Slow passage through resonance for a weakly nonlinear dispersive waves

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Outlines

This lecture is a review of our works concerning the generation of solitons by resonance in nonlinear equations. Main subjects of the review

- perturbed nonlinear equations;
- weak resonances;
- generation of solitons as envelope function for packet of waves;
- connection formulas for parameters of packets and exciting force.
Examples

Let us consider the NLSE perturbed by the small driving force.

\[ i\partial_t \Psi + \partial_x^2 \Psi + |\Psi|^2 \Psi = \varepsilon^2 f(\varepsilon x) e^{iS/(\varepsilon^2)}, \quad 0 < \varepsilon \ll 1. \tag{1} \]

Here \(0 < \varepsilon \ll 1\), \(f(y)\) is smooth and rapidly vanishes as \(y \to \pm \infty\) and \(S = (\varepsilon^2 t)^2\).

The following pictures present some of solutions for this equation.
Soliton exitation
Annihilation of soliton
Scattering of soliton
Problems

These pictures allow us to formulate two problems:

- **Informal problem:**
  How to increase the order of the solution, using the small force?

- **Formal problem:**
  How to find a connection formula for ”small” and ”big” solutions?
Approaches

• One of ways for solution of the informal problem give a local resonance.

• Uniform asymptotic formulas for the solution give a matching method.

Together they give a new approach to obtain asymptotic formulas for bifurcation of the solution near the resonant curve.
Resonance

Consider the linear oscillator under periodic perturbation

\[ \ddot{x} + \omega_0^2 x = F_0 \cos \omega t, \]

where \( F_0 = \text{const} \) is the amplitude of the perturbation and \( \omega = \text{const} \) is the frequency of the perturbation. If \( \omega = \omega_0 \) then amplitude of solution increases under the oscillating perturbation.
Local (weak) resonance

Let the frequency of the perturbation changes slowly $\omega = \omega(\varepsilon t)$ and there exists the moment $t_0$ such that $\omega_0 = \omega(\varepsilon t_0)$. Then the resonant increasing of the solution takes place near the point $t_0$. Typical solution is presented on a following picture:

The resonance of this type is usually called local or weak resonance. The main effect is an appearance of the term of the order of $\varepsilon^{-1/2}$ after the passage through the resonance domain.
Local resonance. References

- J.Kevorkian, SIAM J.Appl. Math. V.20, 1. 1971,
- Neishtadt A.I., DAN USSR, 221, 2. 1975,
- J.Sanders, SIAM J. Math. Anal. V.10, 6. 1979,
- S. Dobrokhotov and others, PMM 51, 5. 1987,
Generation of solitons

There are some ways to obtain the solitary packet of waves.

- One of them is a spontaneous generation from an initial profile of the wave packet. Such method for the soliton generation uses the asymptotic behaviour of the soliton equations. The solitonic envelope is formed as an asymptotic limit for the long time [Manakov, Ablowitz, 1973].

- Another way uses the transverse instability of the waves in the nonlinear medium [Kadomtsev and Petviashvily, 1973].

- The autoresonant excitation of periodic solutions for NLSE was done by L. Friedland and A. Shagalov (1998).
Generation of solitons by weak resonance

We give another approach to generate a soliton. Our approach is based on slow passage through the local resonance. This way allows:

- To obtain a soliton as envelope function for wave packets in solutions of NLSE, KdV and nonlinear Klein-Gordon equation.
- To control the parameters of generated solitons by choosing a shape of small external force.
Generation of solitons in NLSE

Here we consider the NLSE perturbed by the small driving force.

\[ i \partial_t \Psi + \partial_x^2 \Psi + |\Psi|^2 \Psi = \varepsilon^2 f e^{i(\varepsilon^2 t)^2/\varepsilon^2}, \quad 0 < \varepsilon \ll 1. \] (2)

**Theorem 1.** There exist asymptotic solution of perturbed NLSE such that:

\[ \Psi = O(\varepsilon^2), \quad t \ll -\varepsilon^{-1} \]

and

\[ \Psi = \varepsilon u + O(\varepsilon^2), \quad \varepsilon^{-1} \ll t \leq K \varepsilon^{-2}, \quad K = \text{const} > 0. \]

Main term is defined by the Cauchy problem for NLSE

\[ iu_{t_2} + u x_1 x_1 + |u|^2 u = 0, \quad u|_{t_2=0} = (1 - i) \sqrt{\pi} f(x_1), \]

where \( t_2 = t \varepsilon^2 \) and \( x_1 = x \varepsilon. \)
Generation of solitary packets in solutions of nonlinear Klein-Gordon equation

The aim of follow sections is to explain the mechanism of generation for solitary packets by local resonance in nonlinear Klein-Gordon equation. This approach may be applied for the perturbed NLSE also, but it seems more useful for explanations for perturbed NKG as an example.
Let us consider the nonlinear Klein-Gordon equation:

$$\partial_t^2 U - \partial_x^2 U + U + \gamma U^3 = \varepsilon^2 f(\varepsilon x) \exp \left\{ i \frac{S(\varepsilon^2 t, \varepsilon^2 x)}{\varepsilon^2} \right\} + \text{c.c.} \quad (3)$$

Here $0 < \varepsilon \ll 1$, $\gamma = \text{const}$; $f(y)$ is smooth and rapidly vanishes as $y \to \pm \infty$. The function $S(y, z)$ and all derivatives are bounded.

Simplest solution of this equation is forced oscillations. The forced oscillations have an order of perturbation force. Asymptotic behavior of such solution gives by WKB-approximation:

$$U_f \sim -\varepsilon^2 f \frac{f}{l} \exp(iS/\varepsilon^2) + \text{c.c.}, \quad l = (\partial_{t_2} S)^2 - (\partial_{x_2} S)^2 - 1,$$

where $x_j = \varepsilon^j x$, $t_j = \varepsilon^j t$, $j = 1, 2$. 

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There are two details for this WKB-approximation:

- the order of the asymptotic solution is $\varepsilon^2$;
- the formula for this solution has singularity at line $l(x_2, t_2) = 0$.

For this asymptotic solution we have two problem:

- **Informal problem:**
  
  How to increase the order of the solution, using the small force?

- **Formal problem:**
  
  How to obtain asymptotic formulas that are valid close to the singular line?
Numeric simulations
This picture shows the generation of the solitary packet of waves for equation (3) with special perturbation.

This picture shows a profile $\left. U(x, t) \right|_{x=0}$ of the packet. We see a typical picture of weak (local) resonance for the profile of the wave packet.
Asymptotic analysis

All domains where we construct the solution are separated on three pairwise joint domains. The pre-resonant domain corresponds to forced oscillations with the amplitude of the order $\varepsilon^2$. This oscillations break down when the driving force becomes resonant. The resonant layer is a thin domain near the resonant curve $l(x_2, t_2) = 0$. In this layer the amplitude of the oscillations increases up to the order $\varepsilon$. In the post-resonant domain the amplitude of the solution stabilizes on the order of $\varepsilon$. 
Pre-resonant expansion

In the domain $-l \gg \varepsilon$ the formal asymptotic solution of equation (3) modulo $O(\varepsilon^{N+1})$ has the form

$$U = \sum_{n \geq 2} \varepsilon^n \sum_{k \in \Omega_n} U_{n,k}(t_2, x_2, \varepsilon x) \exp \left\{ ik \frac{S(t_2, x_2)}{\varepsilon^2} \right\}. \quad (4)$$

The set $\Omega_n$ for the higher-order term is described by the formula

$$\Omega_n = \begin{cases} 
\{ \pm 1 \}, & n \leq 5; \\
\{ \pm 1, \pm 3, \ldots, \pm (2l + 3) \}, & l = \left\lfloor (n - 6)/4 \right\rfloor, \quad n \geq 6.
\end{cases}$$
The coefficients of the asymptotics $U_{n,k}$ are defined out of algebraic equations, for example

$$U_{2,1} = -\frac{f}{l}, \quad U_{3,1} = 2i\frac{\partial x_1 f \partial x_2 S}{l^2},$$

(5)

Here we obtain the WKB-type of the asymptotic expansion which is valid before the resonance layer. This piece of the solution is presented in the following picture:
Resonant expansion

This part contains the asymptotic construction of the solution for equation (3) in the neighborhood of the curve \( l = 0 \). The domain of validity of this asymptotics intersects with the domain of validity of expansion (4). These expansions are matched.

In the domain \( |l| \ll 1 \) the formal asymptotic solution for equation (3) modulo \( O(\varepsilon^{N+1}) \) has the form

\[
U = \sum_{n \geq 1}^{N} \varepsilon^n W_n(t_1, x_1, t_2, x_2, \varepsilon),
\]

(6)
where

$$W_n = \sum_{k \in \Omega_n} W_{n,k}(x_2, t_2, x_1, t_1) \exp \left\{ ik \frac{S(t_2, x_2)}{\varepsilon^2} \right\},$$  \hspace{1cm} (7)

The function \( W_{n,1} \) is a solution of the problem for differential equations like the equation for the coefficient \( W_{1,1}(x_1, t_1, x_2, t_2) \), which is defined by first order partial differential equation:

$$2i \partial_{t_2} S \partial_{t_1} W_{1,1} - 2i \partial_{x_2} S \partial_{x_1} W_{1,1} - \lambda W_{1,1} = f,$$

with a given asymptotic behaviour:

$$W_{1,1} \sim \frac{-f}{\lambda}, \quad \lambda \rightarrow -\infty, \quad \text{where} \quad \lambda = \frac{l}{\varepsilon}. $$
We obtain:

\[ U(x, t, \varepsilon) \sim \varepsilon W_{1,1}(x_1, t_1, x_2, t_2) \exp\{iS/\varepsilon^2\} + c.c. \]

There is an essential difference between asymptotics (6) and external pre-resonance asymptotics (4). In the first place the leading-order term in (6) has an order \( \varepsilon \) while the leading order term in (4) has an order \( \varepsilon^2 \). In the second place the coefficients of asymptotics (6) depend on fast variables \( x_1 = x_2/\varepsilon \) and \( t_1 = t_2/\varepsilon \).

The resonant layer contains the strip where the solution increases due to the local resonance. This piece of the strip is shown on the following figure:
Post-resonant expansion

In the domain \( l \gg \varepsilon \) the formal asymptotic solution of equation (3) modulo \( O(\varepsilon^{N+1}) \) has a form

\[
U(x, t, \varepsilon) \sim \varepsilon \Psi(x_1, t_1, t_2) \exp\left\{ i\varphi(x_2, t_2)/\varepsilon^2 \right\} + \text{c.c.} \quad (8)
\]

Here the function \( \varphi(x_2, t_2) \) satisfies the eikonal equation

\[
(\partial_{t_2} \varphi)^2 - (\partial_{x_2} \varphi)^2 - 1 = 0 \quad (9)
\]

and initial condition on the curve \( l = 0 \):

\[
\varphi|_{l=0} = S|_{l=0}, \quad \partial_{t_2} \varphi|_{l=0} = \partial_{t_2} S|_{l=0}.
\]
The leading-order term of the asymptotics is a solution of the Cauchy problem for the nonlinear Schrödinger equation

\[ 2i\partial_{t_2}\varphi\partial_{t_2}\Psi_{1,0,\varphi} + \partial_\xi^2\Psi_{1,0,\varphi} + i[\partial_{t_2}\varphi - \partial_{x_2}\varphi]\Psi_{1,0,\varphi} + \gamma|\Psi_{1,0,\varphi}|^2\Psi_{1,0,\varphi} = 0, \]

\[ \Psi_{1,0,\varphi}|_{t=0} = \int_{-\infty}^{\infty} d\sigma f(x_1) \exp(i \int_{0}^{\sigma} d\chi l(x_1, t_1, \varepsilon)), \]

where \( \xi \) is defined by

\[ \frac{dx_1}{d\xi} = \partial_{t_2}\varphi, \quad \frac{dt_1}{d\xi} = \partial_{x_2}\varphi. \]
At last the post-resonant expansion has the following:
Higher-order terms and matching

The structure of constructed asymptotic solution when $l < 0$ and $l > 0$ are sufficiently different. We concentrate on the description of the changing of the solution from the pre-resonant to post-resonant form in the thin layer near the curve $l = 0$. In this transition layer the amplitude of the solution increases due to the resonant pumping. The value of the amplitude is defined by the width of the resonant layer. We found the width of the layer by construction and analysis of the higher-order terms of the asymptotic solution in all domains. This analysis looks very complicated but it is necessary to match the asymptotics of the solution in different domains and obtain formula (11). This formula defines the leading order term of the solution after the slowly passage through the resonance.
Main result

Let us formulate the main result of the work. If the solution of (3) has the form

$$U \sim -\varepsilon^2 \frac{f}{l} \exp(iS(t_2, x_2)/\varepsilon^2) + c.c.,$$

when $l < 0$, then in the domain $l > 0$ this asymptotic solution is

$$U(x, t, \varepsilon) \sim \varepsilon \Psi(x_1, t_1, t_2) \exp\{i\varphi(x_2, t_2)/\varepsilon^2\} + c.c. \quad (10)$$
The phase function \( \varphi \) satisfies the eikonal equation

\[
(\partial_{t_2} \varphi)^2 - (\partial_{x_2} \varphi)^2 - 1 = 0
\]

with conditions

\[
\varphi|_{l=0} = S|_{l=0}, \quad \partial_{t_2} \varphi|_{l=0} = \partial_{t_2} S|_{l=0}.
\]

The envelope function of the leading-order term is a solution of the nonlinear Schrödinger equation

\[
2i\partial_{t_2} \varphi \partial_{t_2} \Psi + \partial^2_\xi \Psi + i[\partial^2_{t_2} \varphi - \partial^2_{x_2} \varphi] \Psi + \gamma|\Psi|^2 \Psi = 0,
\]
where the $\xi$ is defined by

$$\frac{dx_1}{d\xi} = \partial_{t_2}\varphi, \quad \frac{dt_1}{d\xi} = \partial_{x_2}\varphi.$$ 

The initial condition for $\Psi$ is

$$\Psi|_{t=0} = \int_{-\infty}^{\infty} d\sigma f(x_1) \exp(i \int_0^\sigma d\mu l(x_1, t_1, \varepsilon)),$$  \hspace{1cm} (11)$$

The integration in this integral is done in the characteristic direction related with the equation for $W_{1,1}$. 
Applications

Now this is a mathematical model only. The solitary packets are suitable for communication in optical fibers on a large distance. Our approach give opportunity to generate solitary packets of waves and effectively control their parameters by small external force.
Finite amplitude waves

In the previous section the solitary waves appear as envelopes of the wave packets and have order $O(\varepsilon)$. Next question:

- Is it possible to generate waves with finite amplitude after the passage through a weak resonance?

The answer is "YES" but for dispersionless wave equations.

As an example we demonstrate results by S.Glebov, N. Gorbatova and O.Kiselev for the nonlinear wave evaluation with special perturbation:

$$U_{tt} - U_{xx} + (U_x)^2 = \varepsilon^2 f(\varepsilon x) \exp\left\{i S(\varepsilon^2 x, \varepsilon^2 t)/\varepsilon^2\right\} + \text{c.c.}$$
We investigate the simplest case $S = S(\varepsilon^2 t) = (\varepsilon^2 t)^2/2$. Here resonance takes place on the curve $t = 0$. The solution has a form:

$$U(x, t, \varepsilon) \sim \varepsilon^2 \left(-\frac{f}{t^2}\right), \quad t_2 < 0,$$

and

$$U(x, t, \varepsilon) \sim v_0^+(t_1 + x_1, t_2) + v_0^-(t_1 - x_1, t_2), \quad t_2 > 0$$

where

$$\partial_{t_2 \zeta}^2 v_0^+ + \partial_\zeta v_0^+ \partial_\zeta^2 v_0^+ = 0, \quad \zeta = t_1 + x_1;$$

$$\partial_{t_2 \eta}^2 v_0^- + \partial_\eta v_0^- \partial_\eta^2 v_0^- = 0, \quad \eta = t_1 - x_1.$$
By substitution $V^+ = \partial_\zeta v^+_0$ and $V^- = \partial_\eta v^-_0$ we get the pair of the Hopf equations

\[
\begin{align*}
\partial_{t_2} V^+ + V^+ \partial_\zeta V^+ &= 0, \\
\partial_{t_2} V^- + V^- \partial_\eta V^- &= 0.
\end{align*}
\]

Numeric result is given in following picture:
References


Soliton excitation