Constraining three- and four-point functions in CFT's in d > 2 dimensions

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Tours 6/2/2020

We discuss general aspects of CFT’s in $d > 2$ dimensions and the solution of the conformal constraints (conformal Ward identities) for 3-point scalar and tensor correlators in momentum space.

This allows to investigate the role of the conformal anomaly in great generality.

The matching of a general CFT to free-field theory realizations allows to simplify drastically the solution of the conformal Ward identities (CWI's) and the structure of such tensor correlators.

We then turn to 4-point functions, showing the existence of solutions of such identities in the presence of a conformal/dual conformal symmetry. Some phenomenological implications of CFT in physics beyond the Standard Model will also be brifely outlined.

CFT's have been extensively studied in the last 50 years for a variety of reasons

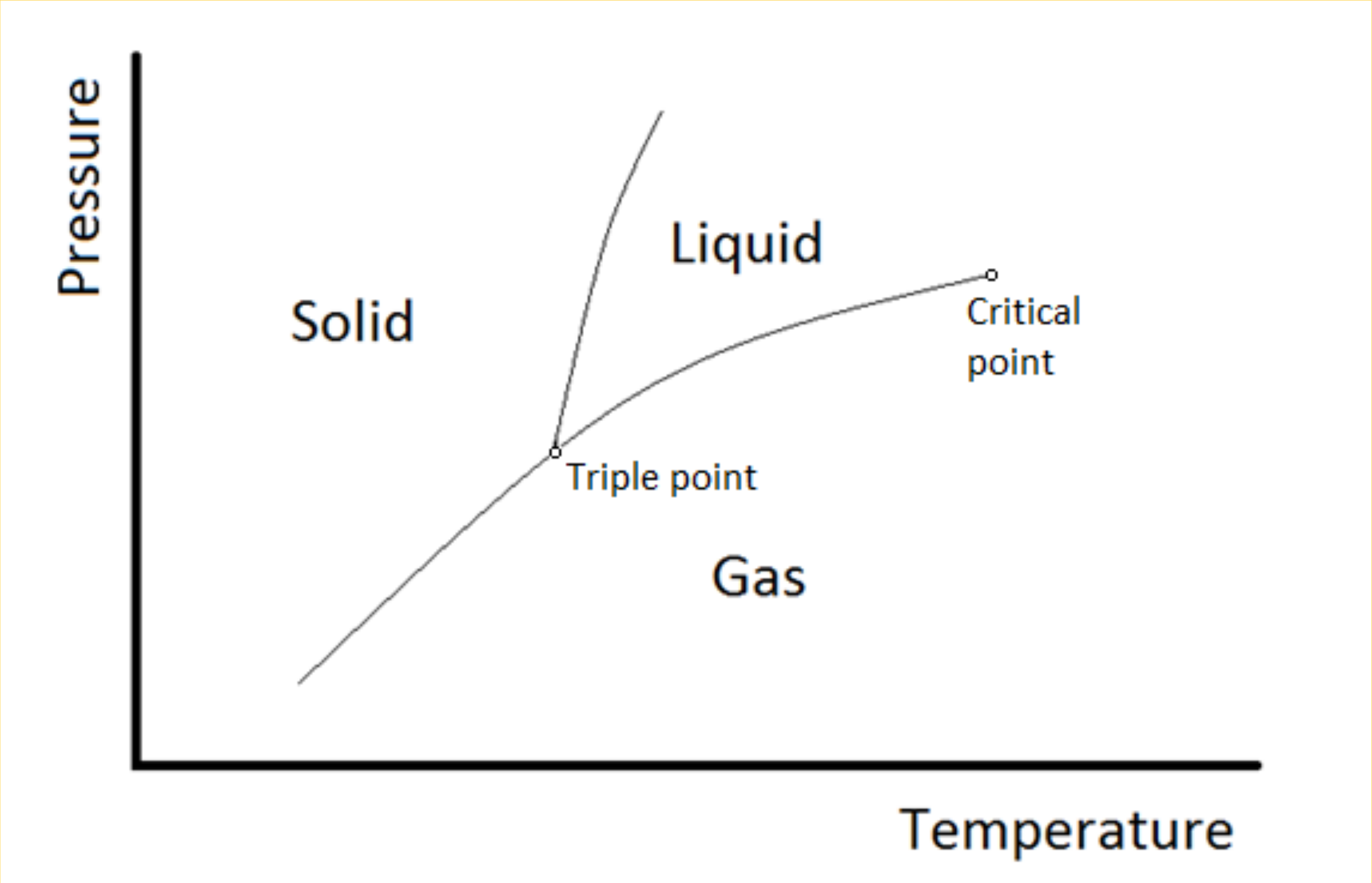
1. string theory
2. critical behaviour of statistical systems
3. Possible applications to particle phenomenology (extensions of the Standard Model with a "possible" conformal phase
4. Early universe.

5. AdS/CFT correspondence. A theory in a conformal phase is dual – in a well defined sense- to a specific gravitational theory. Applications of this correspondence, from ordinary field theories, to cosmology (holography) as well as condensed matter physics have been overwhelming.

In d=2 spacetime dimensions the theory is particularly rich, but much less in higher dimensions. Neverthless, the power of the construction is significant, vene in the presence of only a finite number -rather than infinite- of symmetries.

Our discussions will be focused on theories with d> 2, where most of the activity, both in theory and phenomenology is.

critical behaviour



no mass parameter present, the decay of a correlator is purely algebraic.

a) long range correlation, diverging correlation length as we reach the critical point

2) fluctuations around the critical behaviour characterized by the quantum expectation of values of a set of local operators, identified by the operator product expansion.

Their scaling dimensions enter as specific "parameters of the underlying CFT.

enhancing Poincare' symmetry with 1 dilatation and d special conformal transformations

$$x^\mu(x) \to x'^\mu(x) = x^\mu + v^\mu(x)$$

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x').$$

isometry

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = g_{\mu\nu}(x')$$

$$v^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\sigma}\partial_\nu v^\sigma + g_{\sigma\nu}\partial_\mu v^\sigma = 0.$$

Killing equation

$$dx_\mu dx^\mu \to dx'_\mu dx'^\mu = \Omega(x)^{-2} dx_\mu dx^\mu.$$

conformal

generating the conformal Killing equation (with $\Omega(x) = 1 - \sigma(x)$)

$$v^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\sigma}\partial_\nu v^\sigma + g_{\sigma\nu}\partial_\mu v^\sigma = 2\sigma g_{\mu\nu}.$$

In the flat spacetime limit this becomes

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma\, \eta_{\mu\nu}, \qquad \sigma = \frac{1}{d}\partial \cdot v.$$

One can show that any CT can be written as a local rotation combined with a scaling

$$R^{\mu}_{\alpha} = \Omega \frac{\partial x'^{\mu}}{\partial x^{\alpha}}$$

we can first expand genericaly R around the identity as

$$R = \mathbf{1} + [\epsilon] + \dots$$

with an antisymmetric matrix $[\epsilon]$, which we can re-express in terms of antisymmetric parameters $(\tau_{\rho\sigma})$ and $1/2\, d\,(d-1)$ generators $\Sigma_{\rho\sigma}$ of $SO(d)$ as

$$\begin{aligned} [\epsilon]_{\mu\alpha} &= \frac{1}{2}\tau_{\rho\sigma}\left(\Sigma_{\rho\sigma}\right)_{\mu\alpha} \\ \left(\Sigma_{\rho\sigma}\right)_{\mu\alpha} &= \delta_{\rho\mu}\delta_{\sigma\alpha} - \delta_{\rho\alpha}\delta_{\sigma\mu} \end{aligned}$$

R acts on an vector, tensor, etc. using the appropiate form of SIgma

$$R_{\mu\alpha} = \delta_{\mu\alpha} + \tau_{\mu\alpha} = \delta_{\mu\alpha} + \frac{1}{2}\partial_{[\alpha}v_{\mu]}$$

SIgma can be expressed in terms of the antisymmetric part of the conformal Killing vector v

with $\partial_{[\alpha}v_{\mu]} \equiv \partial_{\alpha}v_{\mu} - \partial_{\mu}v_{\alpha}$.

examples

$$
\begin{aligned}
A'^{\mu}(x') &= \Omega^{\Delta_A} R_{\mu\alpha} A^{\alpha}(x) \\
&= (1-\sigma+\ldots)^{\Delta_A}(\delta_{\mu\alpha}+\frac{1}{2}\partial_{[\alpha}v_{\mu]}+\ldots)A^{\alpha}(x)
\end{aligned}
$$

from which one can easily deduce that

$$\delta A^{\mu}(x) \equiv A'^{\mu}(x) - A^{\mu}(x) = -(v\cdot\partial + \Delta_A\sigma)A^{\mu}(x) + \frac{1}{2}\partial_{[\alpha}v_{\mu]}A^{\alpha}(x),$$

which is defined to be the Lie derivative of A^{μ} in the v direction

$$L_v A^{\mu}(x) \equiv -\delta A^{\mu}(x).$$

As an example, in the case of a generic rank-2 tensor field $(\phi^{I\,K})$ of scaling dimension Δ_ϕ, transforming according to a representation $D^I_J(R)$ of the rotation group $SO(d)$,

$$\phi'^{I\,K}(x') = \Omega^{\Delta_\phi} D^I_{I'}(R) D^K_{K'}(R)\phi^{I'\,K'}(x).$$

In the case of the stress energy tensor ($D(R) = R$), with scaling (mass) dimension Δ_T ($\Delta_T = d$)

$$\begin{aligned} T'^{\mu\nu}(x') &= \Omega^{\Delta_T} R^\mu_\alpha R^\nu_\beta T^{\alpha\beta}(x) \\ &= (1 - \Delta_T \sigma + \ldots)(\delta_{\mu\alpha} + \frac{1}{2}\partial_{[\alpha} v_{\mu]} + \ldots)(\delta_{\mu\alpha} + \frac{1}{2}\partial_{[\alpha} v_{\mu]} + \ldots)\, T^{\alpha\beta}(x) \end{aligned}$$

where $\partial_{[\alpha} v_{\mu]} \equiv \partial_\alpha v_\mu - \partial_\mu v_\alpha$. One gets

$$\delta T^{\mu\nu}(x) = -\Delta_T\, \sigma\, T^{\mu\nu} - v \cdot \partial\, T^{\mu\nu}(x) + \frac{1}{2}\partial_{[\alpha} v_{\mu]}\, T^{\alpha\nu} + \frac{1}{2}\partial_{[\nu} v_{\alpha]} T^{\mu\alpha}.$$

For a special conformal transformation (SCT) one chooses

$$v_\mu(x) = b_\mu x^2 - 2x_\mu b \cdot x$$

$$\delta T^{\mu\nu}(x) = -(b^\alpha x^2 - 2x^\alpha b \cdot x)\, \partial_\alpha T^{\mu\nu}(x) - \Delta_T \sigma T^{\mu\nu}(x) + 2(b_\mu x_\alpha - b_\alpha x_\mu) T^{\alpha\nu} + 2(b_\nu x_\alpha - b_\alpha x_\nu)\, T^{\mu\alpha}(x).$$

$$\begin{aligned} \mathcal{K}^\kappa T^{\mu\nu}(x) &\equiv \delta_\kappa T^{\mu\nu}(x) = \frac{\partial}{\partial b^\kappa}(\delta T^{\mu\nu}) \\ &= -(x^2 \partial_\kappa - 2x_\kappa x \cdot \partial) T^{\mu\nu}(x) + 2\Delta_T x_\kappa T^{\mu\nu}(x) + 2(\delta_{\mu\kappa} x_\alpha - \delta_{\alpha\kappa} x_\mu) T^{\alpha\nu}(x) \\ &\quad +2(\delta_{\kappa\nu} x_\alpha - \delta_{\alpha\kappa} x_\nu) T^{\mu\alpha}. \end{aligned}$$

SCT

$$\begin{aligned}\mathcal{K}^\kappa T^{\mu\nu}(x) &\equiv \delta_\kappa T^{\mu\nu}(x) = \frac{\partial}{\partial b^\kappa}(\delta T^{\mu\nu}) \\ &= -(x^2\partial_\kappa - 2x_\kappa x\cdot\partial)T^{\mu\nu}(x) + 2\Delta_T x_\kappa T^{\mu\nu}(x) + 2(\delta_{\mu\kappa}x_\alpha - \delta_{\alpha\kappa}x_\mu)T^{\alpha\nu}(x) \\ &\quad +2(\delta_{\kappa\nu}x_\alpha - \delta_{\alpha\kappa}x_\nu)T^{\mu\alpha}.\end{aligned}$$

$$\begin{aligned}[][K^\mu, D] &= -iK^\mu, \\ [P^\mu, K^\nu] &= 2i\delta^{\mu\nu}D + 2iJ^{\mu\nu}, \\ [K^\mu, K^\nu] &= 0, \\ [J^{\rho\sigma}, K^\mu] &= i\delta^{\mu\rho}K^\sigma - i\delta^{\mu\sigma}K^\rho.\end{aligned}$$

$$\begin{array}{rl} \text{translations} & L_g = a^\mu\partial_\mu, \\ \text{rotations} & L_g = \dfrac{\omega^{\mu\nu}}{2}[x_\nu\partial_\mu - x_\mu\partial_\nu] - \Sigma_{\mu\nu}, \\ \text{scale transformations} & L_g = \sigma\,[x\cdot\partial + \Delta], \\ \text{special conformal transformations} & L_g = b^\mu[x^2\partial_\mu - 2x_\mu\, x\cdot\partial - 2\Delta\, x_\mu - 2x_\nu\Sigma_\mu{}^\nu]. \end{array}$$

d (transl) + ½(d-1)d (Lorentz) +1 (dil.) + d (sct) = (d+1)(d+2)/2

generators SO(2,D)

include inversion $x_\mu \to x'_\mu = \frac{x_\mu}{x^2}, \qquad \Omega(x) = x^2,$ O(2,d)

$O_i(x) \to O'(x') = \lambda^{-\Delta_i} O(x)$ primaries each of scaling dimension Δ_i

$$\Phi(x_1, x_2, \ldots, x_n) = \langle O_1(x_1) O_2(x_2) \ldots O_n(x_n) \rangle \, .$$

$$K^\kappa(x_i)\Phi(x_1, x_2, \ldots, x_n) = 0$$

$$K^\kappa(x_i) \equiv \sum_{j=1}^{n} \left(2\Delta_j x_j^\kappa - x_j^2 \frac{\partial}{\partial x_j^\kappa} + 2 x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right)$$

$$D(x_i)\Phi(x_1, \ldots x_n) = 0$$

$$D(x_i) \equiv \sum_{i=1}^{n} \left(x_i^\alpha \frac{\partial}{\partial x_i^\alpha} + \Delta_i \right)$$

two and 3-point functions of primary scalar fields, in the scalar case, are easily fixed

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} \delta_{\Delta_1 \Delta_2}.$$

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\; x_{23}^{\Delta_2+\Delta_3-\Delta-1}\; x_{13}^{\Delta_3+\Delta_1-\Delta_2}}.$$

$$\Gamma^{\mu\nu\alpha\beta}(x_1, x_2, x_3) = \langle T^{\mu\nu}(x_1) J^{\alpha}(x_2) J^{\beta}(x_3)\rangle$$

TJJ

$$\begin{aligned}
\mathcal{K}^{\kappa}\Gamma^{\mu\nu\alpha\beta}(x_1,x_2,x_3) \quad &= \quad \sum_{i=1}^{3} K_{i\,scalar}^{\,\kappa}(x_i)\Gamma^{\mu\nu\alpha\beta}(x_1,x_2,x_3) \\
&+2\left(\delta^{\mu\kappa}x_{1\rho} - \delta^{\kappa}_{\rho}x_1^{\mu}\right)\Gamma^{\rho\nu\alpha\beta} + 2\left(\delta^{\nu\kappa}x_{1\rho} - \delta^{\kappa}_{\rho}x_1^{\nu}\right)\Gamma^{\mu\rho\alpha\beta} \\
&2\left(\delta^{\alpha\kappa}x_{2\rho} - \delta^{\kappa}_{\rho}x_2^{\alpha}\right)\Gamma^{\mu\nu\rho\beta} + 2\left(\delta^{\beta\kappa}x_{3\rho} - \delta^{\kappa}_{\rho}x_3^{\beta}\right)\Gamma^{\mu\nu\alpha\rho} = 0,
\end{aligned}$$

analysis of the TT, TTT in coordinate space done long ago by Osborn and Petkou to fix the structure of the primary correlator we are actually using the conformal ward identities

$$K^\kappa(x_i)\Phi(x_1,x_2,\dots,x_n)=0$$

$$K^\kappa(x_i)\equiv\sum_{j=1}^{n}\left(2\Delta_j x_j^\kappa - x_j^2\frac{\partial}{\partial x_j^\kappa}+2x_j^\kappa x_j^\alpha\frac{\partial}{\partial x_j^\alpha}\right)$$

$$D(x_i)\Phi(x_1,\dots x_n)=0$$

$$D(x_i)\equiv\sum_{i=1}^{n}\left(x_i^\alpha\frac{\partial}{\partial x_i^\alpha}+\Delta_i\right)$$

which for 3-point functions reduce the solution to a unique expression, modulo one constant.

In the TJJ case things are more complicated.

Can we proceed autonomously to derive these results from momentum space?

Delle Rose, Mottola, Serino, C.C.
Bzowski, McFadden, Skenderis

2013

$$\Phi(x_1,x_2,\ldots,x_n)=\int dp_1 dp_2 \ldots dp_{n-1} e^{i(p_1x_1+p_2x_2+\ldots p_{n-1}x_{n-1}+\overline{p}_n x_n)}\Phi(p_1,p_2,\ldots,\overline{p}_n).$$

$$\sum_{j=1}^{n}\left(x_j^\alpha\frac{\partial}{\partial x_j^\alpha}+\Delta_j\right)\Phi(x_1,x_2,\ldots,x_n)=0.$$

dilatation WI

$$\left[\sum_{j=1}^{n}\Delta_j-(n-1)d-\sum_{j=1}^{n-1}p_j^\alpha\frac{\partial}{\partial p_j^\alpha}\right]\Phi(p_1,p_2,\ldots,\overline{p}_n)=0.$$

momentum

SC WI

$$\sum_{j=1}^{n}\left(-x_j^2\frac{\partial}{\partial x_j^\kappa}+2x_j^\kappa x_j^\alpha\frac{\partial}{\partial x_j^\alpha}+2\Delta_j x_j^\kappa\right)\Phi(x_1,x_2,\ldots,x_n)=0$$

$$\sum_{j=1}^{n-1}\left(p_j^\kappa\frac{\partial^2}{\partial p_j^\alpha\partial p_j^\alpha}+2(\Delta_j-d)\frac{\partial}{\partial p_j^\kappa}-2p_j^\alpha\frac{\partial^2}{\partial p_j^\kappa\partial p_j^\alpha}\right)\Phi(p_1,\ldots p_{n-1},\bar{p}_n)=0.$$

momentum

Delle Rose, Mottola, Serino, C.C.

2013

Solving the Conformal Constraints for Scalar Operators in Momentum Space and the Evaluation of Feynman's Master Integrals

$$\left(-p_\mu \frac{\partial}{\partial p_\mu} + \eta_1 + \eta_2 - d\right) G^{ij}(p) = 0\,,$$
$$\left(p_\mu \frac{\partial^2}{\partial p^\nu \partial p_\nu} - 2\, p_\nu \frac{\partial^2}{\partial p^\mu \partial p_\nu} + 2(\eta_1 - d)\frac{\partial}{\partial p^\mu} + 2(\Sigma_{\mu\nu})^i_k \frac{\partial}{\partial p_\nu}\right) G^{kj}(p) = 0\,,$$

$$G^{ij}(p) \;\equiv\; \langle \mathcal{O}_1^i(p)\mathcal{O}_2^j(-p)\rangle .$$

The invariance under scale transformations implies that $G_S(p^2)$ is a homogeneous function of degree $\alpha = \frac{1}{2}(\eta_1 + \eta_2 - d)$. At the same time, it is easy to show that the second equation can be satisfied only if $\eta_1 = \eta_2$. Therefore conformal symmetry fixes the structure of the scalar two-point function up to an arbitrary overall constant C as

$$G_S(p^2) = \langle \mathcal{O}_1(p)\mathcal{O}_2(-p)\rangle = \delta_{\eta_1\eta_2}\, C\, (p^2)^{\eta_1 - d/2}\,.$$

If we redefine

$$C = c_{S12}\, \frac{\pi^{d/2}}{4^{\eta_1 - d/2}} \frac{\Gamma(d/2 - \eta_1)}{\Gamma(\eta_1)}$$

$$G_S(p^2) = \delta_{\eta_1\eta_2}\, c_{S12}\, \frac{\pi^{d/2}}{4^{\eta_1 - d/2}} \frac{\Gamma(d/2 - \eta_1)}{\Gamma(\eta_1)} (p^2)^{\eta_1 - d/2}\,,$$

we reobtain the familiar form of coordinate space

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \equiv \mathcal{F.T.}\left[G_S(p^2)\right] = \delta_{\eta_1\eta_2}\, c_{S12} \frac{1}{(x_{12}^2)^{\eta_1}}\,,$$

$$G_V^{\alpha\beta}(p) \equiv \langle V_1^\alpha(p) V_2^\beta(-p)\rangle.$$

$$\partial^\mu V_\mu = 0,$$

$$G_V^{\alpha\beta}(p) = \pi^{\alpha\beta}(p)\, f_V(p^2)\,,$$ with $$\pi^{\alpha\beta}(p) = \eta^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}$$

$$G_V^{\alpha\beta}(p) = \delta_{\eta_1\eta_2}\, c_{V12}\, \frac{\pi^{d/2}}{4^{\eta_1-d/2}} \frac{\Gamma(d/2-\eta_1)}{\Gamma(\eta_1)} \left(\eta^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}\right) (p^2)^{\eta_1-d/2}\,,$$

modulo an overall constant

rank 4

$$T_{\mu\nu} = T_{\nu\mu}\,, \qquad \partial^\mu T_{\mu\nu} = 0\,, \qquad T_\mu{}^\mu = 0\,.$$

$$G_T^{\alpha\beta\mu\nu}(p) = \Pi_d^{\alpha\beta\mu\nu}(p)\, f_T(p^2)$$

$$\Pi_d^{\alpha\beta\mu\nu}(p) = \frac{1}{2}\left[\pi^{\alpha\mu}(p)\pi^{\beta\nu}(p) + \pi^{\alpha\nu}(p)\pi^{\beta\mu}(p)\right] - \frac{1}{d-1}\pi^{\alpha\beta}(p)\pi^{\mu\nu}(p)\,,$$

and the scalar function $f_T(p^2)$ determined as usual, up to a multiplicative constant, by requiring the invariance under dilatations and special conformal transformations. We obtain

$$G_T^{\alpha\beta\mu\nu}(p) = \delta_{\eta_1\eta_2}\, c_{T12}\, \frac{\pi^{d/2}}{4^{\eta_1-d/2}} \frac{\Gamma(d/2-\eta_1)}{\Gamma(\eta_1)}\, \Pi_d^{\alpha\beta\mu\nu}(p)\, (p^2)^{\eta_1-d/2}\,.$$

simple poles for non positive integer arguments, which occur, in our case, when $\eta = d/2 + n$ with $n = 0, 1, 2, \ldots$.

$$d \to d - 2\epsilon,$$

$$\Gamma\left(d/2-\eta\right)\,(p^2)^{\eta-d/2} = \frac{(-1)^n}{n!}\left(-\frac{1}{\epsilon}+\psi(n+1)+O(\epsilon)\right)(p^2)^{n+\epsilon}\,,$$

anomalous variation of scale invariance

Indeed, when $\eta = d/2 + n$, employing dimensional

$$G^{ij}(p^2) = \frac{1}{\epsilon} G^{ij}_{sing}(p^2) + G^{ij}_{finite}(p^2)\,.$$

$$\left(p^2\,\frac{\partial}{\partial p^2} - n\right) G^{ij}(p^2) = G^{ij}_{sing}(p^2)\,,$$

Moving to scalar 3-point functions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{c_{123}}{(x_{12}^2)^{\frac{1}{2}(\eta_1+\eta_2-\eta_3)}\,(x_{23}^2)^{\frac{1}{2}(\eta_2+\eta_3-\eta_1)}\,(x_{31}^2)^{\frac{1}{2}(\eta_3+\eta_1-\eta_2)}}\,.$$

how to reopbtain this result directly from momentum space

$$J(\nu_1,\nu_2,\nu_3) = \int \frac{d^dl}{(2\pi)^d}\frac{1}{(l^2)^{\nu_3}((l+p_1)^2)^{\nu_2}((l-p_2)^2)^{\nu_1}}\,,$$

master Feynman integral

$$\eta_1 = d-\nu_2-\nu_3\,, \qquad \eta_2 = d-\nu_1-\nu_3\,, \qquad \eta_3 = d-\nu_1-\nu_2\,.$$

$$\int \frac{d^dp_1}{(2\pi)^d}\frac{d^dp_2}{(2\pi)^d}\frac{d^dp_3}{(2\pi)^d}(2\pi)^d\delta^{(d)}(p_1+p_2+p_3)\,J(\nu_1,\nu_2,\nu_3)e^{-ip_1\cdot x_1-ip_2\cdot x_2-ip_3\cdot x_3}$$
$$= \frac{1}{4^{\nu_1+\nu_2+\nu_3}\pi^{3d/2}}\frac{\Gamma(d/2-\nu_1)\Gamma(d/2-\nu_2)\Gamma(d/2-\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)}\frac{1}{(x_{12}^2)^{d/2-\nu_3}(x_{23}^2)^{d/2-\nu_1}(x_{31}^2)^{d/2-\nu_2}}\,,$$

We need to solve

n=3

$$\sum_{j=1}^{n-1}\left(p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_j^\alpha} + 2(\Delta_j - d)\frac{\partial}{\partial p_j^\kappa} - 2p_j^\alpha \frac{\partial^2}{\partial p_j^\kappa \partial p_j^\alpha}\right)\Phi(p_1,\dots p_{n-1},\bar{p}_n) = 0.$$

$$\left[\sum_{j=1}^{n}\Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right]\Phi(p_1,p_2,\dots,\bar{p}_n) = 0.$$

perform the change of variables (Delle Rose, Serino, Mottola, CC, 2013)

$$x = \frac{p_1^2}{p_3^2}, \quad y = \frac{p_2^2}{p_3^2},$$

$$\begin{aligned}
\frac{\partial}{\partial p_1^\mu} &= 2(p_{1\,\mu}+p_{2\,\mu})\frac{\partial}{\partial p_3^2} + \frac{2}{p_3^2}\left((1-x)p_{1\,\mu} - x\,p_{2\,\mu}\right)\frac{\partial}{\partial x} - 2(p_{1\,\mu}+p_{2\,\mu})\frac{y}{p_3^2}\frac{\partial}{\partial y}, \\
\frac{\partial}{\partial p_2^\mu} &= 2(p_{1\,\mu}+p_{2\,\mu})\frac{\partial}{\partial p_3^2} - 2(p_{1\,\mu}+p_{2\,\mu})\frac{x}{p_3^2}\frac{\partial}{\partial x} + \frac{2}{p_3^2}\left((1-y)p_{2\,\mu} - y\,p_{1\,\mu}\right)\frac{\partial}{\partial y}.
\end{aligned}$$

and assume an ansatz of the form

$$G_{123}(p_1^2,p_2^2,p_3^2) = (p_3^2)^{-d+\frac{1}{2}(\eta_1+\eta_2+\eta_3)}\,\Phi(x,y)$$

The equations are found to become a hypergeometric system of rank-4

$$\begin{cases} \Big[x(1-x)\frac{\partial^2}{\partial x^2} - y^2\frac{\partial^2}{\partial y^2} - 2\,x\,y\frac{\partial^2}{\partial x \partial y} + [\gamma - (\alpha+\beta+1)x]\,\frac{\partial}{\partial x} \\ \qquad\qquad -(\alpha+\beta+1)y\frac{\partial}{\partial y} - \alpha\,\beta\Big]\Phi(x,y) = 0\,, \\ \Big[y(1-y)\frac{\partial^2}{\partial y^2} - x^2\frac{\partial^2}{\partial x^2} - 2\,x\,y\frac{\partial^2}{\partial x \partial y} + [\gamma' - (\alpha+\beta+1)y]\,\frac{\partial}{\partial y} \\ \qquad\qquad -(\alpha+\beta+1)x\frac{\partial}{\partial x} - \alpha\,\beta\Big]\Phi(x,y) = 0\,, \end{cases}$$

Appell system of equations

(see **Campes de Feriet and Appell's** book)

$$\alpha = \frac{d}{2} - \frac{\eta_1+\eta_2-\eta_3}{2}\,, \qquad \gamma = \frac{d}{2} - \eta_1 + 1\,,$$

$$\beta = d - \frac{\eta_1+\eta_2+\eta_3}{2}\,, \qquad \gamma' = \frac{d}{2} - \eta_2 + 1\,.$$

$$F_4(\alpha,\beta;\gamma,\gamma';x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}\,(\beta)_{i+j}}{(\gamma)_i\,(\gamma')_j}\frac{x^i}{i!}\frac{y^j}{j!}$$

$(\alpha)_i = \Gamma(\alpha+i)/\Gamma(\alpha)$ is the Pochhammer symbol.

Appell's hypergeometric functions $F_1(x,y)$, $F_2(x,y)$, $F_3(x,y)$, $F_4(x,y)$ are defined by the hypergeometric series:

$$F_1\left(\begin{matrix} a;\ b_1, b_2 \\ c \end{matrix}\middle| x,\, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}\,(b_1)_n\,(b_2)_m}{(c)_{n+m}\; n!\, m!}\, x^n\, y^m,$$

$$F_2\left(\begin{matrix} a;\ b_1, b_2 \\ c_1, c_2 \end{matrix}\middle| x,\, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}\,(b_1)_n\,(b_2)_m}{(c_1)_n\,(c_2)_m\; n!\, m!}\, x^n\, y^m,$$

$$F_3\left(\begin{matrix} a_1, a_2;\ b_1, b_2 \\ c \end{matrix}\middle| x,\, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a_1)_n (a_2)_m (b_1)_n (b_2)_m}{(c)_{n+m}\; n!\, m!}\, x^n\, y^m,$$

$$F_4(a,b,c_1,c_2;x,y) \equiv F_4\left(\begin{matrix} a;\ b \\ c_1, c_2 \end{matrix}\middle| x,\, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}\,(b)_{n+m}}{(c_1)_n\,(c_2)_m\; n!\, m!}\, x^n\, y^m$$

and are bivariate generalizations of the Gauss hypergeometric series

$${}_2\mathrm{F}_1\left(\begin{matrix} A,\ B \\ C \end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(A)_n\,(B)_n}{(C)_n\; n!}\, z^n.$$

They extend Euler's hypergeometric equation of ${}_2\mathrm{F}_1$

solution of

$$z(1-z)\,\frac{d^2y(z)}{dz^2} + \big(C-(A+B+1)z\big)\frac{dy(z)}{dz} - A\,B\,y(z) = 0.$$

`The hypergeometric system of equations corrispondign to F4, can also be obtained by first rewriting the special CWI's which are four-vector equations to the scalar form (Bzowsky, McFadden, Skenderis, 2013)

$$K^\kappa(p_i) \equiv \sum_{j=1}^{2} \left(2(\Delta_j - d)\frac{\partial}{\partial p_j^\kappa} + p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_j^\alpha} - 2p_j^\alpha \frac{\partial^2}{\partial p_j^\kappa \partial p_j^\alpha} \right) \Phi(p_1, p_2, \bar{p}_3) = 0,$$

$$\frac{\partial \Phi}{\partial p_i^\mu} = \frac{p_i^\mu}{p_i}\frac{\partial \Phi}{\partial p_i} - \frac{\bar{p}_3^\mu}{p_3}\frac{\partial \Phi}{\partial p_3}.$$

chain rule

$${K_{scalar}}^\kappa \Phi = 0$$

$$K^\kappa_{scalar} = \sum_{i=1}^{3} p_i^\kappa \, K_i$$

$$K_i \equiv \frac{\partial^2}{\partial p_i \partial p_i} + \frac{d+1-2\Delta_i}{p_i}\frac{\partial}{\partial p_i}$$

$$\frac{\partial^2 \Phi}{\partial p_i \partial p_i} + \frac{1}{p_i}\frac{\partial \Phi}{\partial p_i}(d+1-2\Delta_1) - \frac{\partial^2 \Phi}{\partial p_3 \partial p_3} - \frac{1}{p_3}\frac{\partial \Phi}{\partial p_3}(d+1-2\Delta_3) = 0$$

$$K_{ij} \equiv K_i - K_j$$

$$K_{13}^\kappa \Phi = 0 \qquad \text{and} \qquad K_{23}^\kappa \Phi = 0.$$

General solutions

(rank of the system)

One discovers an Appell system under certain conditions

Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex

MM Maglio, C.C.

the transiton to the F4 system is guaranteed if we set to vanish the 1/x, 1/y terms in the change of variables

$$K_{31}\Phi = 0 \qquad K_{21}\Phi = 0$$

$$K_{21}\phi = 4p_1^{\Delta-2d-2}x^a y^b\left(x(1-x)\frac{\partial}{\partial x\partial x} + (Ax+\gamma)\frac{\partial}{\partial x} - 2xy\frac{\partial^2}{\partial x\partial y} - y^2\frac{\partial^2}{\partial y\partial y} + Dy\frac{\partial}{\partial y} + (E+\frac{G}{x})\right) \times F(x,y) = 0$$

$$F_4(\alpha(a,b),\beta(a,b);\gamma(a),\gamma'(b);x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha(a,b),i+j)\,(\beta(a,b),i+j)}{(\gamma(a),i)\,(\gamma'(b),j)}\frac{x^i}{i!}\frac{y^j}{j!}$$

$$a = 0 \equiv a_0 \qquad \text{or} \qquad a = \Delta_2 - \frac{d}{2} \equiv a_1.$$

$$b = 0 \equiv b_0 \qquad \text{or} \qquad b = \Delta_3 - \frac{d}{2} \equiv b_1.$$

$$\gamma(a) = 2a + \frac{d}{2} - \Delta_2 + 1$$

$$\gamma'(b) = 2b + \frac{d}{2} - \Delta_3 + 1$$

$$\alpha(a,b) = a + b + \frac{d}{2} - \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)$$

$$\beta(a,b) = a + b + d - \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3)$$

GENERAL 3-POINT CORRELATOR

$$\langle O(p_1)\,O(p_2)\,O(p_3)\rangle = \left(p_3^2\right)^{-d+\frac{\Delta_t}{2}}\,C(\Delta_1,\Delta_2,\Delta_3,d)$$

$$\Bigg\{\Gamma\left(\Delta_1-\frac{d}{2}\right)\Gamma\left(\Delta_2-\frac{d}{2}\right)\Gamma\left(d-\frac{\Delta_1+\Delta_2+\Delta_3}{2}\right)\Gamma\left(d-\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)$$
$$\times\,F_4\left(\frac{d}{2}-\frac{\Delta_1+\Delta_2-\Delta_3}{2},d-\frac{\Delta_t}{2},\frac{d}{2}-\Delta_1+1,\frac{d}{2}-\Delta_2+1;x,y\right)$$

$$+\,\Gamma\left(\frac{d}{2}-\Delta_1\right)\Gamma\left(\Delta_2-\frac{d}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{d}{2}+\frac{\Delta_1-\Delta_2-\Delta_3}{2}\right)$$
$$\times\,x^{\Delta_1-\frac{d}{2}}\,F_4\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2},\frac{d}{2}-\frac{\Delta_2+\Delta_3-\Delta_1}{2},\Delta_1-\frac{d}{2}+1,\frac{d}{2}-\Delta_2+1;x\right.$$

$$+\,\Gamma\left(\Delta_1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_2\right)\Gamma\left(\frac{-\Delta_1+\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{d}{2}+\frac{-\Delta_1+\Delta_2-\Delta_3}{2}\right)$$
$$\times\,y^{\Delta_2-\frac{d}{2}}\,F_4\left(\frac{\Delta_2-\Delta_1+\Delta_3}{2},\frac{d}{2}-\frac{\Delta_1-\Delta_2+\Delta_3}{2},\frac{d}{2}-\Delta_1+1,\Delta_2-\frac{d}{2}+1;x,y\right)$$

$$+\,\Gamma\left(\frac{d}{2}-\Delta_1\right)\Gamma\left(\frac{d}{2}-\Delta_2\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(-\frac{d}{2}+\frac{\Delta_1+\Delta_2+\Delta_3}{2}\right)$$
$$\times\,x^{\Delta_1-\frac{d}{2}}\,y^{\Delta_2-\frac{d}{2}}F_4\left(-\frac{d}{2}+\frac{\Delta_t}{2},\frac{\Delta_1+\Delta_2-\Delta_3}{2},\Delta_1-\frac{d}{2}+1,\Delta_2-\frac{d}{2}+1;x,y\right)\Bigg\}.$$

linear combination of 4 fundamental solutions

It is important to verify that the symmetric solution above does not have any unphysical singularity in the

physical region, reproducing the expected behaviour in the large momentum limit $p_3 \gg p_1$

(MM Maglio, CC)

If we define

in terms of F4

$$B(\lambda,\mu)=\left(\frac{a}{c}\right)^{\lambda}\left(\frac{b}{c}\right)^{\mu}\Gamma\left(\frac{\alpha+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{\alpha+\lambda+\mu+\nu}{2}\right)\Gamma(-\lambda)\Gamma(-\mu)\times$$
$$\times\,F_4\left(\frac{\alpha+\lambda+\mu-\nu}{2},\frac{\alpha+\lambda+\mu+\nu}{2};\lambda+1,\mu+1;\frac{a^2}{c^2},\frac{b^2}{c^2}\right),$$

Then one obtains an explictly symmetric expression (Bzowski, McFadden, Slkenderis)

$$\int_0^\infty ds\, s^{\alpha-1} K_\lambda(p_1 s) K_\mu(p_2 s) K_\nu(p_3 s) =$$

$$= \frac{2^{\alpha-4}}{c^\alpha} \left[B(\lambda,\mu) + B(\lambda,-\mu) + B(-\lambda,\mu) + B(-\lambda,-\mu)\right],$$

$$\Phi(p_1,p_2,p_3) = C_{123}\, p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \int_0^\infty dx\, x^{\frac{d}{2}-1}\, K_{\Delta_1-\frac{d}{2}}(p_1\, x)\, K_{\Delta_2-\frac{d}{2}}(p_2\, x)\, K_{\Delta_3-\frac{d}{2}}(p_3\, x)$$

The Bessel functions K_ν satisfy the equations

$$\frac{\partial}{\partial p}\left[p^\beta\, K_\beta(p\, x)\right] = -x\, p^\beta\, K_{\beta-1}(px)$$

$$K_{\beta+1}(x) = K_{\beta-1}(x) + \frac{2\beta}{x} K_\beta(x)$$

Tensor correlators | TJJ and TTT correlators

Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex

MM Maglio, CC 2017

The General 3-Graviton Vertex (TTT) of Conformal Field Theories in Momentum Space in $d = 4$

MM Maglio, CC 2018

TTT in CFT: Trace Identities and the Conformal Anomaly Effective Action

MM Maglio, E Mottola, CC 2018

connection with the nonlocal anomaly action

Bzowski, McFadden Skenderis, 2013 BMS

The general reconstruction method is due to BMS

We have provided a simplified analysis of the TJJ and TTT by matching the general reconstruction to free field theory

The general (nonperturbative) result obtained for this and other correlators can be simplified by choosing 3 independent field theory solutions which are conformal at 1-loop (e.g. QED, QCD)

The simplification is drastic and allows to avoid all the complications related to the renomalization of the 3K integrals.

How to proceed

the TTT Case

$$\langle T^{\mu\nu}(x)\rangle = \frac{2}{\sqrt{g(x)}}\frac{\delta\mathcal{W}}{\delta g_{\mu\nu}(x)}$$

$$\mathcal{W} = \frac{1}{\mathcal{N}}\int \mathcal{D}\Phi\; e^{-S}$$

$$\langle T^{\mu_1\nu_1}(x_1)\dots T^{\mu_n\nu_n}(x_n)\rangle \equiv \left[\frac{2}{\sqrt{g(x_1)}}\dots\frac{2}{\sqrt{-g(x_n)}}\frac{\delta^n\mathcal{W}}{\delta g_{\mu_1\nu_1}(x_1)\dots\delta g_{\mu_n\nu_n}(x_n)}\right]_{flat}$$
$$= 2^n \frac{\delta^n\mathcal{W}}{\delta g_{\mu_1\nu_1}(x_1)\dots\delta g_{\mu_n\nu_n}(x_n)}\bigg|_{flat}$$

$$\begin{aligned}\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle = 8\bigg\{ &-\left\langle \frac{\delta S}{\delta g_{\mu_1\nu_1}(x_1)}\frac{\delta S}{\delta g_{\mu_2\nu_2}(x_2)}\frac{\delta S}{\delta g_{\mu_3\nu_3}(x_3)}\right\rangle \\ &+\left\langle \frac{\delta^2 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)}\frac{\delta S}{\delta g_{\mu_3\nu_3}(x_3)}\right\rangle + \left\langle \frac{\delta^2 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_3\nu_3}(x_3)}\frac{\delta S}{\delta g_{\mu_2\nu_2}(x_2)}\right\rangle \\ &+\left\langle \frac{\delta^2 S}{\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\frac{\delta S}{\delta g_{\mu_1\nu_1}(x_1)}\right\rangle - \left\langle \frac{\delta^3 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\right\rangle\bigg\}\end{aligned}$$

$$\partial_\nu \langle T^{\mu\nu}(x_1) T^{\rho\sigma}(x_2) T^{\alpha\beta}(x_3)\rangle = \Big[\langle T^{\rho\sigma}(x_1)T^{\alpha\beta}(x_3)\rangle \partial^\mu \delta(x_1,x_2) + \langle T^{\alpha\beta}(x_1)T^{\rho\sigma}(x_2)\rangle \partial^\mu \delta(x_1,x_3)\Big]$$
$$- \Big[\delta^{\mu\rho}\langle T^{\nu\sigma}(x_1)T^{\alpha\beta}(x_3)\rangle + \delta^{\mu\sigma}\langle T^{\nu\rho}(x_1)T^{\alpha\beta}(x_3)\rangle\Big]\partial_\nu \delta(x_1,x_2)$$
$$- \Big[\delta^{\mu\alpha}\langle T^{\nu\beta}(x_1)T^{\rho\sigma}(x_2)\rangle + \delta^{\mu\beta}\langle T^{\nu\alpha}(x_1)T^{\rho\sigma}(x_2)\rangle\Big]\partial_\nu \delta(x_1,x_3)\,.$$

$$p_{1\nu_1}\,\langle T^{\mu_1\nu_1}(p_1)\,T^{\mu_2\nu_2}(p_2)\,T^{\mu_3\nu_3}(p_3)\rangle = -p_2^{\mu_1}\,\langle T^{\mu_2\nu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle - p_3^{\mu_1}\,\langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_1+p_3)\rangle$$
$$+ p_{2\alpha}\left[\delta^{\mu_1\nu_2}\,\langle T^{\mu_2\alpha}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle + \delta^{\mu_1\mu_2}\,\langle T^{\nu_2\alpha}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle\right]$$
$$+ p_{3\alpha}\left[\delta^{\mu_1\nu_3}\,\langle T^{\mu_3\alpha}(p_1+p_3)T^{\mu_2\nu_2}(p_2)\rangle + \delta^{\mu_1\mu_3}\,\langle T^{\nu_3\alpha}(p_1+p_3)T^{\mu_2\nu_2}(p_2)\rangle\right].$$

while naive scale invariance gives the traceless condition

$$g_{\mu\nu}\,\langle T^{\mu\nu}\rangle = 0.$$

$$\beta_a(S) = -\frac{3\pi^2}{720}\,, \qquad \beta_b(S) = \frac{\pi^2}{720}\,,$$
$$\beta_a(F) = -\frac{9\pi^2}{360}\,, \qquad \beta_b(F) = \frac{11\pi^2}{720}$$
$$\beta_a(G) = -\frac{18\pi^2}{360}\,, \qquad \beta_b(G) = \frac{31\pi^2}{360}$$

After renormalization this equation is modified by the contribution of the conformal anomaly, by the general expression

$$g_{\mu\nu}(z)\langle T^{\mu\nu}(z)\rangle = \sum_{I=F,S,G} n_I\Big[\beta_a(I)\,C^2(z) + \beta_b(I)\,E(z)\Big] + \frac{\kappa}{4} n_G F^{a\,\mu\nu}\,F^a_{\mu\nu}(z)$$
$$\equiv \mathcal{A}(z,g)\,,$$

$$C^2 = R_{abcd}R^{abcd} - \frac{4}{d-2}R_{ab}R^{ab} + \frac{2}{(d-2)(d-1)}R^2, \qquad E = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

$$\begin{aligned} & g_{\mu_1\nu_1} \langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle \\ & \quad = 4\,\mathcal{A}^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) - 2\,\langle T^{\mu_2\nu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle - 2\,\langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_1+p_3)\rangle \end{aligned}$$

$$\begin{aligned} = 4 & \left[\beta_a\,\left[C^2\right]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) + \beta_b\,\left[E\right]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3)\right] \\ & - 2\,\langle T^{\mu_2\nu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle - 2\,\langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_1+p_3)\rangle\,. \end{aligned}$$

special CWI's take the form

$$\begin{aligned} 0 = K^\kappa \langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle &= \sum_{i=1}^{3} K^\kappa_{i\,scalar}(x_i)\,\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle \\ &+ 2\left(\delta^{\mu_1\kappa}x_{1\rho} - \delta^\kappa_\rho x_1^{\mu_1}\right)\langle T^{\rho\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle + 2\left(\delta^{\nu_1\kappa}x_{1\rho} - \delta^\kappa_\rho x_1^{\nu_1}\right)\langle T^{\mu_1\rho}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle \\ &+ 2\left(\delta^{\mu_2\kappa}x_{2\rho} - \delta^\kappa_\rho x_2^{\mu_2}\right)\langle T^{\mu_1\nu_1}(x_1)T^{\rho\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle + 2\left(\delta^{\nu_2\kappa}x_{2\rho} - \delta^\kappa_\rho x_2^{\nu_2}\right)\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\rho}(x_2)T^{\mu_3\nu_3}(x_3)\rangle \\ &+ 2\left(\delta^{\mu_3\kappa}x_{3\rho} - \delta^\kappa_\rho x_3^{\mu_3}\right)\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\rho\nu_3}(x_3)\rangle + 2\left(\delta^{\nu_3\kappa}x_{3\rho} - \delta^\kappa_\rho x_3^{\nu_3}\right)\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\rho}(x_3)\rangle \end{aligned}$$

$$\sum_{j=1}^{2}\left[2(\Delta_j-d)\frac{\partial}{\partial p_j^\kappa}-2p_j^\alpha\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_j^\kappa}+(p_j)_\kappa\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_{j\alpha}}\right]\langle T^{\mu_1\nu_1}(p_1)\,T^{\mu_2\nu_2}(p_2)\,T^{\mu_3\nu_3}(\bar p_3)\rangle$$
$$+\,2\left(\delta^{\kappa(\mu_1}\frac{\partial}{\partial p_1^{\alpha_1}}-\delta^\kappa_{\alpha_1}\delta^{\lambda(\mu_1}\frac{\partial}{\partial p_1^\lambda}\right)\langle T^{\nu_1)\alpha_1}(p_1)\,T^{\mu_2\nu_2}(p_2)\,T^{\mu_3\nu_3}(\bar p_3)\rangle$$
$$+\,2\left(\delta^{\kappa(\mu_2}\frac{\partial}{\partial p_2^{\alpha_2}}-\delta^\kappa_{\alpha_2}\delta^{\lambda(\mu_2}\frac{\partial}{\partial p_2^\lambda}\right)\langle T^{\nu_2)\alpha_2}(p_2)\,T^{\mu_3\nu_3}(\bar p_3)\,T^{\mu_1\nu_1}(p_1)\rangle=0.$$

Reconstruction in the BMS approach

$$T^{\mu\nu}=t^{\mu\nu}+t^{\mu\nu}_{loc}$$

projectors

$$\pi^\mu_\alpha=\delta^\mu_\alpha-\frac{p^\mu p_\alpha}{p^2},\qquad \tilde\pi^\mu_\alpha=\frac{1}{d-1}\pi^\mu_\alpha$$
$$\Pi^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\left(\pi^\mu_\alpha\pi^\nu_\beta+\pi^\mu_\beta\pi^\nu_\alpha\right)-\frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta},$$
$$\mathcal{I}^{\mu\nu}_\alpha=\frac{1}{p^2}\left[2p^{(\mu}\delta^{\nu)}_\alpha-\frac{p_\alpha}{d-1}(\delta^{\mu\nu}+(d-2)\frac{p^\mu p^\nu}{p^2})\right]$$
$$\mathcal{I}^{\mu\nu}_{\alpha\beta}=\mathcal{I}^{\mu\nu}_\alpha p_\beta=\frac{p_\beta}{p^2}\left(p^\mu\delta^\nu_\alpha+p^\nu\delta^\mu_\alpha\right)-\frac{p_\alpha p_\beta}{p^2}\left(\delta^{\mu\nu}+(d-2)\frac{p^\mu p^\nu}{p^2}\right)$$
$$\mathcal{L}^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\left(\mathcal{I}^{\mu\nu}_{\alpha\beta}+\mathcal{I}^{\mu\nu}_{\beta\alpha}\right)\qquad \tau^{\mu\nu}_{\alpha\beta}=\tilde\pi^{\mu\nu}\delta_{\alpha\beta}$$

transverse traceless sector

$$\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle=\Pi_1{}^{\mu_1\nu_1}_{\alpha_1\beta_1}\Pi_2{}^{\mu_2\nu_2}_{\alpha_2\beta_2}\Pi_3{}^{\mu_3\nu_3}_{\alpha_3\beta_3}\,\langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T^{\alpha_3\beta_3}(p_3)\rangle$$

the tt sectors is parameterised in a specific way

BMS

$$
\begin{aligned}
\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3)\\
&\times\Big[A_1\, p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3}p_1^{\beta_3} + A_2\,\delta^{\beta_1\beta_2}p_2^{\alpha_1}p_3^{\alpha_2}p_1^{\alpha_3}p_1^{\beta_3} + A_2\,(p_1\leftrightarrow p_3)\,\delta^{\beta_2\beta_3}p_3^{\alpha_2}p_1^{\alpha_3}p_2^{\alpha_1}p_2^{\beta_1}\\
&\quad + A_2\,(p_2\leftrightarrow p_3)\,\delta^{\beta_3\beta_1}p_1^{\alpha_3}p_2^{\alpha_1}p_3^{\alpha_2}p_3^{\beta_2} + A_3\,\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}p_1^{\alpha_3}p_1^{\beta_3} + A_3(p_1\leftrightarrow p_3)\,\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1}\\
&\quad + A_3(p_2\leftrightarrow p_3)\,\delta^{\alpha_3\alpha_1}\delta^{\beta_3\beta_1}p_3^{\alpha_2}p_3^{\beta_2} + A_4\,\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3}p_2^{\beta_1}p_3^{\beta_2} + A_4(p_1\leftrightarrow p_3)\,\delta^{\alpha_2\alpha_1}\delta^{\alpha_3\beta_1}p_3^{\beta_2}p_1^{\beta_3}\\
&\quad + A_4(p_2\leftrightarrow p_3)\,\delta^{\alpha_3\alpha_2}\delta^{\alpha_1\beta_2}p_1^{\beta_3}p_2^{\beta_1} + A_5\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}\Big]
\end{aligned}
\tag{5.12}
$$

the entire correlator is reconstructed via

$$
\begin{aligned}
\langle T^{\mu_1\nu_1}\,T^{\mu_2\nu_2}\,T^{\mu_3\nu_3}\rangle &= \langle t^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t_{loc}^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t_{loc}^{\mu_3\nu_3}\rangle\\
&\quad + \langle t_{loc}^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t_{loc}^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t_{loc}^{\mu_3\nu_3}\rangle + \langle t_{loc}^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t_{loc}^{\mu_3\nu_3}\rangle
\end{aligned}
$$

at the same time one solves the diolatation WI

$$
\left(\sum_{j=1}^{3}\Delta_j - 2d - \sum_{j=1}^{2} p_j^\alpha\frac{\partial}{\partial p_j^\alpha}\right)\langle T^{\alpha_1\beta_1}T^{\alpha_2\beta_2}T^{\alpha_3\beta_3}\rangle
$$

the intermediate steps are rather technical

BMS

$$
\begin{aligned}
&K_{13}A_1 = 0 && K_{23}A_1 = 0 \\
&K_{13}A_2 = 8A_1 && K_{23}A_2 = 8A_1 \\
&K_{13}A_2(p_1 \leftrightarrow p_3) = -8A_1 && K_{23}A_2(p_1 \leftrightarrow p_3) = 0 \\
&K_{13}A_2(p_2 \leftrightarrow p_3) = 0 && K_{23}A_2(p_2 \leftrightarrow p_3) = -8A_1 \\
&K_{13}A_3 = 2A_2 && K_{23}A_3 = 2A_2 \\
&K_{13}A_3(p_1 \leftrightarrow p_3) = -2A_2(p_1 \leftrightarrow p_3) && K_{23}A_3(p_1 \leftrightarrow p_3) = 0 \\
&K_{13}A_3(p_2 \leftrightarrow p_3) = 0 && K_{23}A_3(p_2 \leftrightarrow p_3) = -2A_2(p_2 \leftrightarrow p_3) \\
&K_{13}A_4 = -4A_2(p_2 \leftrightarrow p_3) && K_{23}A_4 = -4A_2(p_1 \leftrightarrow p_3) \\
&K_{13}A_4(p_1 \leftrightarrow p_3) = 4A_2(p_2 \leftrightarrow p_3) && K_{23}A_4(p_1 \leftrightarrow p_3) = 4A_2(p_2 \leftrightarrow p_3) - 4A_2 \\
&K_{13}A_4(p_2 \leftrightarrow p_3) = 4A_2(p_1 \leftrightarrow p_3) - 4A_2 && K_{23}A_4(p_2 \leftrightarrow p_3) = 4A_2(p_1 \leftrightarrow p_3) \\
&K_{13}A_5 = 2\left[A_4 - A_4(p_1 \leftrightarrow p_3)\right] && K_{23}A_5 = 2\left[A_4 - A_4(p_2 \leftrightarrow p_3)\right]
\end{aligned}
$$

primary WI's

and some secondary WI's which connect 3- and 2-point functions

The primary can be solved in temrs of 3K integrals and deffine a generalised hypergeometric system of Appell type for F4.

$$C_{31} = -\frac{2}{p_1^2}\left[L_6 A_1 + RA_2 - RA_2(p_2 \leftrightarrow p_3)\right]$$

$$C_{32} = -\frac{1}{p_1^2}\left[L_4\, A_2 + 2p_1^2\, A_2 + 4RA_3 - 2RA_4(p_1 \leftrightarrow p_3)\right]$$

$$C_{33} = -\frac{2}{p_1^2}\left[L_4\, A_2(p_1 \leftrightarrow p_3) - R\, A_4 + RA_4(p_2 \leftrightarrow p_3) + 2p_1^2(A_2(p_2 \leftrightarrow p_3) - A_2)\right]$$

$$C_{34} = -\frac{1}{p_1^2}\left[L_4\, A_2(p_2 \leftrightarrow p_3) - 4R\, A_3(p_2 \leftrightarrow p_3) + 2RA_4(p_1 \leftrightarrow p_3) - 2p_1^2 A_2(p_2 \leftrightarrow p_3)\right]$$

$$C_{35} = -\frac{2}{p_1^2}\left[L_2\, A_3(p_1 \leftrightarrow p_3) + p_1^2(A_4 - A_4(p_2 \leftrightarrow p_3)\right]$$

$$C_{36} = -\frac{1}{p_1^2}\left[L_2\, A_4 + 2R\, A_5 + 8p_1^2 A_3(p_2 \leftrightarrow p_3) - 2p_1^2(A_4 + A_4(p_1 \leftrightarrow p_3))\right]$$

$$C_{37} = -\frac{1}{p_1^2}\left[L_2\, A_4(p_2 \leftrightarrow p_3) - 2R\, A_5 - 8p_1^2 A_3 + 2p_1^2(A_4(p_2 \leftrightarrow p_3) + A_4(p_1 \leftrightarrow p_3))\right]$$

Secondary

$$\mathrm{L}_N = p_1(p_1^2 + p_2^2 - p_3^2)\frac{\partial}{\partial p_1} + 2p_1^2 p_2 \frac{\partial}{\partial p_2} + \left[(2d - \Delta_1 - 2\Delta_2 + N)p_1^2 + (2\Delta_1 - d)(p_3^2 - p_2^2)\right]$$

$$\mathrm{R} = p_1 \frac{\partial}{\partial p_1} - (2\Delta_1 - d)\,.$$

Solutions (MM Maglio, CC)

solutions determined independently either in terms of 3K integrals (BMS) or by direct properties of the signle F4's in the corresponding hypergeometric system (Maglio,CC)

examples

$$A_1 = p_3^{d-6} \sum_{a,b} C_1\, f_1(a,b)\, x^a y^b\, F_4(\alpha(a,b)+3, \beta(a,b)+3; \gamma(a), \gamma'(b); x, y)$$

$$f_1\left(0, \frac{d}{2}\right) = f_1\left(\frac{d}{2}, 0\right) = 1$$

$$f_1(0,0) = -\frac{(d-4)(d-2)}{(d+2)(d+4)}$$

$$f_1\left(\frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma\left(-\frac{d}{2}\right)\Gamma\left(d+3\right)}{2\,\Gamma\left(\frac{d}{2}\right)}$$

$$A_2 = p_3^{d-4} \sum_{ab} x^a y^b \Big[C_2\, f_2(a,b)\, F_4(\alpha+2, \beta+2; \gamma, \gamma'; x, y) + \frac{2\,C_1}{(\beta+2)} f_1(a,b)\, F_4(\alpha+3, \beta+2; \gamma, \gamma'; x, y) \Big].$$

$$f_2\,(0,0) = \frac{d-2}{d+2}$$

$$f_2\left(\frac{d}{2}, 0\right) = f_2\left(0, \frac{d}{2}\right) = 1$$

$$f_2\left(\frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma(-d/2)\Gamma(d+2)}{\Gamma(d/2)}$$

the solution is fixed up to 5 independent constants, depending oin the spacetime dimension d

Lagrangian realizations and reconstruction

MM Maglio, CC

$$S_{scalar} = \frac{1}{2}\int d^dx\,\sqrt{-g}\,\left[g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \chi\,R\,\phi^2\right]$$

$$S_{fermion} = \frac{i}{2}\int d^dx\,e\,e^\mu_a\,\left[\bar{\psi}\gamma^a(D_\mu\psi) - (D_\mu\bar{\psi})\gamma^a\psi\right],$$

in d=4 we need 3 sectors to perform the matching

$$S_M = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,F^{\mu\nu}F_{\mu\nu},$$

$$S_{gf} = -\frac{1}{\xi}\int d^4x\,\sqrt{-g}\,(\nabla_\mu A^\mu)^2,$$

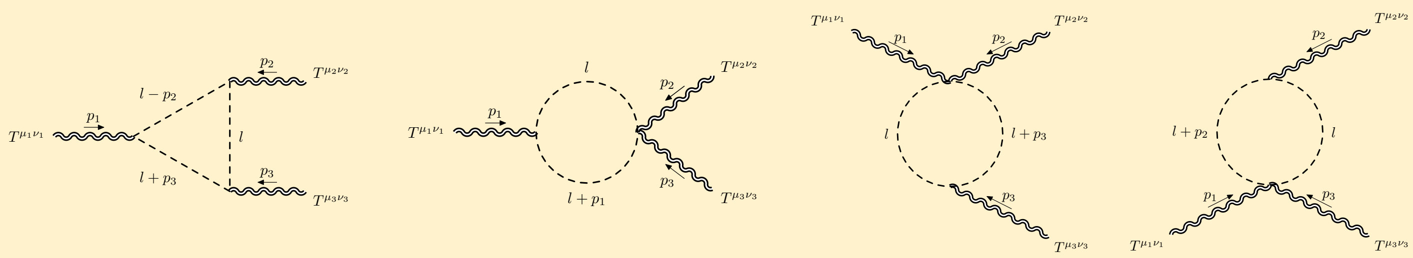
$$S_{gh} = \int d^4x\,\sqrt{-g}\,\partial^\mu\bar{c}\,\partial_\mu\,c.$$

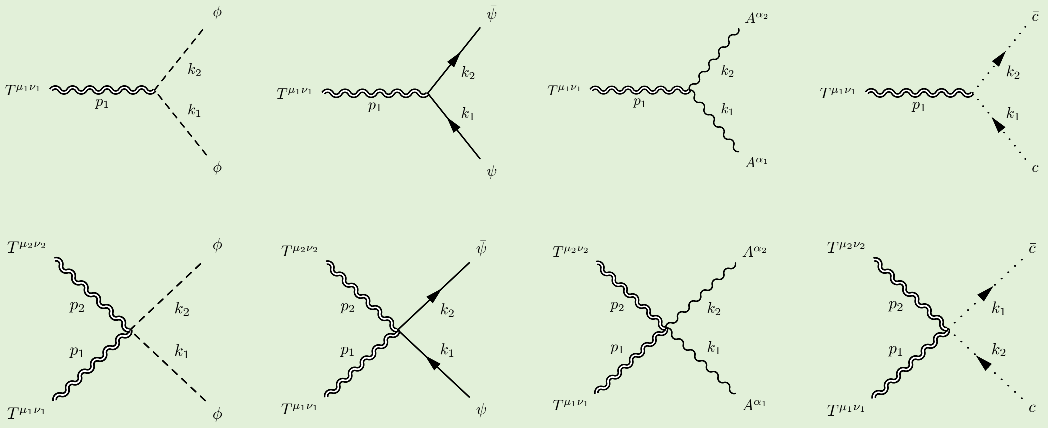
$$D_\mu = \partial_\mu + \Gamma_\mu = \partial_\mu + \frac{1}{2}\Sigma^{ab}\,e^\sigma_a\nabla_\mu\,e_{b\,\sigma}.$$

The Σ^{ab} are the generators of the Lorentz group in the spin 1/2 representation.

$$S_{abelian} = S_M + S_{gf} + S_{gh}$$

where $\chi = (d-2)/(4d-4)$ for a conformally coupled scalar in d dimensions, and R is the Ricci scalar. e^a_μ is the vielbein and e its determinant, with the covariant derivative D_μ given by scalar





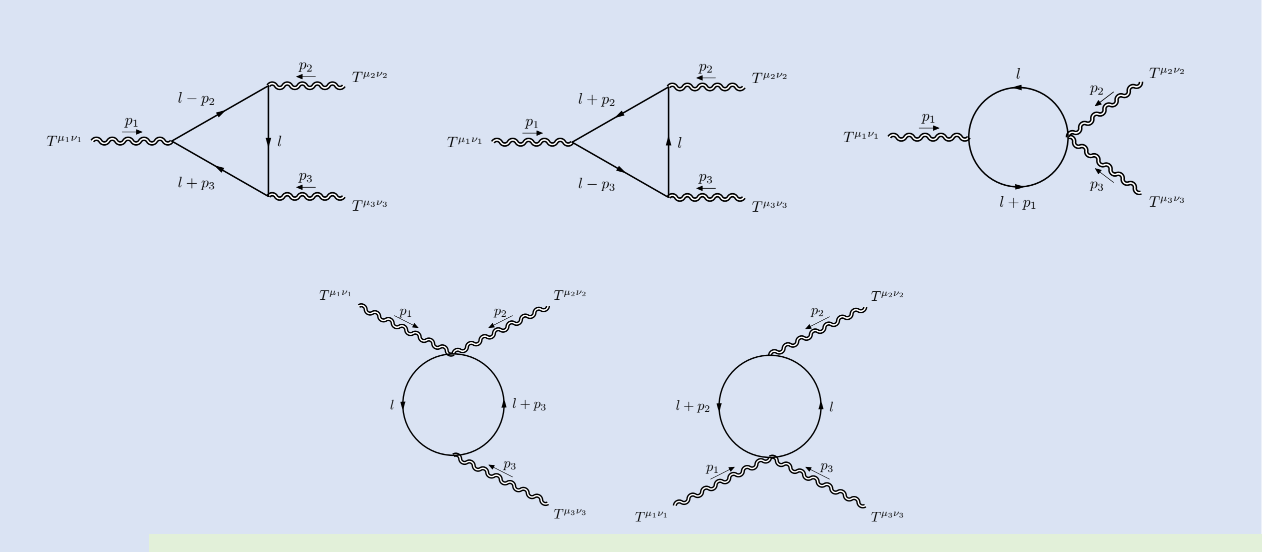
vertices

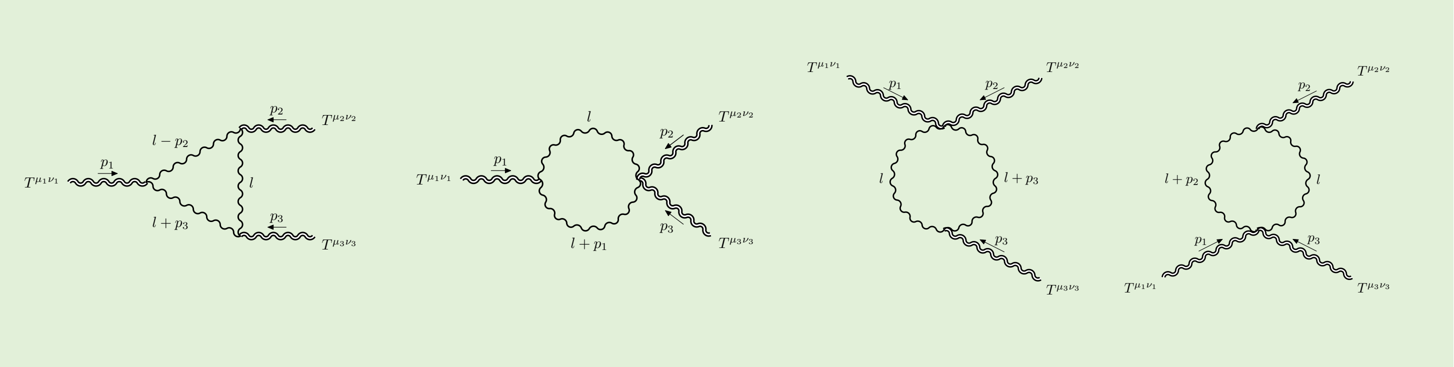
triangle + 3 bubbles

$$\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_S = -V_S^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) + \sum_{i=1}^{3} W_{S,i}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3)$$

$$\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle_S = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3)$$
$$\times\left[-V_S^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3) + \sum_{i=1}^{3} W_{S,i}^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3)\right]$$

tt sector





Can be matched to the complete solution of the CWI's in d=3 we need two sectors (scalar and fermion)

$$A_1^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)}{60(p_1+p_2+p_3)^6}\Big[p_1^3+6p_1^2(p_3+p_2)+(6p_1+p_2+p_3)\big((p_2+p_3)^2+3p_2p_3\big)\Big]$$

$$A_2^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)}{60(p_1+p_2+p_3)^6}\Big[4p_3^2\big(7(p_1+p_2)^2+6p_1p_2\big)+20p_3^3(p_1+p_2)+4p_3^4 + 3(5p_3+p_1+p_2)(p_1+p_2)\big((p_1+p_2)^2+p_1p_2\big)\Big] + \frac{\pi^3\,n_F}{3(p_1+p_2+p_3)^4}\Big[p_1^3+4p_1^2(p_2+p_3)+(4p_1+p_2+p_3)\big((p_2+p_3)^2+p_2p_3\big)\Big]$$

$$A_3^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)\,p_3^2}{240(p_1+p_2+p_3)^4}\Big[28p_3^2(p_1+p_2)+3p_3\big(11(p_1+p_2)^2+6p_1\,p_2\big)+7p_3^3 + 12(p_1+p_2)\big((p_1+p_2)^2+p_1p_2\big)\Big] + \frac{\pi^3 n_F\,p_3^2}{6(p_1+p_2+p_3)^3}\Big[3p_2(p_1+p_2)+2\big((p_1+p_2)^2+p_1p_2\big)+p_3^2\Big] - \frac{\pi^3(n_s+4n_F)}{16(p_1+p_2+p_3)^2}\Big[p_1^3+2p_1^2(p_2+p_3)+(2p_1+p_2+p_3)\big((p_2+p_3)^2-p_2p_3\big)\Big]$$

$$A_4^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)}{120(p_1+p_2+p_3)^4}\Big[(4p_3+p_1+p_2)\big(3(p_1+p_2)^4-3(p_1+p_2)^2p_1p_2+4p_1^2p_2^2\big) + 9p_3^2(p_1+p_2)\big((p_1+p_2)^2-3p_1p_2\big)-3p_3^5-12p_3^4(p_1+p_2)-9p_3^3\big((p_1+p_2)^2+2p_1p_2\big)\Big] + \frac{\pi^3\,n_F}{6(p_1+p_2+p_3)^3}\Big[(p_1+p_2)\big((p_1+p_2)^2-p_1p_2\big)(p_1+p_2+3p_3)-p_3^4-3p_3^3(p_1+p_2) - 6p_1p_2p_3^2\Big] - \frac{\pi^3(n_s+4n_F)}{8(p_1+p_2+p_3)^2}\Big[p_1^3+2p_1^2(p_2+p_3)+(2p_1+p_2+p_3)\big((p_2+p_3)^2-p_2p_3\big)\Big]$$

$$
\begin{aligned}
A_5^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)}{240(p_1+p_2+p_3)^3}\Big[& -3(p_1+p_2+p_3)^6 + 9(p_1+p_2+p_3)^4(p_1p_2+p_2p_3+p_1p_3) \\
& + 12(p_1+p_2+p_3)^2(p_1p_2+p_2p_3+p_3p_1)^2 - 33(p_1+p_2+p_3)^2 p_1p_2p_3 \\
& + 12(p_1+p_2+p_3)(p_1p_2+p_2p_3+p_1p_3)p_1p_2p_3 + 8p_1^2p_2^2p_3^2\Big] \\
& + \frac{\pi^3 n_F}{12(p_1+p_2+p_3)^2}\Big[-(p_1+p_2+p_3)^5 + 3(p_1+p_2+p_3)^3(p_1p_2+p_2p_3+p_1p_3) \\
& + 4(p_1+p_2+p_3)(p_1p_2+p_2p_3+p_1p_3)^2 - 11(p_1+p_2+p_3)^2 p_1p_2p_3 \\
& + 4(p_1p_2+p_2p_3+p_1p_3)p_1p_2p_3\Big] - \frac{\pi^3(n_S+4n_F)}{16}\Big[p_1^3+p_2^3+p_3^3\Big] \qquad (8.2
\end{aligned}
$$

$$
\alpha_1 = \frac{\pi^3(n_S-4n_F)}{480}, \qquad \alpha_2 = \frac{\pi^3 n_F}{6}, \qquad c_T = \frac{3\pi^{5/2}}{128}(n_S+4n_F),
$$

matched to the BMS solution for d=3

d=5

+similar

$$A_1^{d=5}(p_1,p_2,p_3) = \frac{\pi^4(n_S-4n_F)}{560(p_1+p_2+p_3)^7}\Big[(p_1+p_2+p_3)^2\big((p_1+p_2+p_3)^4+(p_1+p_2+p_3)^2(p_1p_2+p_2p_3+p_1p_3) + (p_1p_2+p_2p_3+p_1p_3)^2\big)+(p_1+p_2+p_3)\big((p_1+p_2+p_3)^2+5(p_1p_2+p_2p_3+p_1p_3)\big)p_1p_2p_3+10p_1^2p_2^2p_3^2\Big].$$

$$\alpha_1 = \frac{\pi^4(n_S-4n_F)}{560\times 72}, \qquad \alpha_2 = \frac{\pi^4 n_F}{240}, \qquad c_T = \frac{5\pi^{7/2}}{1024}(n_S+8n_F).$$

matched to BMS

in d=3 and 5 there are no anomalies

in d=4

The correlator in $d=4$ and the trace anomaly

we need 3 sectors and we need to renormalize because the gauge sector is not finite

$$\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_G = -V_G^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) + \sum_{i=1}^{3} W_{G,i}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3)$$

$$\begin{aligned}\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle_G &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3)\\ &\times\left[-V_G^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3) + \sum_{i=1}^{3} W_{G,i}^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3)\right]\end{aligned}$$

$$\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle = \sum_{I=F,G,S} n_I\,\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_I$$

A1 is finite

$$A_2^{Div} = \frac{\pi^2}{45\,\varepsilon}\left[26n_G - 7n_F - 2n_S\right]$$

$$A_3^{Div} = \frac{\pi^2}{90\,\varepsilon}\left[3(s+s_1)\big(6n_F + n_S + 12n_G\big) + s_2(11n_F + 62n_G + n_S)\right]$$

$$A_4^{Div} = \frac{\pi^2}{90\,\varepsilon}\left[(s+s_1)\big(29n_F + 98n_G + 4n_S\big) + s_2(43n_F + 46n_G + 8n_S)\right]$$

$$A_5^{Div} = \frac{\pi^2}{180\,\varepsilon}\Big\{n_F\left(43s^2 - 14s(s_1+s_2) + 43s_1^2 - 14s_1s_2 + 43s_2^2\right)$$

Renormalization of the TTT

$$S_{count} = -\frac{1}{\varepsilon} \sum_{I=F,S,G} n_I \int d^d x \sqrt{-g} \Big(\beta_a(I)\, C^2 + \beta_b(I)\, E \Big)$$

$$\langle T^{\mu_1\nu_1}(p_1) T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_3) \rangle_{count} =$$

$$= -\frac{1}{\varepsilon} \sum_{I=F,S,G} n_I \Big(\beta_a(I)\, V_{C^2}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) + \beta_b(I)\, V_E^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) \Big)$$

$$V_{C^2}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) = 8 \int d^d x_1\, d^d x_2\, d^d x_3\, d^d x \left(\frac{\delta^3(\sqrt{-g}C^2)(x)}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)} \right)_{flat} e^{-i(p_1\, x_1 + p_2\, x_2 + p_3\, x_3)}$$

$$\equiv 8 \left[\sqrt{-g}\, C^2 \right]^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} (p_1,p_2,p_3)$$

Anomalous CWI's in QED (MM Maglio,CC)

$$\mathrm{K}_{13}A_3^{Ren} = 2A_2^{Ren} - \frac{2\pi^2}{45}\,(7n_F - 26n_G + 2n_S)$$
$$\mathrm{K}_{23}A_3^{Ren} = 2A_2^{Ren} - \frac{2\pi^2}{45}\,(7n_F - 26n_G + 2n_S)$$

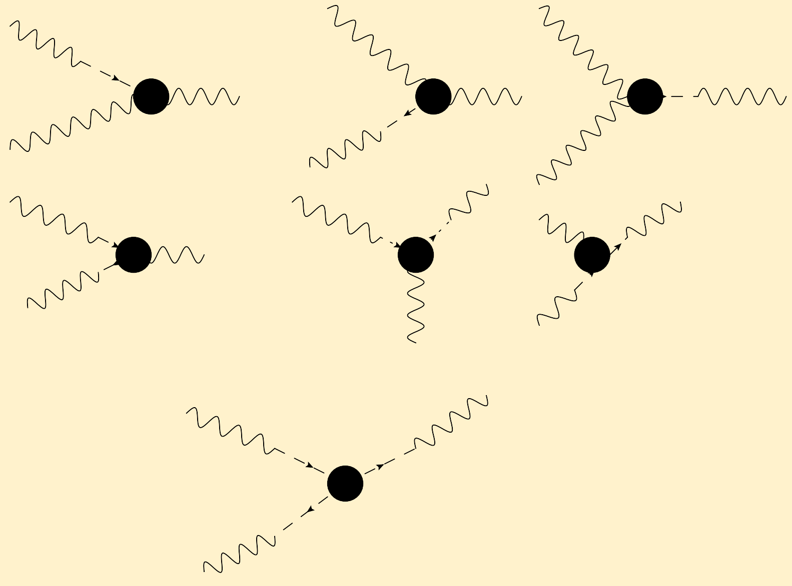
$$\mathrm{K}_{13}A_4^{Ren} = -4A_2^{Ren}(p_2 \leftrightarrow p_3) + \frac{4\pi^2}{45}\,(7n_F - 26n_G + 2n_S)$$
$$\mathrm{K}_{23}A_4^{Ren} = -4A_2^{Ren}(p_1 \leftrightarrow p_3) + \frac{4\pi^2}{45}\,(7n_F - 26n_G + 2n_S)$$

$$\mathrm{K}_{13}A_5^{Ren} = 2\left[A_4^{Ren} - A_4^{Ren}(p_1 \leftrightarrow p_3)\right] - \frac{4\pi^2}{9}(s - s_2)\,(5n_F + 2n_G + n_s)$$
$$\mathrm{K}_{23}A_5^{Ren} = 2\left[A_4^{Ren} - A_4^{Ren}(p_2 \leftrightarrow p_3)\right] - \frac{4\pi^2}{9}(s_1 - s_2)\,(5n_F + 2n_G + n_s)$$

one needs also to investigate the

Secondary anomalous CWI's from free field theory

$$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle_{Ren} = \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}t^{\mu_3\nu_3}\rangle_{Ren} + \langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle_{Ren\,l\,t} + \langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle_{anomaly}$$



(a)

the anomaly part

$$
\begin{aligned}
&\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} = \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)}{3\,p_1^2}\langle T(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} \\
&+ \frac{\hat{\pi}^{\mu_2\nu_2}(p_2)}{3\,p_2^2}\langle T^{\mu_1\nu_1}(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} + \frac{\hat{\pi}^{\mu_3\nu_3}(p_3)}{3\,p_3^2}\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T(p_3)\rangle_{anomaly} \\
&- \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_2\nu_2}(p_2)}{9\,p_1^2p_2^2}\langle T(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} - \frac{\hat{\pi}^{\mu_2\nu_2}(p_2)\hat{\pi}^{\mu_3\nu_3}(p_2)}{9p_2^2p_3^2}\langle T^{\mu_1\nu_1}(p_1)T(p_2)T(p_3)\rangle_{anomaly} \\
&- \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_3\nu_3}(\bar{p}_3)}{9p_1^2p_3^2}\langle T(p_1)T^{\mu_2\nu_2}(p_2)T(p_3)\rangle_{anomaly} + \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_2\nu_2}(p_2)\hat{\pi}^{\mu_3\nu_3}(\bar{p}_3)}{27p_1^2p_2^2p_3^2}\langle T(p_1)T(p_2)T(\bar{p}_3)\rangle_{anomaly}.
\end{aligned}
$$

$$^{(\mathcal{A})}\mathcal{S}_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) = \frac{8}{3}\Big\{\pi^{\mu_1\nu_1}(p_1)\,\Big[b'E^{(2)} + b(C^2)^{(2)}\Big]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) + (\text{cyclic})\Big\}$$
$$-\frac{16b'}{9}\Big\{\pi^{\mu_1\nu_1}(p_1)\,Q^{\mu_2\nu_2}(p_1,p_2,p_3)\,\pi^{\mu_3\nu_3}(p_3) + (\text{cyclic})\Big\}$$
$$+\frac{16b'}{27}\,\pi^{\mu_1\nu_1}(p_1)\,\pi^{\mu_2\nu_2}(p_2)\,\pi^{\mu_3\nu_3}(p_3)\,\Big\{p_3^2\,p_1\cdot p_2 + (\text{cyclic})\Big\}$$

3-point anomaly vertex

TTT

$$Q^{\mu_2\nu_2}(p_1,p_2,p_3) \equiv p_{1\mu}\,[R^{\mu\nu}]^{\mu_2\nu_2}(p_2)\,p_{3\nu}$$
$$= \frac{1}{2}\Big\{(p_1\cdot p_2)(p_2\cdot p_3)\,\eta^{\mu_2\nu_2} + p_2^2\,p_1^{(\mu_2}\,p_3^{\nu_2)} - (p_2\cdot p_3)\,p_1^{(\mu_2}\,p_2^{\nu_2)} - (p_1\cdot p_2)\,p_2^{(\mu_2}\,p_3^{\nu_2)}\Big\}$$

pieces

$$\big[E^{(2)}\big]^{\mu_i\nu_i\mu_j\nu_j} = \big[R^{(1)}_{\mu\alpha\nu\beta}R^{(1)\mu\alpha\nu\beta}\big]^{\mu_i\nu_i\mu_j\nu_j} - 4\,\big[R^{(1)}_{\mu\nu}R^{(1)\mu\nu}\big]^{\mu_i\nu_i\mu_j\nu_j} + \big[\big(R^{(1)}\big)^2\big]^{\mu_i\nu_i\mu_j\nu_j}$$
$$\big[(C^2)^{(2)}\big]^{\mu_i\nu_i\mu_j\nu_j} = \big[R^{(1)}_{\mu\alpha\nu\beta}R^{(1)\mu\alpha\nu\beta}\big]^{\mu_i\nu_i\mu_j\nu_j} - 2\,\big[R^{(1)}_{\mu\nu}R^{(1)\mu\nu}\big]^{\mu_i\nu_i\mu_j\nu_j} + \frac{1}{3}\,\big[(R^{(1)})^2\big]^{\mu_i\nu_i\mu_j\nu_j}$$

exactly reproduced by the

$$\mathcal{S}^{NL}_{\text{anom}}[g] = \frac{1}{4}\int dx\sqrt{-g_x}\,\Big(E - \tfrac{2}{3}\Box R\Big)_x \int dx'\sqrt{-g_{x'}}\,D_4(x,x')\Big[\frac{b'}{2}\,\big(E - \tfrac{2}{3}\Box R\big) + b\,C^2\Big]_{x'}$$

Riegert action

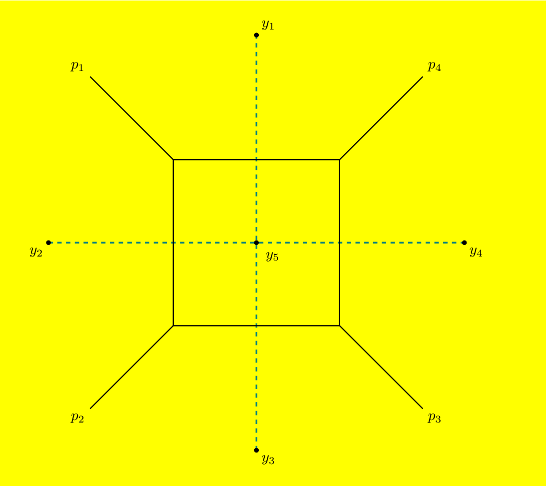
$$\Delta_4 \equiv \nabla_\mu\left(\nabla^\mu\nabla^\nu + 2R^{\mu\nu} - \tfrac{2}{3}Rg^{\mu\nu}\right)\nabla_\nu = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \tfrac{2}{3}R\Box + \tfrac{1}{3}(\nabla^\mu R)\nabla_\mu$$

$$D_4(x,x') = (\Delta_4^{-1})_{xx'}$$

4-point functions (MM Maglio, CC)

On Some Hypergeometric Solutions of the Conformal Ward Identities of Scalar 4-point Functions in Momentum Space

$$
\begin{aligned}
C_{13} = \Bigg\{ & \frac{\partial^2}{\partial p_1^2} + \frac{(d-2\Delta_1+1)}{p_1}\frac{\partial}{\partial p_1} - \frac{\partial^2}{\partial p_3^2} - \frac{(d-2\Delta_3+1)}{p_3}\frac{\partial}{\partial p_3} \\
& + \frac{1}{s}\frac{\partial}{\partial s}\left(p_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2} - p_3\frac{\partial}{\partial p_3} - p_4\frac{\partial}{\partial p_4}\right) + \frac{(\Delta_3+\Delta_4-\Delta_1-\Delta_2)}{s}\frac{\partial}{\partial s} \\
& + \frac{1}{t}\frac{\partial}{\partial t}\left(p_1\frac{\partial}{\partial p_1} + p_4\frac{\partial}{\partial p_4} - p_2\frac{\partial}{\partial p_2} - p_3\frac{\partial}{\partial p_3}\right) + \frac{(\Delta_2+\Delta_3-\Delta_1-\Delta_4)}{t}\frac{\partial}{\partial t} \\
& + \frac{(p_1^2-p_3^2)}{st}\frac{\partial^2}{\partial s \partial t}\Bigg\}\Phi(p_1,p_2,p_3,p_4,s,t) = 0.
\end{aligned}
$$



+ ....

dual conformal symmetry

$$k = y_{51}, \qquad p_1 = y_{12}, \qquad p_2 = y_{23}, \qquad p_3 = y_{34}$$

$$\Phi_{Box}(p_1,p_2,p_3,p_4) = \int \frac{d^d k}{k^2(k+p_1)^2(k+p_1+p_2)^2(k+p_1+p_2+p_3)^2}$$

by requiring conformal invariance in momentum/coordinate space and in dual coordinate space

one btains a unique solution

DCC solutions (dual conformal/conformal) probably related ti a Yangian symmetry

$$\begin{cases}\left[\frac{\partial^2}{\partial p_1^2}+\frac{(d-2\Delta+1)}{p_1}\frac{\partial}{\partial p_1}-\frac{\partial^2}{\partial p_3^2}-\frac{(d-2\Delta+1)}{p_3}\frac{\partial}{\partial p_3}+\frac{(p_1^2-p_3^2)}{st}\frac{\partial^2}{\partial s\partial t}\right]I_{\tilde{\alpha}\{\beta_1,\beta_2,\beta_3\}}=0\\ \left[\frac{\partial^2}{\partial p_2^2}+\frac{(d-2\Delta+1)}{p_2}\frac{\partial}{\partial p_2}-\frac{\partial^2}{\partial p_4^2}-\frac{(d-2\Delta+1)}{p_4}\frac{\partial}{\partial p_4}+\frac{(p_2^2-p_4^2)}{st}\frac{\partial^2}{\partial s\partial t}\right]I_{\tilde{\alpha}\{\beta_1,\beta_2,\beta_3\}}=0\\ \left[\frac{\partial^2}{\partial p_3^2}+\frac{(d-2\Delta+1)}{p_3}\frac{\partial}{\partial p_3}-\frac{\partial^2}{\partial p_4^2}-\frac{(d-2\Delta+1)}{p_4}\frac{\partial}{\partial p_4}+\frac{(p_2^2-p_1^2)}{st}\frac{\partial^2}{\partial s\partial t}\right]I_{\tilde{\alpha}\{\beta_1,\beta_2,\beta_3\}}=0\end{cases}$$

new hypergeometric systems

of variables

$$\begin{aligned}&\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle=\\ &=\sum_{a,b}c(a,b)\Bigg[(s^2\,t^2)^{\Delta-\frac{3}{4}d}\left(\frac{p_1^2p_3^2}{s^2t^2}\right)^a\left(\frac{p_2^2p_4^2}{s^2t^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_1^2p_3^2}{s^2t^2},\frac{p_2^2p_4^2}{s^2t^2}\right)\\ &\quad+\,(s^2\,u^2)^{\Delta-\frac{3}{4}d}\left(\frac{p_2^2p_3^2}{s^2u^2}\right)^a\left(\frac{p_1^2p_4^2}{s^2u^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_2^2p_3^2}{s^2u^2},\frac{p_1^2p_4^2}{s^2u^2}\right)\\ &\quad+\,(t^2\,u^2)^{\Delta-\frac{3}{4}d}\left(\frac{p_1^2p_2^2}{t^2u^2}\right)^a\left(\frac{p_3^2p_4^2}{t^2u^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_1^2p_2^2}{t^2u^2},\frac{p_3^2p_4^2}{t^2u^2}\right)\Bigg]\end{aligned}$$

particular solutons of these systems are Lauricella functions

$$\left\{ \begin{array}{l} x_j(1-x_j)\dfrac{\partial^2 F}{\partial x_j^2} + \displaystyle\sum_{s\neq j} x_r \sum_{\text{for } r=j} x_s \dfrac{\partial^2 F}{\partial x_r \partial x_s} + [\gamma_j - (\alpha+\beta+1)x_j]\dfrac{\partial F}{\partial x_j} - (\alpha+\beta+1)\sum_{k\neq j} x_k \dfrac{\partial F}{\partial x_k} - \alpha\,\beta\,F = 0 \\ (j=1,2,3) \end{array} \right.$$

$$x = \frac{p_1^2}{p_4^2}, \quad y = \frac{p_2^2}{p_4^2}, \quad z = \frac{p_3^2}{p_4^2}$$

$$\begin{aligned}
S_1(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= F_C\big(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z\big), \\
S_2(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= x^{1-\gamma}\, F_C\big(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\gamma',\gamma'',x,y,z\big)\,, \\
S_3(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= y^{1-\gamma'}\, F_C\big(\alpha-\gamma'+1,\beta-\gamma'+1,\gamma,2-\gamma',\gamma'',x,y,z\big)\,, \\
S_4(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= z^{1-\gamma''}\, F_C\big(\alpha-\gamma''+1,\beta-\gamma''+1,\gamma,\gamma',2-\gamma'',x,y,z\big), \\
S_5(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= x^{1-\gamma}y^{1-\gamma'}\, F_C\big(\alpha-\gamma-\gamma'+2,\beta-\gamma-\gamma'+2,2-\gamma,2-\gamma',\gamma'',x,y,z\big)\,, \\
S_6(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= x^{1-\gamma}z^{1-\gamma''}\, F_C\big(\alpha-\gamma-\gamma''+2,\beta-\gamma-\gamma''+2,2-\gamma,\gamma',2-\gamma'',x,y,z\big)\,, \\
S_7(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= y^{1-\gamma'}z^{1-\gamma''}\, F_C\big(\alpha-\gamma'-\gamma''+2,\beta-\gamma'-\gamma''+2,\gamma,2-\gamma',2-\gamma'',x,y,z\big)\,, \\
S_8(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= x^{1-\gamma}y^{1-\gamma'}z^{1-\gamma''} \\
&\quad \times\, F_C\big(\alpha-\gamma-\gamma'-\gamma''+2,\beta-\gamma-\gamma'-\gamma''+2,2-\gamma,2-\gamma',2-\gamma'',x,y,z\big)\,.
\end{aligned}$$

Giuseppe Lauricella (1867–1913)

Lauricella lived in Pisa and Sicily, he was sicilian

$$I_{\alpha-1\{\nu_1,\nu_2,\nu_3,\nu_4\}}(a_1,a_2,a_3,a_4) = \int_0^\infty dx\, x^{\alpha-1} \prod_{i=1}^{4}(a_i)^{\nu_i}\, K_{\nu_i}(a_i\, x)$$

4K integrals (introduced in Maglio, CC 2019)

Conclusions

CFT in momentum space is a fast developing field

Many connections to phenomenology

3 point functions understood

$$\frac{d\alpha_s}{d\,logQ^2} = -\frac{\beta_0}{4\pi}\alpha_s^2$$

PHASES OF A GAUGE THEORY

⊗ Yang-Mills theory

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \log(-Q^2/\Lambda^2)} \qquad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$$

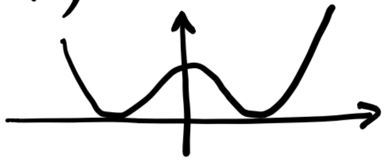
$$\frac{d\alpha_s(Q^2)}{d\log(-Q^2/\Lambda^2)} = -\frac{\beta_0}{4\pi}\alpha_s^2 \qquad \beta(\alpha_s) = -\frac{\beta_0}{4\pi}\alpha_s^2$$

negative $\beta(\alpha)$ $\Rightarrow$ confinement

⊗ Abelian theory e.g. scalar SSB electrodynamics

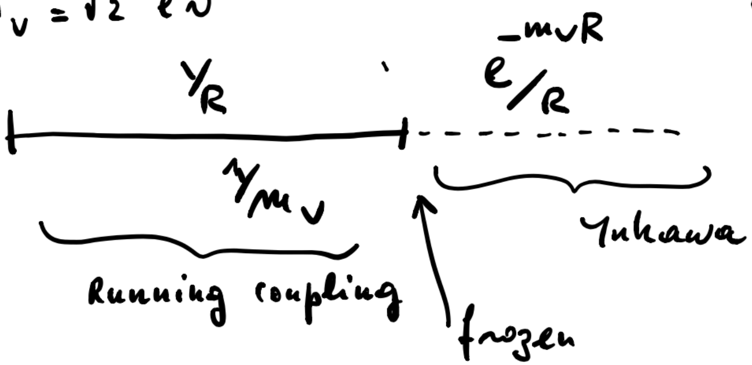
$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - U(\phi)$$

$\langle\phi\rangle = v$



$D_\mu = \partial_\mu - ieA_\mu$

$A_\mu \to m_V = \sqrt{2}\, e v$



Ordinary QED (with fermions)

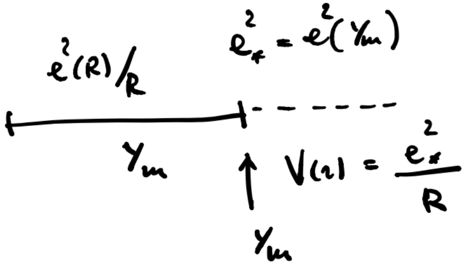
m = fermion mass

$$V(r) = \frac{e^2(R)}{R}$$

$$e^2(R) \sim \frac{1}{\ln R}$$

log running in $0 < R < 1/m$

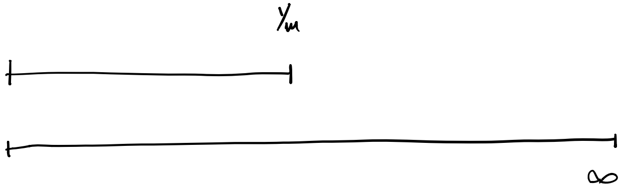
frozen at $R = 1/m$



Coulomb phase

IR free phase

$m = 0$



$$V(r) = \frac{e^2(R)}{R} \qquad e(R) \sim \frac{1}{\log R}$$

no charge at ∞ distance (Landau zero-charge phase) conformal phase

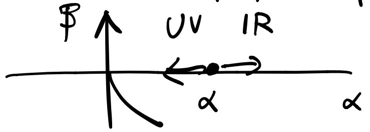
Conformal phase

$$T^{\mu}{}_{\mu} = \beta(\alpha)\, G^{a}_{\mu\nu} G^{a}_{\mu\nu}$$

$$\beta(\alpha) = -\beta_0 \frac{\alpha^2}{2\pi} - \beta_1 \frac{\alpha^3}{4\pi^2} \qquad \alpha = \frac{g^2}{4\pi}$$

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f \qquad \beta_1 = \frac{17}{3} N_c^2 - \frac{n_f}{6 N_c}\left(13 N_c^2 - 3\right)$$

$$\beta(\alpha) = \frac{d\alpha(\mu)}{\partial \log \mu} \qquad \beta \stackrel{\alpha}{=} -\beta_0 < 0 \Rightarrow \text{asymptotic freedom}$$



now take a # of flavours $\qquad$ border value $\underline{16}$ $\quad SU(3)_c$

$$0 < \nu \ll \overbrace{\frac{11}{2} N_c}$$

$$N_f = \frac{11}{2} N_c - \nu$$

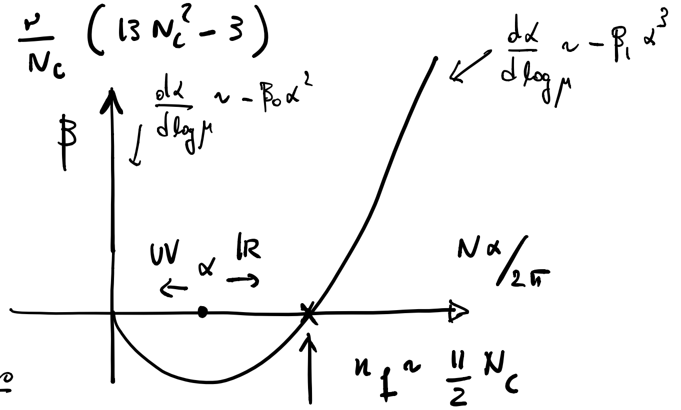
then

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3}\overbrace{\left(\frac{11}{2} N_c - \nu\right)}^{n_f} = \frac{2}{3}\nu \quad \text{(small)}$$

$$\beta_1 = -\frac{25}{4} N_c^2 + \frac{11}{4} + \frac{1}{6}\frac{\nu}{N_c}\left(13 N_c^2 - 3\right)$$

$$\boxed{\beta_1 < 0 \qquad \frac{d\alpha}{d\log\mu} \sim -\beta_1 \alpha^3}$$

if β_1 dominates



Condition for equality between the two contributions $\qquad \alpha = \frac{2\pi\beta_0}{-\beta_1}$

$$\frac{8}{75}\frac{\nu}{N} \ll 1$$

$$\beta(\alpha) = -\beta_0 \frac{\alpha^2}{2\pi} - \beta_1 \frac{\alpha^3}{4\pi^2} + \cdots$$

$$\frac{\beta_0 \alpha^2}{2\pi} = \frac{-\beta_1 \alpha^3}{4\pi^2} \Rightarrow \alpha = 2\pi \frac{\beta_0}{-\beta_1} = 2\pi \frac{\overbrace{\frac{11}{3} N_c}^{\beta_0}}{\frac{25}{4} N_c^2} = \underbrace{\frac{22}{25}\left(\frac{4}{3}\right)}_{\sim 1} \frac{\pi}{N_c}$$

$$\boxed{\frac{N\alpha^*}{\pi} \sim 1}$$

the second term takes over but α^* can be large, strongly coupled phase

Now choose $N_f \sim \frac{11}{2} N_c$, then from $\boxed{\alpha^* = 2\pi \frac{\beta_0}{(-\beta_1)}}$ $\quad \beta_0 = \frac{2}{3}\nu$

and we discover that

$$\beta_1 = \frac{-25}{4}N^2 + \frac{11}{4} + \frac{1}{6}\frac{\nu}{N}(13N^2-3)$$

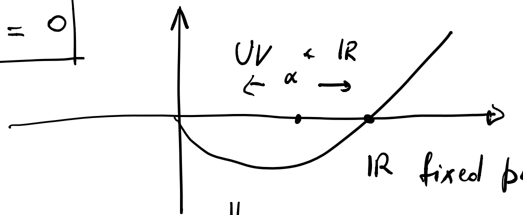
$$\frac{N\alpha^*}{2\pi} = \frac{N\beta_0}{(-\beta_1)} = \frac{N(\frac{2}{3}\nu)}{\frac{25}{4}N^2\Big[\underbrace{1 - \frac{11}{4}\frac{4}{25N^2} - \frac{1}{6}\frac{\nu}{N}\frac{4}{25N^2}\left(13N^2-3\right)}_{f(\nu,N)}\Big]}$$

$$= \frac{8}{75}\frac{\nu}{N f(\nu,N)}$$

then $\boxed{\frac{N\alpha^*}{2\pi} \ll 1}$

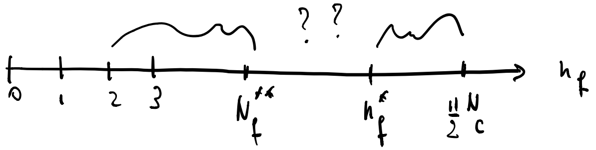
Therefore, if $n_f^* \simeq \frac{11}{2}N \simeq 16$, then the 2nd term catches up with the 1st one prematurely

and $\boxed{\beta(\alpha^*) = 0}$

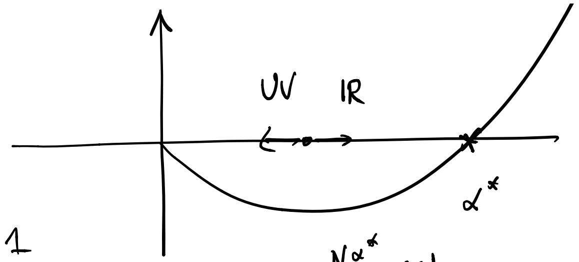


⇓

Conformal phase



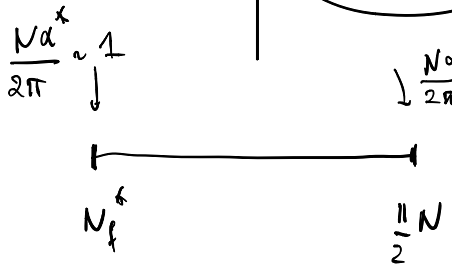
in the conformal window α^* is small,



but α^* stays small

$N_f^* \le n_f \le \frac{11}{2}N_c$

Conformal window



We have a theory which is conformal in the IR

but it can be → weakly or → strongly coupled

$\boxed{\text{depending on } n_f}$