence of the second vertex on p cannot be ascribed to neither of them.

Two typical terms encountered in (E.6) are

$$\int \frac{d^dl}{(2\pi)^d}\,\frac{(l+p)^\rho\,(l+p)^\sigma(l-q)^\alpha\,(l-q)^\beta}{(l-q)^2\,(l+p)^2}\,,$$
$$\int \frac{d^dl}{(2\pi)^d}\,\frac{(l+p)^\rho\,(l+p)^\sigma(l-q)^\alpha\,p^\beta}{(l-q)^2\,(l+p)^2}\,. \tag{E.8}$$

The first one is treated at once restricting the procedure used for the three point function to the case of two propagators.

For the second one, the following relation is immediately checked:

$$\int \frac{d^dl}{(2\pi)^d}\frac{(l+p)^\rho(l+p)^\sigma(l-q)^\alpha p^\beta}{(l-q)^2(l+p)^2} =$$
$$C(1)^2\int\frac{d^dl}{(2\pi)^d}d^dx_{12}d^dx_{23}d^dx_{31}\delta^{(d)}(x_{23})\frac{(-i)^4\partial^\rho_{31}\partial^\sigma_{31}\partial^\alpha_{12}(\partial_{31}-\partial_{23})^\beta e^{i[l\cdot x_{23}+(l-q)\cdot x_{12}+(l+p)\cdot x_{31}]}}{(x_{12}^2)^{d/2-1}(x_{31}^2)^{d/2-1}}. \tag{E.9}$$

Notice that an integration by parts brings in a derivative on the delta functions giving

$$C(1)^2\int\frac{d^dl}{(2\pi)^d}d^dx_{12}d^dx_{23}d^dx_{31}e^{i[l\cdot x_{23}+(l-q)\cdot x_{12}+(l+p)\cdot x_{31}]}$$
$$(i)^4\partial^\rho_{31}\partial^\sigma_{31}\partial^\alpha_{12}(\partial_{31}-\partial_{23})^\beta\frac{\delta^d(x_{23})}{(x_{12}^2)^{d/2-1}(x_{31}^2)^{d/2-1}}. \tag{E.10}$$

This approach has been followed in all the derivations of the expressions given in (4.2).

The integration on l brings about a $\delta^{(d)}(x_{12}+x_{23}+x_{31})$, so that it is natural to chose the parameterization

$$x_{12}=x_1-x_2\,,\quad x_{23}=x_2-x_3\,,\quad x_{31}=x_3-x_1\,. \tag{E.11}$$

A more inviolved example is the 4-particle vertex. For instance the $V_{TT\phi\phi}(i\,\partial_{31},-i\,\partial_{12},i\,(\partial_{12}-\partial_{23}))$ is obtained from $V_{TT\phi\phi}(p,q,l)$ with the functional replacements

$$p\to\hat p=i\,\partial_{31},\qquad q\to\hat q=-i\,\partial_{12}\qquad l\to\hat l=i\,(\partial_{12}-\partial_{23}) \tag{E.12}$$

giving

$$\begin{aligned}
&V^{\mu\nu\rho\sigma}_{TT\phi\phi}(i\,\partial_{31},-i\,\partial_{12},i\,(\partial_{12}-\partial_{23}))=\\
&\frac{1}{2}\,i\,\partial_{31}\cdot(-i)\,\partial_{12}s^{\mu\nu\rho\sigma}-\frac{1}{4}\,G^{\mu\nu\rho\sigma}(i\,\partial_{31},-i\,\partial_{12})+\frac{1}{4}\,\delta^{\rho\sigma}\,i\,\partial_{31\,\alpha}\,(-i)\,\partial_{12\,\beta}\,C^{\mu\nu\alpha\beta}\\
&+\chi\left\{\left[\left(\delta^{\mu\lambda}\,\delta^{\alpha\kappa}\,\delta^{\nu\beta}+\delta^{\mu\alpha}\,\delta^{\nu\kappa}\,\delta^{\beta\lambda}-\delta^{\mu\kappa}\,\delta^{\nu\lambda}\,\delta^{\alpha\beta}-\delta^{\mu\nu}\,\delta^{\alpha\lambda}\,\delta^{\beta\kappa}\right)s^{\rho\sigma}_{\lambda\kappa}\right.\right.\\
&\left.+\frac{1}{2}\delta^{\rho\sigma}\left(\delta^{\mu\alpha}\delta^{\nu\beta}-\delta^{\mu\nu}\delta^{\alpha\beta}\right)\right](i\partial_{31\alpha}(-i)\partial_{12\beta}+i\partial_{31\beta}(-i)\partial_{12\alpha}+i\partial_{31\alpha}i\partial_{31\beta}+(-i)\partial_{12\alpha}(-i)\partial_{12\beta})\\
&-\left[\left(\delta^{\mu\alpha}\,\delta^{\nu\beta}-\delta^{\mu\nu}\,\delta^{\alpha\beta}\right)\left(\left[\Gamma^\lambda_{\alpha\beta}\right]^{\rho\sigma}\,(i\,(\partial_{12}-\partial_{23}))\right)(-i)\,\left(i\,\partial_{31\,\lambda}+\,(-i)\,\partial_{12\,\lambda}\right)\right.\\
&\left.\left.+\frac{1}{2}\left(\delta^{\mu\alpha}\,\delta^{\nu\beta}-\frac{1}{2}\,\delta^{\mu\nu}\,\delta^{\alpha\beta}\right)\left(\left[R_{\alpha\beta}\right]^{\rho\sigma}\,(i\,(\partial_{12}-\partial_{23}))\right)\right]\right\}.
\end{aligned} \tag{E.13}$$

F Regularizations and distributional identities

We add few more comments and examples which illustrate the regularization that we have applied in the computation of the various correlators.

The computation of the logarithmic integrals requires some care due to the distributional nature of some of these formulas. As an example we consider the integrals

$$H_1 = \int d^d l \, e^{il\cdot x} \, \frac{\mu^{2\omega}}{[l^2]^{1+\omega}} \qquad H_2 = \int d^d l \, e^{il\cdot x} \, \frac{\mu^{2\omega}}{[l^2]^{\omega}} \qquad H_3 = \int d^d l \, e^{il\cdot x} \, \log\left(\frac{l^2}{\mu^2}\right) \tag{F.1}$$

We can relate them in the form

$$H_3 = -\frac{\partial}{\partial\omega} H_2\bigg|_{\omega=0} = \Box\left(\frac{\partial}{\partial\omega} H_1\bigg|_{\omega=0}\right) \tag{F.2}$$

In the two cases we get, using (8.1)

$$-\frac{\partial}{\partial\omega} H_2\bigg|_{\omega=0} = -\frac{(4\,\pi)^{d/2}\,\Gamma(d/2)}{(x^2)^{d/2}} \tag{F.3}$$

and

$$\frac{\partial}{\partial\omega} H_1\bigg|_{\omega=0} = \frac{2^{d-2}\pi^{d/2}\Gamma(d/2-1)}{[x^2]^{d/2-1}}\left(\log(x^2\mu^2) + \gamma - \log 4 - \psi\left(\frac{d-2}{2}\right)\right) \tag{F.4}$$

By redefining the regularization scale μ with eq. (8.20) we clearly obtain from (F.4)

$$\int d^d l \frac{\log(l^2/\mu^2) e^{il\cdot x}}{l^2} = 2^{d-2}\pi^{d/2}\Gamma(d/2-1)\frac{\log x^2\bar{\mu}^2}{[x^2]^{d/2-1}} \tag{F.5}$$

and

$$H_3 = \Box\left(\frac{\partial}{\partial\omega} H_1\bigg|_{\omega=0}\right) = 2^{d-2}\pi^{d/2}\Gamma(d/2-1)\Box\left(\frac{\log x^2\bar{\mu}^2}{[x^2]^{d/2-1}}\right) \tag{F.6}$$

The use of H_2 instead gives

$$H_3 = -\frac{\partial}{\partial\omega} H_2\bigg|_{\omega=0} = -\frac{2^d\pi^{d/2}\Gamma(d/2)}{[x^2]^{d/2}} \tag{F.7}$$

Notice that this second relation coincides with (F.6) away from the point $x = 0$, but differs from it right on the singularity, since

$$\Box\frac{\log x^2\mu^2}{[x^2]^{d/2-1}} = -2\,(d-2)\left(\frac{\pi^{d/2}}{\Gamma(d/2)}\,\log(x^2\mu^2)\,\delta^d(x) + \frac{1}{[x^2]^{d/2}}\right) \tag{F.8}$$

For this reason we take (F.6) as the regularized expression of H_3, in agreement with the standard approach of differential regularization.

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F.1 Evaluation of the single log integrals

The direct method discussed in the second part of the paper, though very general and applicable to any correlator, introduces in momentum space some logarithmic integrals which are more difficult to handle. They take the role of the ordinary master integrals of perturbation theory. The scalar integrals needed for the tensor reduction of the logarithmic contributions in the text are defined in (F.17). After a shift of the momentum in the argument of the logarithm, a standard tensor reduction gives

$$\begin{aligned} IL_\mu(0,p_1,p_2) &= CL_1(p_1,p_2)\, p_{1\,\mu} + CL_2(p_1,p_2)\, p_{2\,\mu}\,, \\ CL_1(p_1,p_2) &= \frac{({p_1}^2 - p_1\cdot p_2){p_2}^2\, IL(0,p_1,p_2) + ({p_2}^2 - p_1\cdot p_2)\, IL^\mu{}_\mu(0,p_1,p_2)}{2\,(p_1\cdot p_2)^2 - {p_1}^2\,{p_2}^2}\,, \\ CL_2(p_1,p_2) &= \frac{({p_2}^2 - p_1\cdot p_2){p_1}^2\, IL(0,p_1,p_2) + ({p_1}^2 - p_1\cdot p_2)\, IL^\mu{}_\mu(0,p_1,p_2)}{2\,(p_1\cdot p_2)^2 - {p_1}^2\,{p_2}^2}\,. \end{aligned} \tag{F.9}$$

To complete the computation of the VVV correlator we need the explicit form of the logarithmic integrals in terms of ordinary logarithmic and polylogarithmic functions. We define

$$\mathcal{I} \equiv \int d^d l \frac{\log\left(l^2/\mu^2\right)}{(l+p_1)^2(l-p_2)^2} = -\frac{\partial}{\partial\lambda}\int d^d l\, \frac{\mu^{2\lambda}}{(l^2)^\lambda\,(l+p_1)^2\,(l-p_2)^2}_{\lambda=0}\,. \tag{F.10}$$

The logarithmic integral is identified from the term of $\mathrm{O}(\lambda)$ in the series expansion of the previous expression. Because the coefficient in front of the parametric integral starts at this order, we just need to know the zeroth order expansion of the integrand, which we separate into two terms. The first one is integrable

$$I_1 = \int_0^1 dt \frac{t^{-\epsilon}(yt)^{1-\epsilon-\lambda}}{A(t)^{1-\epsilon}} = \int_0^1 dt \frac{t^{-\epsilon}(yt)^{1-\epsilon}}{A(t)^{1-\epsilon}} + O(\lambda) \equiv I_1^{(0)} + O(\lambda)\,, \tag{F.11}$$

while the last term has a singularity in $t=0$ which must be factored out and re-expressed in terms of a pole in λ

$$\begin{aligned} I_2 &= -\int_0^1 dt \frac{t^{-\epsilon}(x/t)^{1-\epsilon-\lambda}}{A(t)^{1-\epsilon}} = -\frac{x^{1-\epsilon-\lambda}}{\lambda}\int_0^1 dt \frac{1}{A(t)^{1-\epsilon}}\frac{d}{dt}t^\lambda \\ &= -\frac{x^{1-\epsilon-\lambda}}{\lambda}\left[1-(\epsilon-1)\int_0^1 dt\frac{t^\lambda}{A(t)^{1-\epsilon}}\left(\frac{1}{t-t_1}+\frac{1}{t-t_2}\right)\right] \\ &= \frac{x^{1-\epsilon}}{\lambda}\left\{-1+(\epsilon-1)\int_0^1 dt\frac{1}{A(t)^{1-\epsilon}}\left(\frac{1}{t-t_1}+\frac{1}{t-t_2}\right)\right]\right\} \\ &\quad + x^{1-\epsilon}\left[\log x + (\epsilon-1)\int_0^1 dt \frac{\log\left(t/x\right)}{A(t)^{1-\epsilon}}\left(\frac{1}{t-t_1}+\frac{1}{t-t_2}\right)\right] + O(\lambda) \\ &\equiv \frac{1}{\lambda}I_2^{(-1)} + I_2^{(0)} + O(\lambda), \end{aligned} \tag{F.12}$$

where t_1 and t_2 are the two roots of $A(t) = yt^2 + (1-x-y)t + x$. We are now able to write down the full λ-expansion of $J(1,1,\lambda)$ and to extract the logarithmic integral $\mathcal{I}$

$$\mathcal{I} = -\frac{\pi^{2-\epsilon}i^{1+2\epsilon}}{(p_3^2)^\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{\epsilon-1}\left\{I_1^{(0)} + I_2^{(0)}\right\}. \tag{F.13}$$

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The previous expression can be expanded in $d = 4-2\epsilon$ dimensions in which it manifests a $1/\epsilon$ pole of ultraviolet origin

$$\mathcal{I} = \frac{\pi^{2-\epsilon} i^{1+2\epsilon}}{(p_3^2)^\epsilon} \left(-\frac{1}{\epsilon} + \gamma \right) \Big[A(x,y) + \epsilon\, B(x,y) \Big] + O(\epsilon) \,, \tag{F.14}$$

where $A(x,y)$ and $B(x,y)$ are defined from the ϵ-expansion of the two integrals $I_1^{(0)}$ and $I_2^{(0)}$ as

$$A(x,y) \;=\; x \log x + \int_0^1 \frac{dt}{A(t)} \left[yt - x \log\left(t/x\right) \left(\frac{1}{t-t_1} + \frac{1}{t-t_2} \right) \right] , \tag{F.15}$$

$$\begin{aligned} B(x,y) \;=\; & -x \log^2 x + \int_0^1 \frac{dt}{A(t)} \Bigg[yt \left(\log\left(t-t_1\right) + \log\left(t-t_2\right) - 2\log t \right) \\ & - x \log\left(t/x\right) \left(\frac{1}{t-t_1} + \frac{1}{t-t_2} \right) \left(\log\left(t-t_1\right) + \log\left(t-t_2\right) - \log\left(x/y\right) - 1 \right) \Bigg] . \end{aligned} \tag{F.16}$$

F.2 List of momentum space integrals

To set the stage for the explicit examples of three point functions treated in section 8, we introduce here a systematic short-hand notation to denote the momentum-space integrals. We define

$$\begin{aligned} I_{\mu_1,\dots,\mu_n}(p) \;&=\; \int d^d l \, \frac{l_{\mu_1} \dots l_{\mu_n}}{l^2 \, (l+p)^2} \,, \\ J_{\mu_1,\dots,\mu_n}(p_1,p_2) \;&=\; \int d^d l \, \frac{l_{\mu_1} \dots l_{\mu_n}}{l^2 \, (l+p_1)^2 (l+p_2)^2} \,, \\ IL_{\mu_1\dots\mu_n}(p_1,p_2,p_3) \;&=\; \int d^d l \, \frac{l_{\mu_1} \dots l_{\mu_n} \, \log\left((l+p_1)^2/\mu^2\right)}{(l+p_2)^2 (l+p_3)^2} \,, \\ ILL_{\mu_1\dots\mu_n}(p_1,p_2,p_3,p_4) \;&=\; \int d^d l \frac{l_{\mu_1} \dots l_{\mu_n} \, \log\left((l+p_1)^2/\mu^2\right) \log\left((l+p_2)^2/\mu^2\right)}{(l+p_3)^2 (l+p_4)^2} \,. \end{aligned} \tag{F.17}$$

For correlators which are finite, the double logarithmic contributions will appear in combinations that can be re-expressed in terms of ordinary Feynman integrals.

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