Correlators of stress energy tensors in D=4 CFT's from Solutions of the Conformal Ward identities in Momentum Space

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It has been shown in the past that a nonlocal (Riegert) action reproduces perturbative results related to the trace anomaly and the breaking of conformal symmetry in D=4 conformal field theories.
The issue whether such an action is consistent or not has been widely debated in the literature.
I report on recent progress in the study of this problem using a reconstruction method that builds correlators of 3 point functions by solving the conformal constraints in momentum space

with special thanks to Prof. **Dino Vaira** (Univ. di Bologna)

The analysis of Conformal Field Theories in D>2 dimensions finds enormous applications in contemporary physics

ADS-CFT
Physics the Early Universe in its De Sitter phase, just to mention a few.

An important role is played by the conformal anomaly, which appears in even spacetime dimensions.

Beside the gravitational domain, the trace anomaly may have a role to play in the search for physics beyond the Standard Model, being associated with the appearance of a DILATON and a possible Higgs/dilaton mixing, where the dilaton appears as a dynamical state.

But one of the toughest issues to investigate concerns the breaking of this simmetry.
A non-conformal phase (spontaneously broken) phase needs an "explicit" breaking, since we cannot introduce dimensionfull constants in a SM-like Lagrangian. The view is still quite foggy.

One possibility is to consider the anomalous breaking of this symmetry, where the anomaly is introduced by renormalization.

previous studies at U. of Salento

Graviton Vertices and the Mapping of Anomalous Correlators to Momentum Space for a General Conformal Field Theory

Delle Rose, Serino, Mottola, C.C

Dilaton Interactions and the Anomalous Breaking of Scale Invariance of the Standard Model

Delle Rose and Serino, C.C.

Three and Four Point Functions of Stress Energy Tensors in $D = 3$ for the Analysis of Cosmological Non-Gaussianities

Delle Rose, Serino, C.C.

Solving the Conformal Constraints for Scalar Operators in Momentum Space and the Evaluation of Feynman's Master Integrals

Delle Rose, Serino, C.C.

Conformal anomalies and the gravitational effective action: The TJJ correlator for a Dirac fermion

Armillis, Delle Rose, C.C.

$$T^{\mu}_{\mu} = -\frac{1}{8}\left[2b\,C^2 + 2b'\left(E - \frac{2}{3}\Box R\right) + 2c\,F^2\right]$$

where b, b' and c are parameters that for a single fermion in the theory result $b = 1/320\,\pi^2$, $b' = -11/5760\,\pi^2$, and $c = -e^2/24\,\pi^2$; furthermore C^2 denotes the Weyl tensor squared and E is the Euler density given by

$$\begin{aligned} C^2 &= C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3} \\ E &= {}^*\!R_{\lambda\mu\nu\rho}\,{}^*\!R^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \end{aligned}$$

The effective action is identified by solving the following variational equation by inspection

$$-\frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta\Gamma}{\delta g_{\mu\nu}} = T^{\mu}_{\mu}.$$

$$S_{anom}[g,A] = \frac{1}{8}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\left(E-\frac{2}{3}\Box R\right)_x \Delta_4^{-1}(x,x')\left[2b\,F+b'\left(E-\frac{2}{3}\Box R\right)+2\,c\,F_{\mu\nu}F^{\mu\nu}\right]_{x'}$$

Obtained by Riegert long ago, the action was derived by solving the variational equation satisfied by the trace of the energy momentum tensor. $\Delta_4^{-1}(x,x')$ denotes the Green's function inverse of the conformally covariant differential operator of fourth order, defined by

$$\Delta_4 \equiv \nabla_\mu\left(\nabla^\mu\nabla^\nu+2R^{\mu\nu}-\frac{2}{3}Rg^{\mu\nu}\right)\nabla_\nu = \Box^2+2R^{\mu\nu}\nabla_\mu\nabla_\nu+\frac{1}{3}(\nabla^\mu R)\nabla_\mu-\frac{2}{3}R\Box\,.$$

Expanding around flat space, the local formulation of Riegert's action,

$$S_{anom}[g,A] \to -\frac{c}{6}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,R_x\,\Box^{-1}_{x,x'}\,[F_{\alpha\beta}F^{\alpha\beta}]_{x'}\,,$$

Mottola, Vaulin
Giannotti Mottola

We consider the standard QED lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\,\bar{\psi}\gamma^\mu(\partial_\mu - i\,e\,A_\mu)\psi - m\bar{\psi}\psi,$$

$$T_f^{\mu\nu} = -i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi + g^{\mu\nu}(i\bar{\psi}\gamma^\lambda\overleftrightarrow{\partial}_\lambda\psi - m\bar{\psi}\psi),$$

$$T_{fp}^{\mu\nu} = -\,eJ^{(\mu}A^{\nu)} + eg^{\mu\nu}J^\lambda A_\lambda\,,$$

In the coupling to gravity of the total energy momentum tensor

$$T^{\mu\nu} \equiv T_f^{\mu\nu} + T_{fp}^{\mu\nu} + T_{ph}^{\mu\nu}$$

$$T_{ph}^{\mu\nu} = F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho},$$

which satisfies the inhomogeneous equation

partial T_p

$$T_p^{\mu\nu} \equiv T_f^{\mu\nu} + T_{fp}^{\mu\nu}$$

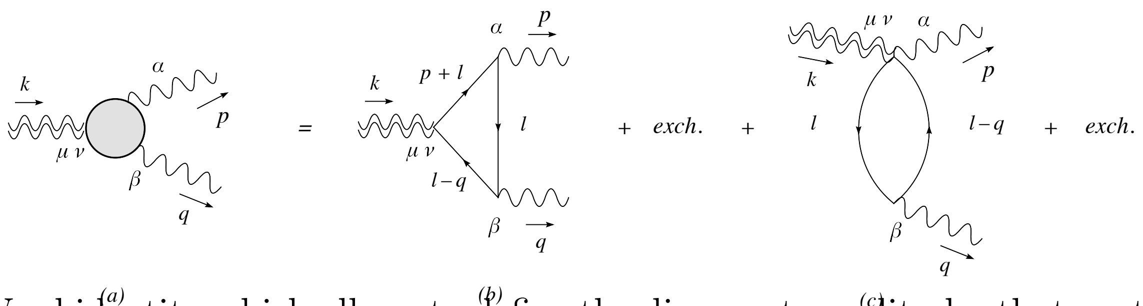
$$\partial_\nu T_p^{\mu\nu} = -\partial_\nu T_{ph}^{\mu\nu}.$$

$$\begin{aligned} \langle T_p^{\mu\nu}(z)\rangle_A &\equiv \int D\psi D\bar{\psi}\; T_p^{\mu\nu}(z)\; e^{i\int d^4x\mathcal{L} + \int J\cdot A(x)d^4x} \\ &= \langle T_p^{\mu\nu}\, e^{i\int d^4x\, J\cdot A(x)}\rangle \end{aligned}$$

$$\Gamma^{\mu\nu\alpha\beta}(z;x,y) \equiv \frac{\delta^2 \langle T_p^{\mu\nu}(z)\rangle_A}{\delta A_\alpha(x)\delta A_\beta(y)}\bigg|_{A=0} = V^{\mu\nu\alpha\beta} + W^{\mu\nu\alpha\beta}$$

$$\begin{aligned} W^{\mu\nu\alpha\beta} &= \frac{\delta^2 \langle T_{fp}^{\mu\nu}(z)(J\cdot A)\rangle}{\delta A_\alpha(x)\delta A_\beta(y)}\bigg|_{A=0} \\ &= \delta^4(x-z)g^{\alpha(\mu}\Pi^{\nu)\beta}(z,y) + \delta^4(y-z)g^{\beta(\mu}\Pi^{\nu)\alpha}(z,x) - g^{\mu\nu}[\delta^4(x-z) - \delta^4(y-z)]\Pi^{\alpha\beta}(x,y) \end{aligned}$$

$$V^{\mu\nu\alpha\beta} = (i\,e)^2 \, \langle T_f^{\mu\nu}(z)J^\alpha(x)J^\beta(y)\rangle_{A=0}$$



The bare Ward identity which allows to define the divergent amplitudes that contribute to the anomaly in Γ in terms of the remaining finite ones is obtained by re-expressing the classical equation

$$\partial_\nu T_{ph}^{\mu\nu} = -F^{\mu\nu} J_\nu$$

as an equation of generating functionals in the background electromagnetic field

$$\partial_\nu \langle T_{ph}^{\mu\nu}\rangle_A = -F^{\mu\nu}\langle J_\nu\rangle_A,$$

which can be expanded perturbatively as

$$\partial_\nu \langle T^{\mu\nu}_{ph} \rangle_A = -F^{\mu\nu} \langle J_\nu \int d^4w (ie) J \cdot A(w) \rangle_+ \ldots .$$

$$\partial_\nu \Gamma^{\mu\nu\alpha\beta} = \frac{\delta^2 \left(F^{\mu\lambda}(z) \langle J_\lambda(z) \rangle_A \right)}{\delta A_\alpha(x) \delta A_\beta(y)} \Bigg|_{A=0}$$

$$\begin{aligned} k_\nu \, \Gamma^{\mu\nu\alpha\beta}(p,q) \quad = \quad & \left(q^\mu p^\alpha p^\beta - q^\mu g^{\alpha\beta} p^2 + g^{\mu\beta} q^\alpha p^2 - g^{\mu\beta} p^\alpha p \cdot q \right) \Pi(p^2) \\ & + \left(p^\mu q^\alpha q^\beta - p^\mu g^{\alpha\beta} q^2 + g^{\mu\alpha} p^\beta q^2 - g^{\mu\alpha} q^\beta p \cdot q \right) \Pi(q^2) \, . \end{aligned}$$

There is, however, a more general way to proceed diffeomorphism invariance

$$\delta g^{\mu\nu} = -(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu),$$
$$\delta A^a_\mu = \xi^\nu \nabla_\nu A^a_\mu + \nabla_\mu \xi^\nu A^a_\nu,$$
$$\delta \phi^I_0 = \xi^\nu \partial_\nu \phi^I_0,$$

$$Z[\phi^I_0, A^a_\mu, g^{\mu\nu}] = \int \mathcal{D}\Phi \exp\left(-S_{CFT}[A^a_\mu, g^{\mu\nu}] - \sum_j \int d^d x \sqrt{g}\, \phi^{(j)}_0 \mathcal{O}_j\right).$$

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g(x)}} \frac{\delta}{\delta g^{\mu\nu}(x)} Z,$$
$$\langle J^{\mu a}(x)\rangle = -\frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta A^a_\mu(x)} Z,$$
$$\langle \mathcal{O}_j(x)\rangle = -\frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta \phi^{(j)}_0(x)} Z.$$

gauge invariance

$$\delta g^{\mu\nu} = 0,$$
$$\delta A^a_\mu = -D^{ac}_\mu \alpha^c = -\partial_\mu \alpha^a - f^{abc} A^b_\mu \alpha^c,$$
$$\delta \phi^I_0 = -i\alpha^a (T^a_R)^{IJ} \phi^J_0,$$

$$\delta_g Z[g_{\mu\nu}, A_\mu, \phi^{(j)}_0] = 0,$$

we get the first Ward Identity related to the gauge symmetry, expressed as

$$0 = D^{ab}_\mu \langle J^{\mu a}\rangle + \Gamma^\lambda_{\mu\lambda} \langle J^{\mu a}(x)\rangle - i(T^a_R)^{IJ} \phi^J_0 \langle \mathcal{O}_I\rangle$$
$$= \nabla_\mu \langle J^{\mu a}\rangle + f^{abc} A^b_\mu \langle J^{\mu c}\rangle - i(T^a_R)^{IJ} \phi^J_0 \langle \mathcal{O}_I\rangle$$

$$\delta_\xi = \int d^d x \left[-(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu)\frac{\delta}{\delta g^{\mu\nu}} + (\xi^\nu \nabla_\nu A^a_\mu + \nabla_\mu \xi^\nu A^a_\nu)\frac{\delta}{\delta A^a_\mu} + \xi^\nu \partial_\nu \phi^I_0 \frac{\delta}{\delta \phi^I_0}\right],$$
$$\delta_\alpha = -\int d^d x \left[(\partial_\mu \alpha^a - f^{abc} A^b_\mu \alpha^c)\frac{\delta}{\delta A^a_\mu} + i\alpha^a (T^a_R)^{IJ} \phi^J_0 \frac{\delta}{\delta \phi^I_0}\right],$$

conservation WI

$$\nabla^\mu \langle T_{\mu\nu}\rangle - F^a_{\mu\nu} \langle J^{\mu a}\rangle + D^{IJ}_\nu \phi^J_0 \cdot \langle \mathcal{O}^I\rangle . = 0$$

gauge WI

$$0 = D^{ac}_\mu \langle J^{\mu a}\rangle - \mathrm{i}(T^a_R)^{IJ} \phi^J_0 \langle \mathcal{O}^I\rangle$$
$$= \nabla_\mu \langle J^{\mu a}\rangle + f^{abc} A^b_\mu \langle J^{\mu c}\rangle - \mathrm{i}(T^a_R)^{IJ} \phi^J_0 \langle \mathcal{O}^I\rangle,$$

$$V^{\mu\nu\alpha\beta}(p,q) = -(-ie)^2 i^3 \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{tr}\{V'^{\mu\nu}(l+p, l-q)(\not{l} - \not{q} + m)\gamma^{\beta}(\not{l} + m)\gamma^{\alpha}(\not{l} + \not{p} + m)\}}{[l^2 - m^2][(l-q)^2 - m^2][(l+p)^2 - m^2]},$$

$$W^{\mu\nu\alpha\beta}(p,q) = -(ie^2)i^2 \int \frac{d^4 l}{(2\pi)^4} \times \frac{\mathrm{tr}\{W'^{\mu\nu\alpha}(\not{l} + m)\gamma^{\beta}(\not{l} - \not{q} + m)\}}{[l^2 - m^2][(l-q)^2 - m^2]},$$

TABLE I. The 43 tensor monomials named $l_i^{\mu\nu\alpha\beta}(p,q)$ in Eq. (43) built up from the metric tensor and the two independent momenta p and q into which a general fourth rank tensor can be expanded.

$p^\mu p^\nu p^\alpha p^\beta$	$p^\mu p^\nu p^\alpha q^\beta$	$p^\mu p^\nu q^\alpha q^\beta$	$p^\mu q^\nu q^\alpha p^\beta$	$p^\mu q^\nu q^\alpha q^\beta$	$g^{\mu\nu} g^{\alpha\beta}$
$q^\mu q^\nu q^\alpha q^\beta$	$p^\mu p^\nu q^\alpha p^\beta$	$p^\mu q^\nu p^\alpha q^\beta$	$q^\mu p^\nu q^\alpha p^\beta$	$q^\mu p^\nu q^\alpha q^\beta$	$g^{\alpha\mu} g^{\beta\nu}$
	$p^\mu q^\nu p^\alpha p^\beta$	$q^\mu p^\nu p^\alpha q^\beta$	$q^\mu q^\nu p^\alpha p^\beta$	$q^\mu q^\nu p^\alpha q^\beta$	$g^{\alpha\nu} g^{\beta\mu}$
	$q^\mu p^\nu p^\alpha p^\beta$			$q^\mu q^\nu q^\alpha p^\beta$	
$p^\mu p^\nu g^{\alpha\beta}$	$p^\beta p^\nu g^{\alpha\mu}$	$p^\beta p^\mu g^{\alpha\nu}$	$p^\alpha p^\nu g^{\beta\mu}$	$p^\mu p^\alpha g^{\beta\nu}$	$p^\alpha p^\beta g^{\mu\nu}$
$p^\mu q^\nu g^{\alpha\beta}$	$p^\beta q^\nu g^{\alpha\mu}$	$p^\beta q^\mu g^{\alpha\nu}$	$p^\alpha q^\nu g^{\beta\mu}$	$p^\mu q^\alpha g^{\beta\nu}$	$p^\alpha q^\beta g^{\mu\nu}$
$q^\mu p^\nu g^{\alpha\beta}$	$q^\beta p^\nu g^{\alpha\mu}$	$q^\beta p^\mu g^{\alpha\nu}$	$q^\alpha p^\nu g^{\beta\mu}$	$q^\mu p^\alpha g^{\beta\nu}$	$q^\alpha p^\beta g^{\mu\nu}$
$q^\mu q^\nu g^{\alpha\beta}$	$q^\beta q^\nu g^{\alpha\mu}$	$q^\beta q^\mu g^{\alpha\nu}$	$q^\alpha q^\nu g^{\beta\mu}$	$q^\mu q^\alpha g^{\beta\nu}$	$q^\alpha q^\beta g^{\mu\nu}$

TABLE II. Basis of 13 fourth rank tensors satisfying the vector current conservation on the external lines with momenta p and q.

i	$t_i^{\mu\nu\alpha\beta}(p,q)$
1	$(k^2 g^{\mu\nu} - k^\mu k^\nu) u^{\alpha\beta}(p\cdot q)$
2	$(k^2 g^{\mu\nu} - k^\mu k^\nu) w^{\alpha\beta}(p\cdot q)$
3	$(p^2 g^{\mu\nu} - 4p^\mu p^\nu) u^{\alpha\beta}(p\cdot q)$
4	$(p^2 g^{\mu\nu} - 4p^\mu p^\nu) w^{\alpha\beta}(p\cdot q)$
5	$(q^2 g^{\mu\nu} - 4q^\mu q^\nu) u^{\alpha\beta}(p\cdot q)$
6	$(q^2 g^{\mu\nu} - 4q^\mu q^\nu) w^{\alpha\beta}(p\cdot q)$
7	$[p\cdot q g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)] u^{\alpha\beta}(p\cdot q)$
8	$[p\cdot q g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)] w^{\alpha\beta}(p\cdot q)$
9	$(p\cdot q p^\alpha - p^2 q^\alpha)[p^\beta(q^\mu p^\nu + p^\mu q^\nu) - p\cdot q(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu)]$
10	$(p\cdot q q^\beta - q^2 p^\beta)[q^\alpha(q^\mu p^\nu + p^\mu q^\nu) - p\cdot q(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu)]$
11	$(p\cdot q p^\alpha - p^2 q^\alpha)[2q^\beta q^\mu q^\nu - q^2(g^{\beta\nu}q^\mu + g^{\beta\mu}q^\nu)]$
12	$(p\cdot q q^\beta - q^2 p^\beta)[2p^\alpha p^\mu p^\nu - p^2(g^{\alpha\nu}p^\mu + g^{\alpha\mu}p^\nu)]$
13	$(p^\mu q^\nu + p^\nu q^\mu)g^{\alpha\beta} + p\cdot q(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu}) - g^{\mu\nu}u^{\alpha\beta} - (g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu)q^\alpha - (g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu)p^\beta$

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{13} F_i(s; s_1, s_2, m^2)\, t_i^{\mu\nu\alpha\beta}(p,q),$$

Giannotti and Mottola

Armillis, Delle Rose, C.C.

$$u^{\alpha\beta}(p,q) \equiv (p\cdot q) g^{\alpha\beta} - q^\alpha p^\beta,$$

$$F_1(s,0,0,0) = -\frac{e^2}{18\pi^2 s},$$

$$F_{13,R}(s,0,0,0) = -\frac{e^2}{144\pi^2}\left[12\log\left(-\frac{s}{\mu^2}\right) - 35\right],$$

$$F_3(s,0,0,0) = F_5(s,0,0,0) = -\frac{e^2}{144\pi^2 s},$$

$$F_7(s,0,0,0) = -4F_3(s,0,0,0),$$

We present here the expressions of the invariant amplitudes in the massless limit. We obtain

$$F_1 = F_{1\mathrm{pole}} + \tilde{F}_1$$

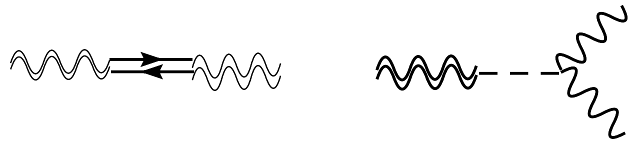
$$\underline{\underline{\mathbf{F_1(s;s_1,s_2,0)}}} = -\frac{e^2}{18\pi^2 s}, \tag{E1}$$

$$\underline{\underline{\mathbf{F_2(s;s_1,s_2,0)}}} = 0, \tag{E2}$$

$$\mathcal{S} = \mathcal{S}_{\mathrm{pole}} + \tilde{\mathcal{S}}.$$

$$\begin{aligned}
\underline{\underline{\mathbf{F_3(s;s_1,s_2,0)}}} = \underline{\underline{\mathbf{F_5(s;s_2,s_1,0)}}} \\
= -\frac{e^2}{144\pi^2 s\sigma^3}[s^6 - 3(s_1 - 4s_2)s^5 + 6(3s_1 - 7s_2)s_2 s^4 + 2(5s_1^3 - 69s_2 s_1^2 + 117s_2^2 s_1 + 23s_2^3)s^3 \\
- 3(5s_1^4 - 62s_2 s_1^3 + 72s_2^2 s_1^2 + 50s_2^3 s_1 + 7s_2^4)s^2 + 3(s_1 - s_2)^2(3s_1^3 - 24s_2 s_1^2 - 33s_2^2 s_1 + 2s_2^3)s \\
- 2(s_1 - s_2)^6] - \frac{e^2 s_1}{48\pi^2\sigma^4}\mathcal{D}_1(s,s_1,0)[(s - s_1)^6 + 2(14s + 11s_1)s_2(s - s_1)^4 - (23s^2 - 214s_1 s + 19s_1^2)s_2^2 \\
\times (s - s_1)^2 - 21s_2^6 + 2(5s_1 - 2s)s_2^5 + (107s^2 - 318s_1 s + 71s_1^2)s_2^4 + 8(-11s^3 + 18s_1 s^2 + 17s_1^2 s - 8s_1^3)s_2^3] \\
- \frac{e^2 s_2}{48\pi^2\sigma^4}\mathcal{D}_2(s,s_2,0)[s_2^6 - 2(s - 14s_1)s_2^5 + (s^2 + 120s_1 s - 37s_1^2)s_2^4 - 4(s^3 + 49s_1 s^2 - 69s_1^2 s + 13s_1^3)s_2^3 \\
+ (s - s_1)(11s^3 - 69s_1 s^2 + 309s_1^2 s - 83s_1^3)s_2^2 - 2(s - s_1)^3(5s^2 - 49s_1 s - 4s_1^2)s_2 + 3(s - s_1)^5(s + 5s_1)] \\
- \frac{e^2}{16\pi^2}\mathcal{C}_0(s,s_1,s_2,0)\Bigg[\frac{2s_1 s_2}{\sigma^4}[2s^6 + 3(s_2 - 3s_1)s^5 + (15s_1^2 + 6s_2 s_1 - 13s_2^2)s^4 + 2(-5s_1^3 - 19s_2 s_1^2 \\
+ 29s_2^2 s_1 + s_2^3)s^3 + 12s_2(4s_1^3 - 4s_2 s_1^2 - 3s_2^2 s_1 + s_2^3)s^2 + (s_1 - s_2)^2(3s_1^3 - 15s_2 s_1^2 - 31s_2^2 s_1 - 5s_2^3)s \\
- (s_1 - s_2)^4(s_1 + s_2)^2]\Bigg],
\end{aligned} \tag{E3}$$

$$\mathcal{S}_{\text{pole}} = -\frac{e^2}{36\pi^2}\int d^4x d^4y (\Box h(x) - \partial_\mu \partial_\nu h^{\mu\nu}(x)) \Box^{-1}_{xy} F_{\alpha\beta}(x) F^{\alpha\beta}(y).$$



(a) (b)

$$S_{anom}[g,A] = \frac{1}{8}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\left(E-\frac{2}{3}\Box R\right)_x \Delta_4^{-1}(x,x')\left[2b\,F + b'\left(E-\frac{2}{3}\Box R\right) + 2\,c\,F_{\mu\nu}F^{\mu\nu}\right]_{x'}$$

$$T^\mu_\mu = b\,F + b'\left(E - \frac{2}{3}\Box R\right) + b''\Box R + c\,F^{a\,\mu\nu}F^a_{\mu\nu},$$

A SIMILAR RESULT HOLDS FOR QCD

Armillis, Delle Rose, CC

The anomaly is mediated by the exchanged of 1 massless effective scalar state (an anomaly pole).

This feature is correctly described by the Riegert action.

Important point:

1. An anomaly action is not unique, and it is not supposed to reproduce the entire result of the perturbative computation, but just the anomaly part.

2. The double pole structure of the action appears to be inconsistent with the tenets of QFT.

There is no simple way out of this situation. At the moment we have only partial results.
Can we do something more, by going "non perturbative" ?

and in the Standard Model

Delle Rose, Serino, C.C.

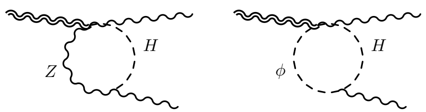
Gravity and the Neutral Currents: Effective Interactions from the Trace Anomaly

$$S = S_G + S_{SM} + S_I = -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\,R + \int d^4x\sqrt{-g}\mathcal{L}_{SM} + \frac{1}{6}\int d^4x\sqrt{-g}\,R\,\mathcal{H}^\dagger\mathcal{H}\,,$$

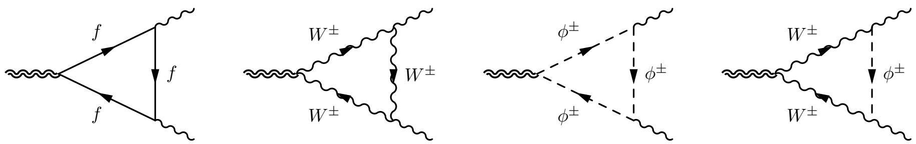
$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}}\frac{\delta[S_{SM}+S_I]}{\delta g^{\mu\nu}(x)},$$

$$\mathcal{L}_{grav}(x) = -\frac{\kappa}{2}T^{\mu\nu}(x)h_{\mu\nu}(x).$$

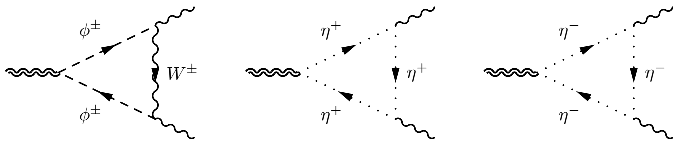
$$T^{Min}_{\mu\nu} = T^{f.s.}_{\mu\nu} + T^{ferm.}_{\mu\nu} + T^{Higgs}_{\mu\nu} + T^{Yukawa}_{\mu\nu} + T^{g.fix.}_{\mu\nu} + T^{ghost}_{\mu\nu}.$$



(a) (b)



(a) (b) (c) (d)



and many more

$$
\begin{aligned}
\Phi_{1\,F}(s,\,0,\,0,\,m_f^2) &= -i\frac{\kappa}{2}\,\frac{\alpha}{3\pi\,s}\,Q_f^2\Bigg\{-\frac{2}{3}+\frac{4\,m_f^2}{s}-2\,m_f^2\,\mathcal{C}_0(s,0,0,m_f^2,m_f^2,m_f^2)\bigg[1-\frac{4m_f^2}{s}\bigg]\Bigg\}\,,\\
\Phi_{2\,F}(s,\,0,\,0,\,m_f^2) &= -i\frac{\kappa}{2}\,\frac{\alpha}{3\pi\,s}\,Q_f^2\Bigg\{-\frac{1}{12}-\frac{m_f^2}{s}-\frac{3\,m_f^2}{s}\mathcal{D}_0(s,0,0,m_f^2,m_f^2)\\
&- m_f^2\mathcal{C}_0(s,0,0,m_f^2,m_f^2,m_f^2)\,\bigg[1+\frac{2\,m_f^2}{s}\bigg]\Bigg\}\,,\\
\Phi_{3\,F}(s,\,0,\,0,\,m_f^2) &= -i\frac{\kappa}{2}\,\frac{\alpha}{3\pi\,s}\,Q_f^2\Bigg\{\frac{11\,s}{12}+3m_f^2+\mathcal{D}_0(s,0,0,m_f^2,m_f^2)\,\big[5m_f^2+s\big]\\
&+ s\,\mathcal{B}_0(0,m_f^2,m_f^2)+3\,m_f^2\,\mathcal{C}_0(s,0,0,m_f^2,m_f^2,m_f^2)\,\big[s+2m_f^2\big]\Bigg\}\,.
\end{aligned}
$$

Only 1 form factor is responsible for the conformal anomaly.

Neat separation of breakings due to anomaly and explicit (mass) effects

$$
\begin{aligned}
\Phi_{1\,F}(s,\,0,\,0,\,m_f^2) &= -i\frac{\kappa}{2}\,\frac{\alpha}{3\pi\,s}\,Q_f^2\Bigg\{-\frac{2}{3}+\frac{4\,m_f^2}{s}-2\,m_f^2\,\mathcal{C}_0(s,0,0,m_f^2,m_f^2,m_f^2)\bigg[1-\frac{4m_f^2}{s}\bigg]\Bigg\}\,,\\
\Phi_{2\,F}(s,\,0,\,0,\,m_f^2) &= -i\frac{\kappa}{2}\,\frac{\alpha}{3\pi\,s}\,Q_f^2\Bigg\{-\frac{1}{12}-\frac{m_f^2}{s}-\frac{3\,m_f^2}{s}\mathcal{D}_0(s,0,0,m_f^2,m_f^2)\\
&- m_f^2\mathcal{C}_0(s,0,0,m_f^2,m_f^2,m_f^2)\,\bigg[1+\frac{2\,m_f^2}{s}\bigg]\Bigg\}\,,\\
\Phi_{3\,F}(s,\,0,\,0,\,m_f^2) &= -i\frac{\kappa}{2}\,\frac{\alpha}{3\pi\,s}\,Q_f^2\Bigg\{\frac{11\,s}{12}+3m_f^2+\mathcal{D}_0(s,0,0,m_f^2,m_f^2)\,\big[5m_f^2+s\big]\\
&+ s\,\mathcal{B}_0(0,m_f^2,m_f^2)+3\,m_f^2\,\mathcal{C}_0(s,0,0,m_f^2,m_f^2,m_f^2)\,\big[s+2m_f^2\big]\Bigg\}\,.
\end{aligned}
$$

Notice that as s→ infinity, mass corrections are subleading and we recover the anomalous breaking related to the 1/s.

this appears to be in agreement with Riegert's action, but what about the "double poles" predicted by this action ?

Supersymmetry. The case of the Ferrara Zumino supercurrent

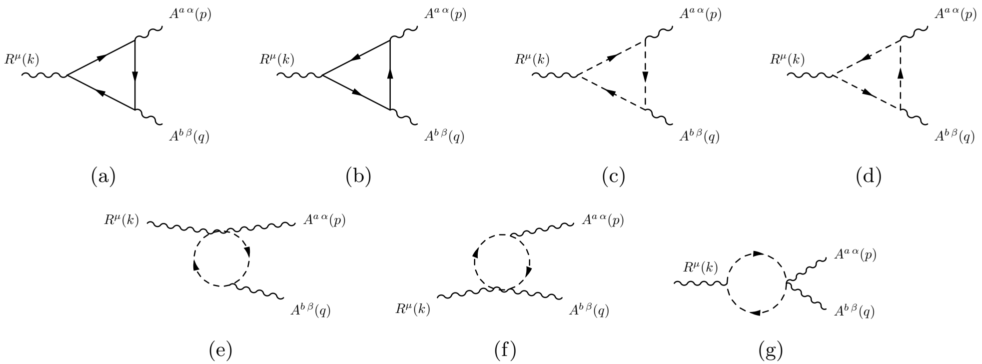
Superconformal Sum Rules and the Spectral Density Flow of the Composite Dilaton (ADD) Multiplet in $\mathcal{N}=1$ Theories Costantini, Delle Rose, Serino, C.C.,

N=1 SYM theory shares a similar behaviour. it is clearly universal

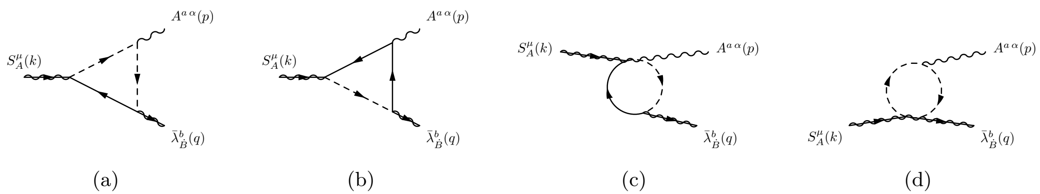
th supercurren combines a stress energy tensor, a chiral and a susy anomaly

$$
\begin{aligned}
R^\mu &= \bar\lambda^a\bar\sigma^\mu\lambda^a + \frac{1}{3}\left(-\bar\chi_i\bar\sigma^\mu\chi_i + 2i\phi_i^\dagger\mathcal{D}^\mu_{ij}\phi_j - 2i(\mathcal{D}^\mu_{ij}\phi_j)^\dagger\phi_i\right), \\
S^\mu_A &= i(\sigma^{\nu\rho}\sigma^\mu\bar\lambda^a)_A F^a_{\nu\rho} - \sqrt{2}(\sigma_\nu\bar\sigma^\mu\chi_i)_A(\mathcal{D}^\nu_{ij}\phi_j)^\dagger - i\sqrt{2}(\sigma^\mu\bar\chi_i)\mathcal{W}_i^\dagger(\phi^\dagger) \\
&- ig(\phi_i^\dagger T^a_{ij}\phi_j)(\sigma^\mu\bar\lambda^a)_A + S^\mu_{I\,A}, \\
T^{\mu\nu} &= -F^{a\,\mu\rho}F^{a\,\nu}{}_\rho + \frac{i}{4}\left[\bar\lambda^a\bar\sigma^\mu(\delta^{ac}\overrightarrow{\partial^\nu} - g\,t^{abc}A^{b\,\nu})\lambda^c + \bar\lambda^a\bar\sigma^\mu(-\delta^{ac}\overleftarrow{\partial^\nu} - g\,t^{abc}A^{b\,\nu})\lambda^c + (\mu\leftrightarrow\nu)\right] \\
&+ (\mathcal{D}^\mu_{ij}\phi_j)^\dagger(\mathcal{D}^\nu_{ik}\phi_k) + (\mathcal{D}^\nu_{ij}\phi_j)^\dagger(\mathcal{D}^\mu_{ik}\phi_k) + \frac{i}{4}\left[\bar\chi_i\bar\sigma^\mu(\delta_{ij}\overrightarrow{\partial^\nu} + igT^a_{ij}A^{a\,\nu})\chi_j\right. \\
&+ \left.\bar\chi_i\bar\sigma^\mu(-\delta_{ij}\overleftarrow{\partial^\nu} + igT^a_{ij}A^{a\,\nu})\chi_j + (\mu\leftrightarrow\nu)\right] - \eta^{\mu\nu}\mathcal{L} + T^{\mu\nu}_I,
\end{aligned}
$$

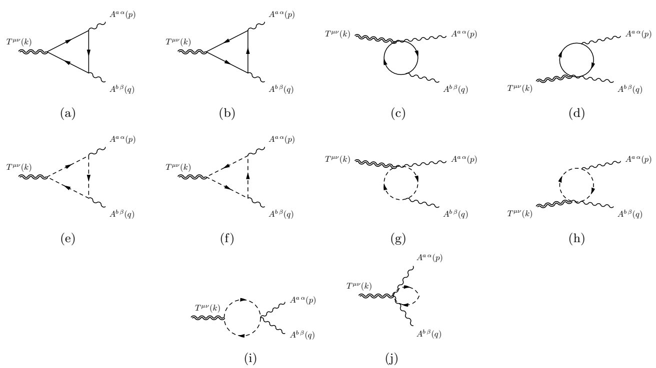
$$
\begin{aligned}
\partial_\mu R^\mu &= \frac{g^2}{16\pi^2}\left(T(A) - \frac{1}{3}T(R)\right)F^{a\,\mu\nu}\tilde F^a_{\mu\nu}, \\
\bar\sigma_\mu S^\mu_A &= -i\frac{3\,g^2}{8\pi^2}\left(T(A) - \frac{1}{3}T(R)\right)\left(\bar\lambda^a\bar\sigma^{\mu\nu}\right)_A F^a_{\mu\nu}, \\
\eta_{\mu\nu}T^{\mu\nu} &= -\frac{3\,g^2}{32\pi^2}\left(T(A) - \frac{1}{3}T(R)\right)F^{a\,\mu\nu}F^a_{\mu\nu}.
\end{aligned}
$$



R current and vector currents



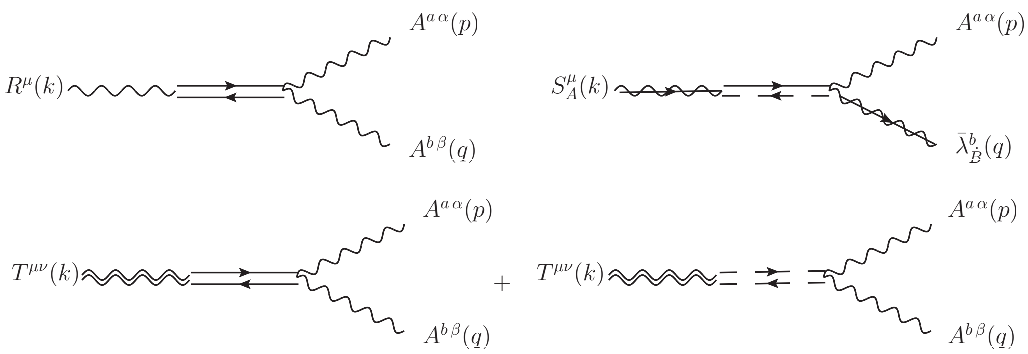
S current and vector currents



Similar behaviour

TVV

Delle Rose, Serino, CC

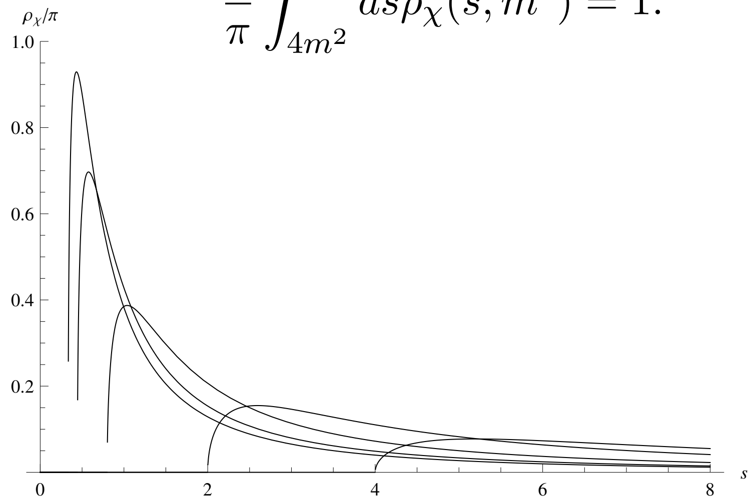


A nonlocal action
is responsible for this behaviour

$$\lim_{m\to 0}\rho_\chi(s,m^2)=\lim_{m\to 0}\frac{2\pi m^2}{s^2}\log\left(\frac{1+\sqrt{\tau(s,m^2)}}{1-\sqrt{\tau(s,m^2)}}\right)\theta(s-4m^2)=\pi\delta(s)$$

$$\frac{1}{\pi}\int_{4m^2}^\infty ds\rho_\chi(s,m^2)=1.$$

In each sector, only 1 form factor is responsible for the anomaly.



Dispersion relation for the anomaly for factor, away from the conformal limit. As m→0, the branch cut turns into a pole.

$$F(Q^2,m^2)=\frac{1}{\pi}\int_0^\infty ds\frac{\rho(s,m^2)}{s+Q^2}\,,$$

$$\lim_{Q^2\to\infty}Q^2F(Q^2,m^2)=f.$$

There are anomaly actions without any pole. **The dynamical dilaton**

Conformal Trace Relations from the Dilaton Wess-Zumino Action Delle Rose, Marzo, Serino,C.C

Weyl gauging

The Wess-Zumino action by the Noether method

$$\mathcal{J}_n \sim \frac{1}{\Lambda^{2(n-2)}} \int d^4x \sqrt{\hat{g}} \hat{R}^n,$$

$$\Gamma_0[\hat{g}] \sim \sum_n \mathcal{J}_n[\hat{g}].$$

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = g_{\mu\nu}\, e^{-2\,\Omega(x)},$$

where $\Omega(x) \equiv \tau(x)/\Lambda$ defines the local Weyl scaling of the background metric, with $\tau(x)$ being the dilaton field

The change in $g_{\mu\nu}$ due to a Weyl transformation is compensated by the shift of τ,

$$\Gamma_{WZ}[g,\tau] = \int d^4x\,\sqrt{g}\left\{\beta_a\left[\frac{\tau}{\Lambda}\left(F-\frac{2}{3}\Box R\right)+\frac{2}{\Lambda^2}\left(\frac{R}{3}\,\partial^\lambda\tau\,\partial_\lambda\tau+(\Box\tau)^2\right)-\frac{4}{\Lambda^3}\,\partial^\lambda\tau\,\partial_\lambda\tau\,\Box\tau+\frac{2}{\Lambda^4}\,\left(\partial^\lambda\tau\,\partial_\lambda\tau\right)^2\right]\right.$$
$$\left.+\beta_b\left[\frac{\tau}{\Lambda}\,G-\frac{4}{\Lambda^2}\left(R^{\alpha\beta}-\frac{R}{2}\,g^{\alpha\beta}\right)\partial_\alpha\tau\,\partial_\beta\tau-\frac{4}{\Lambda^3}\,\partial^\lambda\tau\,\partial_\lambda\tau\,\Box\tau+\frac{2}{\Lambda^4}\,\left(\partial^\lambda\tau\,\partial_\lambda\tau\right)^2\right]\right\}. \quad (49)$$

The dilaton is a dynamical field. No pole in the action, but an auxiliary degree of freedom is introduced.

How to resolve this issue?

Non perturbative solutions

Use conformal Ward Identities.
Move to the TTT and TTTT instead
of the TVV

We start from the scalar case (Delle Rose, Serino, C.C.)
Bzowski-McFadden-Skenderis (BMS)

The TTT discussed perturbatively in (Delle Rose, Serino, C.C.)

$$
\begin{aligned}
\langle T^{\mu_1\nu_1}(x_1)...T^{\mu_n\nu_n}(x_n)\rangle &= \left[\frac{2}{\sqrt{-g_{x_1}}}...\frac{2}{\sqrt{-g_{x_n}}}\,\frac{\delta^n \mathcal{W}}{\delta g_{\mu_1\nu_1}(x_1)...\delta g_{\mu_n\nu_n}(x_n)}\right]\Bigg|_{g_{\mu\nu}=\delta_{\mu\nu}} \\
&= 2^n\,\frac{\delta^n \mathcal{W}}{\delta g_{\mu_1\nu_1}(x_1)...\delta g_{\mu_n\nu_n}(x_n)}\Bigg|_{g_{\mu\nu}=\delta_{\mu\nu}},
\end{aligned}
$$

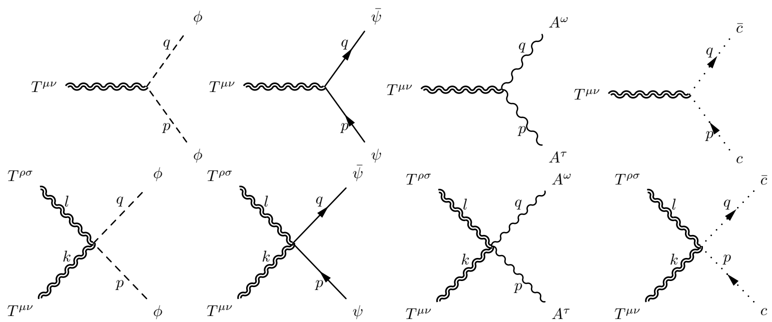
$$
\begin{aligned}
g_{\mu\nu}(z)\langle T^{\mu\nu}(z)\rangle &= \sum_{I=f,s,V} n_I\left[\beta_a(I)\,F(z)+\beta_b(I)\,G(z)+\beta_c(I)\,\Box R(z)+\beta_d(I)\,R^2(z)\right]+\frac{\kappa}{4}n_V F^{a\,\mu\nu}\,F^a_{\mu\nu}(z) \\
&\equiv \mathcal{A}(z,g)\,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_\nu\langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)T^{\alpha\beta}(x_3)\rangle &= \Big[\langle T^{\rho\sigma}(x_1)T^{\alpha\beta}(x_3)\rangle\partial^\mu\delta(x_1,x_2)+\langle T^{\alpha\beta}(x_1)T^{\rho\sigma}(x_2)\rangle\partial^\mu\delta(x_1,x_3)\Big] \\
&- \Big[\delta^{\mu\rho}\langle T^{\nu\sigma}(x_1)T^{\alpha\beta}(x_3)\rangle+\delta^{\mu\sigma}\langle T^{\nu\rho}(x_1)T^{\alpha\beta}(x_3)\rangle\Big]\partial_\nu\delta(x_1,x_2) \\
&- \Big[\delta^{\mu\alpha}\langle T^{\nu\beta}(x_1)T^{\rho\sigma}(x_2)\rangle+\delta^{\mu\beta}\langle T^{\nu\alpha}(x_1)T^{\rho\sigma}(x_2)\rangle\Big]\partial_\nu\delta(x_1,x_3)\,,
\end{aligned}
$$

Conservation WI

$$
\begin{aligned}
\delta_{\mu\nu}\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}\rangle(p,q) &= 4\,\mathcal{A}^{\alpha\beta\rho\sigma}(p,q)-2\,\langle T^{\alpha\beta}T^{\rho\sigma}\rangle(p)-2\,\langle T^{\rho\sigma}T^{\alpha\beta}\rangle(q) \\
&= 4\left[\beta_a\left(\left[F\right]^{\alpha\beta\rho\sigma}(p,q)-\frac{2}{3}\left[\sqrt{-g}\Box\,R\right]^{\alpha\beta\rho\sigma}(p,q)\right)+\beta_b\left[G\right]^{\alpha\beta\rho\sigma}(p,q)\right] \\
&- 2\,\langle T^{\alpha\beta}T^{\rho\sigma}\rangle(p)-2\,\langle T^{\rho\sigma}T^{\alpha\beta}\rangle(q)\,,
\end{aligned}
$$

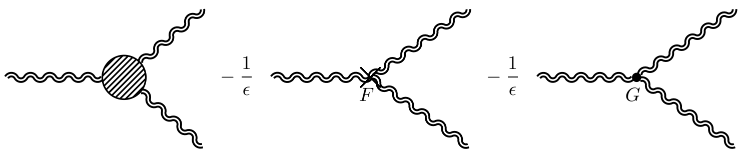
Anomalous (trace WI)



Delle Rose, Serino, C C

Renormalization of the TTT

$$S_{counter} = -\frac{1}{\epsilon} \sum_{I=f,s,V} n_I \int d^d x \sqrt{-g} \Big(\beta_a(I) F + \beta_b(I) G \Big)$$



i	$\Omega_i^S(s) \times 720\,\pi^2$	$\Omega_i^F(s) \times 240\,\pi^2$	$\Omega_i^V(s) \times 1152\,\pi^2$
1	$-\frac{1}{2\,s}$	$-\frac{1}{s}$	$\frac{72}{5\,s}$
2	$-\frac{1}{s}$	$-\frac{1}{3\,s}$	$\frac{64}{5\,s}$
3	$-\frac{7+30\,\mathcal{B}_0(s)}{120}$	$\frac{13-30\,\mathcal{B}_0(s)}{60}$	$\frac{82-120\,\mathcal{B}_0(s)}{25}$
4	$-\frac{2+5\,\mathcal{B}_0(s)}{10}$	$\frac{7-70\,\mathcal{B}_0(s)}{120}$	$\frac{2\,(482+130\,\mathcal{B}_0(s))}{25}$
5	$\frac{1}{6}$	$-\frac{-1+10\,\mathcal{B}_0(s)}{48}$	$-\frac{79+50\,\mathcal{B}_0(s)}{5}$
6	$\frac{23+20\,\mathcal{B}_0(s)}{20}$	$\frac{33+70\,\mathcal{B}_0(s)}{60}$	$-\frac{104\,(22+5\,\mathcal{B}_0(s))}{25}$
7	$-\frac{s\,(16+15\,\mathcal{B}_0(s))}{20}$	$-\frac{3\,s\,(2+5\,\mathcal{B}_0(s))}{10}$	$-\frac{s\,(-11+10\,\mathcal{B}_0(s))}{80}$
8	$-\frac{s\,(47+30\,\mathcal{B}_0(s))}{80}$	$-\frac{3\,s\,(9+10\,\mathcal{B}_0(s))}{40}$	$\frac{s\,(2+5\,\mathcal{B}_0(s))}{40}$
9	$\frac{s\,(2+5\,\mathcal{B}_0(s))}{40}$	$-\frac{7s\,(1-10\,\mathcal{B}_0(s))}{480}$	$-\frac{s\,(487+130\,\mathcal{B}_0(s))}{50}$
10	$\frac{s\,(9+10\,\mathcal{B}_0(s))}{20}$	$\frac{s\,(137+430\,\mathcal{B}_0(s))}{480}$	$-\frac{s\,(883-230\,\mathcal{B}_0(s))}{50}$
11	$-\frac{s\,(7+5\,\mathcal{B}_0(s))}{20}$	$-\frac{7\,s\,(9+10\,\mathcal{B}_0(s))}{240}$	$\frac{s\,(467+130\,\mathcal{B}_0(s))}{25}$
12	$-\frac{s\,(121+90\,\mathcal{B}_0(s))}{240}$	$-\frac{s\,(97+130\,\mathcal{B}_0(s))}{240}$	$\frac{2\,s\,(299+35\,\mathcal{B}_0(s))}{25}$
13	$\frac{5\,s^2\,(3+2\,\mathcal{B}_0(s))}{32}$	$\frac{5\,s^2\,(9+10\,\mathcal{B}_0(s))}{96}$	$-s^2\,(13-\mathcal{B}_0(s))$

$$
\begin{aligned}
t_1^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^\mu p^\nu + q^\mu q^\nu\right) p^\rho p^\sigma q^\alpha q^\beta \\
t_2^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^\mu q^\nu + p^\nu q^\mu\right) p^\rho p^\sigma q^\alpha q^\beta
\end{aligned}
$$

$$
\begin{aligned}
t_3^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^\mu p^\nu + q^\mu q^\nu\right)\left(p^\sigma q^\beta \delta^{\alpha\rho} + p^\sigma q^\alpha \delta^{\beta\rho} + p^\rho q^\beta \delta^{\alpha\sigma} + p^\rho q^\alpha \delta^{\beta\sigma}\right) \\
t_4^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= p^\rho p^\sigma \left(q^\beta q^\nu \delta^{\alpha\mu} + q^\beta q^\mu \delta^{\alpha\nu} + q^\alpha q^\nu \delta^{\beta\mu} + q^\alpha q^\mu \delta^{\beta\nu}\right) \\
&+ q^\alpha q^\beta \left(p^\nu p^\sigma \delta^{\mu\rho} + p^\nu p^\rho \delta^{\mu\sigma} + p^\mu p^\sigma \delta^{\nu\rho} + p^\mu p^\rho \delta^{\nu\sigma}\right) \\
t_5^{\mu\nu\alpha\beta\rho\sigma}(p,q) &= \left(p^\mu q^\nu + q^\mu p^\nu\right)\left(p^\rho\left(q^\alpha \delta^{\beta\sigma} + q^\beta \delta^{\alpha\sigma}\right) + p^\sigma\left(q^\alpha \delta^{\beta\rho} + q^\beta \delta^{\alpha\rho}\right)\right)
\end{aligned}
$$

and more ... (13)

1/s in one channel,
but double poles very difficult to disentangle in the perturbative approach. We need to reduce the number of form factors.

Conformal WI' s

scaling + sct

Solving the Conformal Constraints for Scalar Operators in Momentum Space and the Evaluation of Feynman's Master Integrals

Delle Rose, Serino, C.C.

Implications of conformal invariance in momentum space

Scalar Case

$$dx_\mu dx^\mu \to dx'_\mu dx'^\mu = \Omega(x)^{-2} dx_\mu dx^\mu \,.$$

$$\Omega(x) = 1 - \sigma(x) \qquad \text{and} \qquad \sigma(x) = \lambda - 2b \cdot x \,.$$

In the infinitesimal form, for $d > 2$, the conformal transformations are given by

$$x'_\mu(x) = x_\mu + a_\mu + \omega_\mu{}^\nu x_\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu b \cdot x$$

translations:	$L_g = a^\mu \partial_\mu \,,$
rotations:	$L_g = \frac{\omega^{\mu\nu}}{2} \left[x_\nu \partial_\mu - x_\mu \partial_\nu - \Sigma_{\mu\nu}\right] \,,$
scale transformations :	$L_g = \lambda \left[x \cdot \partial + \eta\right] \,,$
special conformal transformations. :	$L_g = b^\mu \left[x^2 \partial_\mu - 2x_\mu\, x \cdot \partial - 2\eta\, x_\mu - 2x_\nu \Sigma_\mu{}^\nu\right] \,.$

Conformal invariant correlation functions of quasi primary fields can be defined by requiring that

$$\sum_{r=1}^{n}\langle\mathcal{O}_1^{i_1}(x_1)\ldots\delta_g\mathcal{O}_r^{i_r}(x_r)\ldots\mathcal{O}_n^{i_n}(x_n)\rangle=0\,. \tag{8}$$

$$\sum_{r=1}^{n}\left(x_r\cdot\partial^{x_r}+\eta_r\right)\langle\mathcal{O}_1^{i_1}(x_1)\ldots\mathcal{O}_r^{i_r}(x_r)\ldots\mathcal{O}_n^{i_n}(x_n)\rangle=0\,,$$
$$\sum_{r=1}^{n}\left(x_r^2\partial_\mu^{x_r}-2x_{r\,\mu}\,x_r\cdot\partial^{x_r}-2\eta_r\,x_{r\,\mu}-2x_{r\,\nu}({\Sigma_\mu^{(r)}}^{\nu})^{i_r}_{j_r}\right)\langle\mathcal{O}_1^{i_1}(x_1)\ldots\mathcal{O}_r^{j_r}(x_r)\ldots\mathcal{O}_n^{i_n}(x_n)\rangle=0\,.$$

$$\left[-\sum_{r=1}^{n-1}\left(p_{r\,\mu}\,\frac{\partial}{\partial p_{r\,\mu}}+d\right)+\sum_{r=1}^{n}\eta_r\right]\langle\mathcal{O}_1^{i_1}(p_1)\ldots\mathcal{O}_r^{i_r}(p_r)\ldots\mathcal{O}_n^{i_n}(p_n)\rangle=0\,,$$

delle Rose, Serino, C C

$$\sum_{r=1}^{n-1}\left(p_{r\,\mu}\,\frac{\partial^2}{\partial p_r^\nu\partial p_{r\,\nu}}-2\,p_{r\,\nu}\,\frac{\partial^2}{\partial p_r^\mu\partial p_{r\,\nu}}+2(\eta_r-d)\frac{\partial}{\partial p_r^\mu}+2(\Sigma_{\mu\nu}^{(r)})^{i_r}_{j_r}\frac{\partial}{\partial p_{r\,\nu}}\right)$$
$$\times\langle\mathcal{O}_1^{i_1}(p_1)\ldots\mathcal{O}_r^{j_r}(p_r)\ldots\mathcal{O}_n^{i_n}(p_n)\rangle=0\,,$$

BMS

$$G_{123}(p_1,p_2)=\langle\mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(-p_1-p_2)\rangle\,.$$

$$G_{123}(p_1^2,p_2^2,p_3^2)=(p_3^2)^{-d+\frac{1}{2}(\eta_1+\eta_2+\eta_3)}\,\Phi(x,y)\qquad \text{with}\qquad x=\frac{p_1^2}{p_3^2}\,,\quad y=\frac{p_2^2}{p_3^2}\,,$$

We observe that scale invariance, the first equation in Eq.(11), implies that G_{123} is a homogeneous function of degree $\alpha=-d+\frac{1}{2}(\eta_1+\eta_2+\eta_3)$. Therefore it can be written in the form

$$G_{123}(p_1^2,p_2^2,p_3^2)=(p_3^2)^{-d+\frac{1}{2}(\eta_1+\eta_2+\eta_3)}\,\Phi(x,y)\qquad \text{with}\qquad x=\frac{p_1^2}{p_3^2}\,,\quad y=\frac{p_2^2}{p_3^2}\,,\tag{33}$$

$$\begin{aligned}
\frac{\partial}{\partial p_1^\mu} &= 2(p_{1\,\mu}+p_{2\,\mu})\frac{\partial}{\partial p_3^2}+\frac{2}{p_3^2}\left((1-x)p_{1\,\mu}-x\,p_{2\,\mu}\right)\frac{\partial}{\partial x}-2(p_{1\,\mu}+p_{2\,\mu})\frac{y}{p_3^2}\frac{\partial}{\partial y}\,,\\
\frac{\partial}{\partial p_2^\mu} &= 2(p_{1\,\mu}+p_{2\,\mu})\frac{\partial}{\partial p_3^2}-2(p_{1\,\mu}+p_{2\,\mu})\frac{x}{p_3^2}\frac{\partial}{\partial x}+\frac{2}{p_3^2}\left((1-y)p_{2\,\mu}-y\,p_{1\,\mu}\right)\frac{\partial}{\partial y}.
\end{aligned}$$

$$\begin{cases}
\Big[x(1-x)\frac{\partial^2}{\partial x^2}-y^2\frac{\partial^2}{\partial y^2}-2\,x\,y\frac{\partial^2}{\partial x\partial y}+[\gamma-(\alpha+\beta+1)x]\,\frac{\partial}{\partial x} \\
\qquad\qquad -(\alpha+\beta+1)y\frac{\partial}{\partial y}-\alpha\,\beta\Big]\Phi(x,y)=0\,,\\
\Big[y(1-y)\frac{\partial^2}{\partial y^2}-x^2\frac{\partial^2}{\partial x^2}-2\,x\,y\frac{\partial^2}{\partial x\partial y}+[\gamma'-(\alpha+\beta+1)y]\,\frac{\partial}{\partial y} \\
\qquad\qquad -(\alpha+\beta+1)x\frac{\partial}{\partial x}-\alpha\,\beta\Big]\Phi(x,y)=0\,,
\end{cases}$$

with the parameters $\alpha,\beta,\gamma,\gamma'$ defined in terms of the scale dimensions of the three scalar operators as

$$\begin{aligned}
\alpha &= \frac{d}{2}-\frac{\eta_1+\eta_2-\eta_3}{2}\,, &\qquad \gamma &= \frac{d}{2}-\eta_1+1\,,\\
\beta &= d-\frac{\eta_1+\eta_2+\eta_3}{2}\,, &\qquad \gamma' &= \frac{d}{2}-\eta_2+1\,.
\end{aligned}$$

It is interesting to observe that the system of equations in (35), coming from the invariance under special conformal transformations, is exactly the system of partial differential equations defining the hypergeometric Appell's function of two variables, $F_4(\alpha, \beta; \gamma, \gamma'; x, y)$, with coefficients given in Eq.(36). The Appell's function F_4 is defined as the double series (see, e.g., [20, 21, 22] for thorough discussions of the hypergeometric functions and their properties)

$$F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)_{i+j}\,(\beta)_{i+j}}{(\gamma)_i\,(\gamma')_j} \frac{x^i}{i!} \frac{y^j}{j!} \tag{37}$$

$$\begin{aligned}
S_2(\alpha, \beta; \gamma, \gamma'; x, y) &= x^{1-\gamma}\, F_4(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma, \gamma'; x, y)\,, \\
S_3(\alpha, \beta; \gamma, \gamma'; x, y) &= y^{1-\gamma'}\, F_4(\alpha - \gamma' + 1, \beta - \gamma' + 1; \gamma, 2 - \gamma'; x, y)\,, \\
S_4(\alpha, \beta; \gamma, \gamma'; x, y) &= x^{1-\gamma}\, y^{1-\gamma'}\, F_4(\alpha - \gamma - \gamma' + 2, \beta - \gamma - \gamma' + 2; 2 - \gamma, 2 - \gamma'; x, y)\,.
\end{aligned}$$

The conformal invariant correlation function of three scalar quasi primary fields with arbitrary scale dimensions is then given by

Delle Rose, Serino, C.C.

$$
\begin{aligned}
G_{123}(p_1^2,p_2^2,p_3^2) \;=\; & \frac{c_{123}\;\pi^d\,4^{d-\frac{1}{2}(\eta_1+\eta_2+\eta_3)}\,(p_3^2)^{-d+\frac{1}{2}(\eta_1+\eta_2+\eta_3)}}{\Gamma\left(\frac{\eta_1}{2}+\frac{\eta_2}{2}-\frac{\eta_3}{2}\right)\Gamma\left(\frac{\eta_1}{2}-\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\Gamma\left(-\frac{\eta_1}{2}+\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\Gamma\left(-\frac{d}{2}+\frac{\eta_1}{2}+\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)}\Bigg\{ \\
& \Gamma\left(\eta_1-\frac{d}{2}\right)\Gamma\left(\eta_2-\frac{d}{2}\right)\Gamma\left(d-\frac{\eta_1}{2}-\frac{\eta_2}{2}-\frac{\eta_3}{2}\right)\Gamma\left(\frac{d}{2}-\frac{\eta_1}{2}-\frac{\eta_2}{2}+\frac{\eta_3}{2}\right) \\
& \times F_4\left(\frac{d}{2}-\frac{\eta_1+\eta_2-\eta_3}{2}, d-\frac{\eta_1+\eta_2+\eta_3}{2}; \frac{d}{2}-\eta_1+1, \frac{d}{2}-\eta_2+1; x, y\right) \\
& +\Gamma\left(\frac{d}{2}-\eta_1\right)\Gamma\left(\eta_2-\frac{d}{2}\right)\Gamma\left(\frac{\eta_1}{2}-\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\Gamma\left(\frac{d}{2}+\frac{\eta_1}{2}-\frac{\eta_2}{2}-\frac{\eta_3}{2}\right) \\
& \times x^{\eta_1-\frac{d}{2}}\,F_4\left(\frac{d}{2}-\frac{\eta_2+\eta_3-\eta_1}{2}, \frac{\eta_1+\eta_3-\eta_2}{2}; -\frac{d}{2}+\eta_1+1, \frac{d}{2}-\eta_2+1; x, y\right) \\
& +\Gamma\left(\eta_1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\eta_2\right)\Gamma\left(-\frac{\eta_1}{2}+\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\Gamma\left(\frac{d}{2}-\frac{\eta_1}{2}+\frac{\eta_2}{2}-\frac{\eta_3}{2}\right) \\
& \times y^{\eta_2-\frac{d}{2}}\,F_4\left(\frac{d}{2}-\frac{\eta_1+\eta_3-\eta_2}{2}, \frac{\eta_2+\eta_3-\eta_1}{2}; \frac{d}{2}-\eta_1+1, -\frac{d}{2}+\eta_2+1; x, y\right) \\
& +\Gamma\left(\frac{d}{2}-\eta_1\right)\Gamma\left(\frac{d}{2}-\eta_2\right)\Gamma\left(\frac{\eta_1}{2}+\frac{\eta_2}{2}-\frac{\eta_3}{2}\right)\Gamma\left(-\frac{d}{2}+\frac{\eta_1}{2}+\frac{\eta_2}{2}+\frac{\eta_3}{2}\right) \\
& \times x^{\eta_1-\frac{d}{2}}y^{\eta_2-\frac{d}{2}}\,F_4\left(-\frac{d}{2}+\frac{\eta_1+\eta_2+\eta_3}{2}, \frac{\eta_1+\eta_2-\eta_3}{2}; -\frac{d}{2}+\eta_1+1, -\frac{d}{2}+\eta_2+1; x, y\right)\Bigg\}.
\end{aligned}
$$

Notice that this corresponds to the solution of a master integral

$$
J(\nu_1,\nu_2,\nu_3) = \int \frac{d^d l}{(2\pi)^d}\frac{1}{(l^2)^{\nu_3}((l+p_1)^2)^{\nu_2}((l-p_2)^2)^{\nu_1}}\,,
$$

$$
\begin{aligned}
& \int \frac{d^d p_1}{(2\pi)^d}\frac{d^d p_2}{(2\pi)^d}\frac{d^d p_3}{(2\pi)^d}\,(2\pi)^d\delta^{(d)}(p_1+p_2+p_3)\,J(\nu_1,\nu_2,\nu_3)e^{-ip_1\cdot x_1-ip_2\cdot x_2-ip_3\cdot x_3} \\
& = \frac{1}{4^{\nu_1+\nu_2+\nu_3}\pi^{3d/2}}\frac{\Gamma(d/2-\nu_1)\Gamma(d/2-\nu_2)\Gamma(d/2-\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)}\frac{1}{(x_{12}^2)^{d/2-\nu_3}(x_{23}^2)^{d/2-\nu_1}(x_{31}^2)^{d/2-\nu_2}}\,,
\end{aligned}
$$

Recurrence relations from conformal invariance

Integration by parts method just correspond to scale invariance

$$\int \frac{d^d l}{(2\pi)^d} \frac{\partial}{\partial l_\mu} \left\{ \frac{l_\mu}{(l^2)^{\nu_3}((l+p_1)^2)^{\nu_2}((l-p_2)^2)^{\nu_1}} \right\} = 0.$$

constraint of scale invariance. In fact, the scale transformation acts on $J(\nu_1, \nu_2, \nu_3)$ in the form

$$\left[d - 2\,(\nu_1 + \nu_2 + \nu_3) - p_1 \cdot \frac{\partial}{\partial p_1} - p_2 \cdot \frac{\partial}{\partial p_2}\right] \int d^d l \frac{1}{(l^2)^{\nu_3}\,((l+p_1)^2)^{\nu_2}\,((l-p_2)^2)^{\nu_1}} = 0\,. \quad (62)$$

Other recursive relations can be found

$$\left\{ p_{1\,\mu} \frac{\partial^2}{\partial p_1 \cdot \partial p_1} - 2\, p_{1\,\nu} \frac{\partial^2}{\partial p_1^\mu \partial p_{1\,\nu}} - 2\,(\nu_2 + \nu_3) \frac{\partial}{\partial p_1^\mu} + (1 \leftrightarrow 2) \right\} J(\nu_1, \nu_2, \nu_3) = 0\,.$$

$$\begin{aligned}
&\nu_2\, p_{1\,\mu} \Big[(1 + \nu_2 + \nu_3 - d/2)\, J(\nu_1, \nu_2 + 1, \nu_3) + (\nu_2 + 1) \left(J(\nu_1, \nu_2 + 2, \nu_3 - 1) - p_1^2\, J(\nu_1, \nu_2 + 2, \nu_3) \right) \Big] \\
&+ \nu_1\, p_{2\,\mu} \Big[(1 + \nu_1 + \nu_3 - d/2)\, J(\nu_1 + 1, \nu_2, \nu_3) + (\nu_1 + 1) \left(J(\nu_1 + 2, \nu_2, \nu_3 - 1) - p_2^2\, J(\nu_1 + 2, \nu_2, \nu_3) \right) \Big] \\
&+ \nu_2 \Big[(\nu_3 - 1)\, J_\mu(\nu_1, \nu_2 + 1, \nu_3) + (\nu_2 + 1) \left(J_\mu(\nu_1, \nu_2 + 2, \nu_3 - 1) - p_1^2\, J_\mu(\nu_1, \nu_2 + 2, \nu_3) \right) \Big] \\
&- \nu_1 \Big[(\nu_3 - 1)\, J_\mu(\nu_1 + 1, \nu_2, \nu_3) + (\nu_1 + 1) \left(J_\mu(\nu_1 + 2, \nu_2, \nu_3 - 1) - p_2^2\, J_\mu(\nu_1 + 2, \nu_2, \nu_3) \right) \Big] = 0\,, \quad (65)
\end{aligned}$$

We can completely derive, independently of any IBP method, but just using scale and conformal invariance the "step" relations of contiguity for the master integrals

$$J(\nu_1+2,\nu_2,\nu_3) = \frac{1}{\nu_1\,(\nu_1+1)\,(p_1^2+p_2^2-p_3^2)\,p_2^2\,p_3^2}\sum_{(a,b,c)}\mathcal{C}_{(a,b,c)}J(\nu_1+a,\nu_2+b,\nu_3+c)\,,$$

Notice: we have not used any Mellin-Barnes method

We have only one indetermined constant which characterizes the conformal class

TTT (BMS method)

$$\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle = p_3^{\Delta_t - 2d}\left(\frac{p_1^2}{p_3^2}\right)^{\mu}\left(\frac{p_2^2}{p_3^2}\right)^{\lambda} F\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right),$$

$$\begin{aligned}&\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle \\ &\quad = C_{123} p_1^{\Delta_1 - \frac{d}{2}} p_2^{\Delta_2 - \frac{d}{2}} p_3^{\Delta_3 - \frac{d}{2}} \int_0^\infty \mathrm{d}x\, x^{\frac{d}{2}-1} K_{\Delta_1 - \frac{d}{2}}(p_1 x) K_{\Delta_2 - \frac{d}{2}}(p_2 x) K_{\Delta_3 - \frac{d}{2}}(p_3 x),\end{aligned}$$

$$0 = \mathrm{K}_{12}\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle = \mathrm{K}_{23}\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle,$$

$$\mathrm{K}_{ij} = \mathrm{K}_i - \mathrm{K}_j, \qquad \mathrm{K}_j = \frac{\partial^2}{\partial p_j^2} + \frac{d+1-2\Delta_j}{p_j}\frac{\partial}{\partial p_j}, \qquad i,j = 1,2,3.$$

$$\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = \frac{1}{2}\left(\pi^{\mu}_{\alpha}(\boldsymbol{p})\pi^{\nu}_{\beta}(\boldsymbol{p}) + \pi^{\mu}_{\beta}(\boldsymbol{p})\pi^{\nu}_{\alpha}(\boldsymbol{p})\right) - \frac{1}{d-1}\pi^{\mu\nu}(\boldsymbol{p})\pi_{\alpha\beta}(\boldsymbol{p})$$

$$\langle\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) t^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3) X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3},$$

$$
\begin{aligned}
X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} = {} & A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3} \\
& + A_2 \delta^{\beta_1\beta_2} p_2^{\alpha_1} p_3^{\alpha_2} p_1^{\alpha_3} p_1^{\beta_3} + A_2(p_1 \leftrightarrow p_3) \delta^{\beta_2\beta_3} p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} \\
& + A_2(p_2 \leftrightarrow p_3) \delta^{\beta_1\beta_3} p_2^{\alpha_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} \\
& + A_3 \delta^{\alpha_1\alpha_2} \delta^{\beta_1\beta_2} p_1^{\alpha_3} p_1^{\beta_3} + A_3(p_1 \leftrightarrow p_3) \delta^{\alpha_2\alpha_3} \delta^{\beta_2\beta_3} p_2^{\alpha_1} p_2^{\beta_1} \\
& + A_3(p_2 \leftrightarrow p_3) \delta^{\alpha_1\alpha_3} \delta^{\beta_1\beta_3} p_3^{\alpha_2} p_3^{\beta_2} \\
& + A_4 \delta^{\alpha_1\alpha_3} \delta^{\alpha_2\beta_3} p_2^{\beta_1} p_3^{\beta_2} + A_4(p_1 \leftrightarrow p_3) \delta^{\alpha_1\alpha_3} \delta^{\alpha_2\beta_1} p_3^{\beta_2} p_1^{\beta_3} \\
& + A_4(p_2 \leftrightarrow p_3) \delta^{\alpha_1\alpha_2} \delta^{\alpha_3\beta_2} p_2^{\beta_1} p_1^{\beta_3} \\
& + A_5 \delta^{\alpha_1\beta_2} \delta^{\alpha_2\beta_3} \delta^{\alpha_3\beta_1} \, ,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{K}_{12}\, A_1 = 0, && \mathrm{K}_{13}\, A_1 = 0, \\
&\mathrm{K}_{12}\, A_2 = 0, && \mathrm{K}_{13}\, A_2 = 8A_1, \\
&\mathrm{K}_{12}\, A_3 = 0, && \mathrm{K}_{13}\, A_3 = 2A_2, \\
&\mathrm{K}_{12}\, A_4 = 4\left[A_2(p_1 \leftrightarrow p_3) - A_2(p_2 \leftrightarrow p_3)\right], && \mathrm{K}_{13}\, A_4 = -4A_2(p_2 \leftrightarrow p_3), \\
&\mathrm{K}_{12}\, A_5 = 2\left[A_4(p_2 \leftrightarrow p_3) - A_4(p_1 \leftrightarrow p_3)\right], && \mathrm{K}_{13}\, A_5 = 2\left[A_4 - A_4(p_1 \leftrightarrow p_3)\right].
\end{aligned}
$$

The method can be thought of as the Helmholtz decomposition applied to CWI's

BMS solution for the TTT

$$
\begin{aligned}
\langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\rangle &= \langle\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)t^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\rangle \\
&+ \sum_{\sigma}\mathcal{L}^{\mu_{\sigma(1)}\nu_{\sigma(1)}\mu_{\sigma(2)}\nu_{\sigma(2)}\mu_{\sigma(3)}\nu_{\sigma(3)}}(\boldsymbol{p}_{\sigma(1)},\boldsymbol{p}_{\sigma(2)},\boldsymbol{p}_{\sigma(3)}) \\
&- \left[\mathscr{T}^{\mu_3\nu_3}_{\alpha_3}(\boldsymbol{p}_3)p_{3\beta_3} + \frac{\pi^{\mu_3\nu_3}(\boldsymbol{p}_3)}{d-1}\delta_{\alpha_3\beta_3}\right]\mathcal{L}^{\mu_1\nu_1\mu_2\nu_2\alpha_3\beta_3}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3) \\
&- [(\mu_1,\nu_1,\boldsymbol{p}_1)\mapsto(\mu_2,\nu_2,\boldsymbol{p}_2)\mapsto(\mu_3,\nu_3,\boldsymbol{p}_3)\mapsto(\mu_1,\nu_1,\boldsymbol{p}_1)] \\
&- [(\mu_1,\nu_1,\boldsymbol{p}_1)\mapsto(\mu_3,\nu_3,\boldsymbol{p}_3)\mapsto(\mu_2,\nu_2,\boldsymbol{p}_2)\mapsto(\mu_1,\nu_1,\boldsymbol{p}_1)],
\end{aligned}
$$

$$
\mathscr{T}^{\mu\nu}_{\alpha}(\boldsymbol{p}) = \frac{1}{p^2}\left[2p^{(\mu}\delta^{\nu)}_{\alpha} - \frac{p_\alpha}{d-1}\left(\delta^{\mu\nu} + (d-2)\frac{p^\mu p^\nu}{p^2}\right)\right].
$$

Primary conformal Ward identities. The primary CWIs are

$$\begin{aligned}
&\mathrm{K}_{12}\, A_1 = 0, && \mathrm{K}_{13}\, A_1 = 0,\\
&\mathrm{K}_{12}\, A_2 = 0, && \mathrm{K}_{13}\, A_2 = 8A_1,\\
&\mathrm{K}_{12}\, A_3 = 0, && \mathrm{K}_{13}\, A_3 = 2A_2,\\
&\mathrm{K}_{12}\, A_4 = 4\left[A_2(p_1 \leftrightarrow p_3) - A_2(p_2 \leftrightarrow p_3)\right], && \mathrm{K}_{13}\, A_4 = -4A_2(p_2 \leftrightarrow p_3),\\
&\mathrm{K}_{12}\, A_5 = 2\left[A_4(p_2 \leftrightarrow p_3) - A_4(p_1 \leftrightarrow p_3)\right], && \mathrm{K}_{13}\, A_5 = 2\left[A_4 - A_4(p_1 \leftrightarrow p_3)\right].
\end{aligned} \tag{9.11.13}$$

Secondary conformal Ward identities. The independent secondary CWIs are

$$\begin{aligned}
(*)\ \ &\mathrm{L}_6\, A_1 + \mathrm{R}\left[A_2 - A_2(p_2 \leftrightarrow p_3)\right] = \\
&\qquad = 2d \cdot \text{coeff. of } p_2^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} p_1^{\mu_3} p_1^{\nu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle, && (9.11.19)\\
&\mathrm{L}_6\, A_2 + 2\,\mathrm{R}\left[2A_3 - A_4(p_1 \leftrightarrow p_3)\right] = \\
&\qquad = 8d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_3^{\nu_2} p_1^{\mu_3} p_1^{\nu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle, && (9.11.20)
\end{aligned}$$

$$\begin{aligned}
A_1 &= \alpha_1 J_{6\{000\}},\\
A_2 &= 4\alpha_1 J_{5\{001\}} + \alpha_2 J_{4\{000\}},\\
A_3 &= 2\alpha_1 J_{4\{002\}} + \alpha_2 J_{3\{001\}} + \alpha_3 J_{2\{000\}},\\
A_4 &= 8\alpha_1 J_{4\{110\}} - 2\alpha_2 J_{3\{001\}} + \alpha_4 J_{2\{000\}},\\
A_5 &= 8\alpha_1 J_{3\{111\}} + 2\alpha_2 \left(J_{2\{110\}} + J_{2\{101\}} + J_{2\{011\}}\right) + \alpha_5 J_{0\{000\}},
\end{aligned}$$

A re-analysis of these equations is on the way (Matteo Maglio, C.C.).

Implications for Riegert's action

TTT in CFT:
Trace Identities and the Conformal Anomaly Effective Action Matteo Maglio, Mottola, CC

1. Our analysis uses different correlators, they differ by compact terms

2. We write down anomalous CWI's directly in D=4.

This bypass the very difficult process of regularization (adding the counterterm to generate the anomaly)

$$
\begin{aligned}
\tilde{\mathcal{S}}_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = {}& ^{(\perp)}\tilde{\mathcal{S}}_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} + (\Lambda+\Theta)^{\mu_1\nu_1}{}_{\alpha_1\beta_1}(p_1)\,\tilde{\mathcal{S}}_3^{\alpha_1\beta_1\mu_2\nu_2\mu_3\nu_3} \\
& + (\Lambda+\Theta)^{\mu_2\nu_2}{}_{\alpha_2\beta_2}(p_2)\,\tilde{\mathcal{S}}_3^{\mu_1\nu_1\alpha_2\beta_2\mu_3\nu_3} + (\Lambda+\Theta)^{\mu_3\nu_3}{}_{\alpha_3\beta_3}(p_3)\,\tilde{\mathcal{S}}_3^{\mu_1\nu_1\mu_2\nu_2\alpha_3\beta_3} \\
& \quad - (\Lambda+\Theta)^{\mu_2\nu_2}{}_{\alpha_2\beta_2}(p_2)\,(\Lambda+\Theta)^{\mu_3\nu_3}{}_{\alpha_3\beta_3}(p_3)\,\tilde{\mathcal{S}}_3^{\mu_1\nu_1\alpha_2\beta_2\alpha_3\beta_3} \\
& \quad - (\Lambda+\Theta)^{\mu_1\nu_1}{}_{\alpha_1\beta_1}(p_1)\,(\Lambda+\Theta)^{\mu_3\nu_3}{}_{\alpha_3\beta_3}(p_3)\,\tilde{\mathcal{S}}_3^{\alpha_1\beta_1\mu_2\nu_2\alpha_3\beta_3} \\
& \quad - (\Lambda+\Theta)^{\mu_1\nu_1}{}_{\alpha_1\beta_1}(p_1)\,(\Lambda+\Theta)^{\mu_2\nu_2}{}_{\alpha_2\beta_2}(p_2)\,\tilde{\mathcal{S}}_3^{\alpha_1\beta_1\alpha_2\beta_2\mu_3\nu_3} \\
& + (\Lambda+\Theta)^{\mu_1\nu_1}{}_{\alpha_1\beta_1}(p_1)\,(\Lambda+\Theta)^{\mu_2\nu_2}{}_{\alpha_2\beta_2}(p_2)\,(\Lambda+\Theta)^{\mu_3\nu_3}{}_{\alpha_3\beta_3}(p_3)\,\tilde{\mathcal{S}}_3^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}
\end{aligned}
$$

exact solution.
It predicts an anomaly part after renormalization

Riegert action

$$
\begin{aligned}
\mathcal{S}^{(3)}_{\text{anom}} = {}& -\frac{b'}{18}\int dx\left\{R^{(1)}\frac{1}{\overline{\square}}\left(\sqrt{-g}\,\square^2\right)^{(1)}\frac{1}{\overline{\square}}R^{(1)}\right\} + \frac{b'}{9}\int dx\left\{\partial_\mu R^{(1)}\frac{1}{\overline{\square}}\left(R^{(1)\mu\nu} - \frac{1}{3}\eta^{\mu\nu}R^{(1)}\right)\frac{1}{\overline{\square}}\partial_\nu R^{(1)}\right\} \\
& -\frac{1}{6}\int dx\left(b'\,E^{(2)} + b\,[C^2]^{(2)}\right)\frac{1}{\overline{\square}}R^{(1)} + \frac{b'}{9}\int dx\,R^{(1)}\frac{1}{\overline{\square}}\left(\sqrt{-g}\,\square\right)^{(1)}R^{(1)} + \frac{b'}{9}\int dx\,R^{(2)}R^{(1)} \qquad (8.9)
\end{aligned}
$$

This action exactly reproduces the anomaly action of the nonperturbative TTT

The anomaly part of BMS (TTT)

$$
\begin{aligned}
&{}^{(\mathcal{A})}S_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = \frac{1}{3}\,\pi^{\mu_1\nu_1}(p_1)\,\eta_{\alpha_1\beta_1}\,{}^{(\mathcal{A})}S_3^{\alpha_1\beta_1\mu_2\nu_2\mu_3\nu_3} + \frac{1}{3}\,\pi^{\mu_2\nu_2}(p_2)\,\eta_{\alpha_2\beta_2}\,{}^{(\mathcal{A})}S_3^{\mu_1\nu_1\alpha_2\beta_2\mu_3\nu_3} \\
&+\frac{1}{3}\,\pi^{\mu_3\nu_3}(p_3)\,\eta_{\alpha_3\beta_3}\,{}^{(\mathcal{A})}S_3^{\mu_1\nu_1\mu_2\nu_2\alpha_3\beta_3} - \frac{1}{9}\,\pi^{\mu_1\nu_1}(p_1)\,\pi^{\mu_3\nu_3}(p_3)\,\eta_{\alpha_1\beta_1}\eta_{\alpha_3\beta_3}\,{}^{(\mathcal{A})}S_3^{\alpha_1\beta_1\mu_2\nu_2\alpha_3\beta_3} \\
&\frac{1}{9}\,\pi^{\mu_2\nu_2}(p_2)\pi^{\mu_3\nu_3}(p_3)\,\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}\,{}^{(\mathcal{A})}S_3^{\mu_1\nu_1\alpha_2\beta_2\alpha_3\beta_3} - \frac{1}{9}\,\pi^{\mu_1\nu_1}(p_1)\pi^{\mu_2\nu_2}(p_2)\eta_{\alpha_1\beta_1}\,\eta_{\alpha_2\beta_2}\,{}^{(\mathcal{A})}S_3^{\alpha_1\beta_1\alpha_2\beta_2\mu_3\nu_3} \\
&+\frac{1}{27}\,\pi^{\mu_1\nu_1}(p_1)\pi_2^{\mu_2\nu_2}(p_2)\pi^{\mu_3\nu_3}(p_3)\,\eta_{\alpha_1\beta_1}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}\,{}^{(\mathcal{A})}S_3^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\,. \qquad (9.16)
\end{aligned}
$$

exactly reproduced by Riegert's action. Single poles are present.
The reconstruction in the TTTT is on the way

Conclusions

We are moving to the reconstruction of the TTTT.

This will show definitively whether double poles are present.

Thanks