Nonlinear Physics: Theory and Experiment IV

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Recurrence in the KdV Equation?

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Korteweg-de Vries equation (KdV)

$$u_t + uu_x + \delta^2 u_{xxx} = 0$$

a particular continuum limit of an FPU lattice

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} y = (y_{n-1} - 2y_n + y_{n+1}) \left[1 - \alpha \left(y_{n-1} - y_{n+1}\right)\right]$$

Repeated near-recurrences are observed in FPU lattices with

$$y_n(0) = \sin(\tfrac{n\pi}{N}), \qquad \frac{\mathrm{d}}{\mathrm{d}t} y_n(0) = 0.$$

[FPU, 1955]

Look for (near) recurrence in KDV by simulation
[Zabusky & Kruskal,1965]

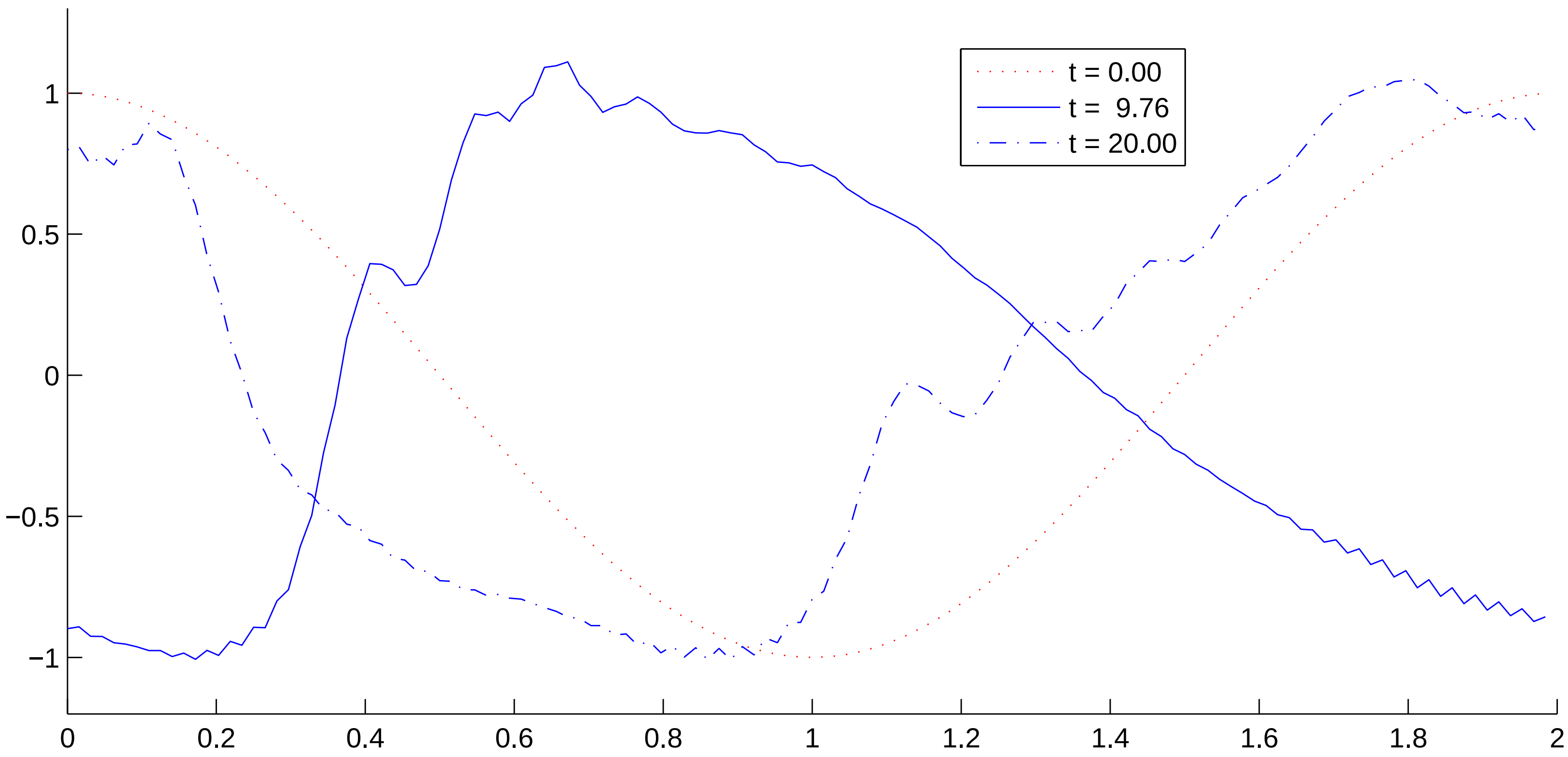
$$u_t + uu_x + \delta^2 u_{xxx} = 0$$

$$\delta = .022^2 << 1 \qquad u(x,0) = \sin(\pi x)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n = \frac{\left(u_{n-1} + u_n + u_{n+1}\right)\left(u_{n-1} - u_{n+1}\right)}{3h} + \delta^2 \frac{u_{n-2} - 2u_{n-1} + 2u_{n+1} - u_{n+2}}{2h^3}$$

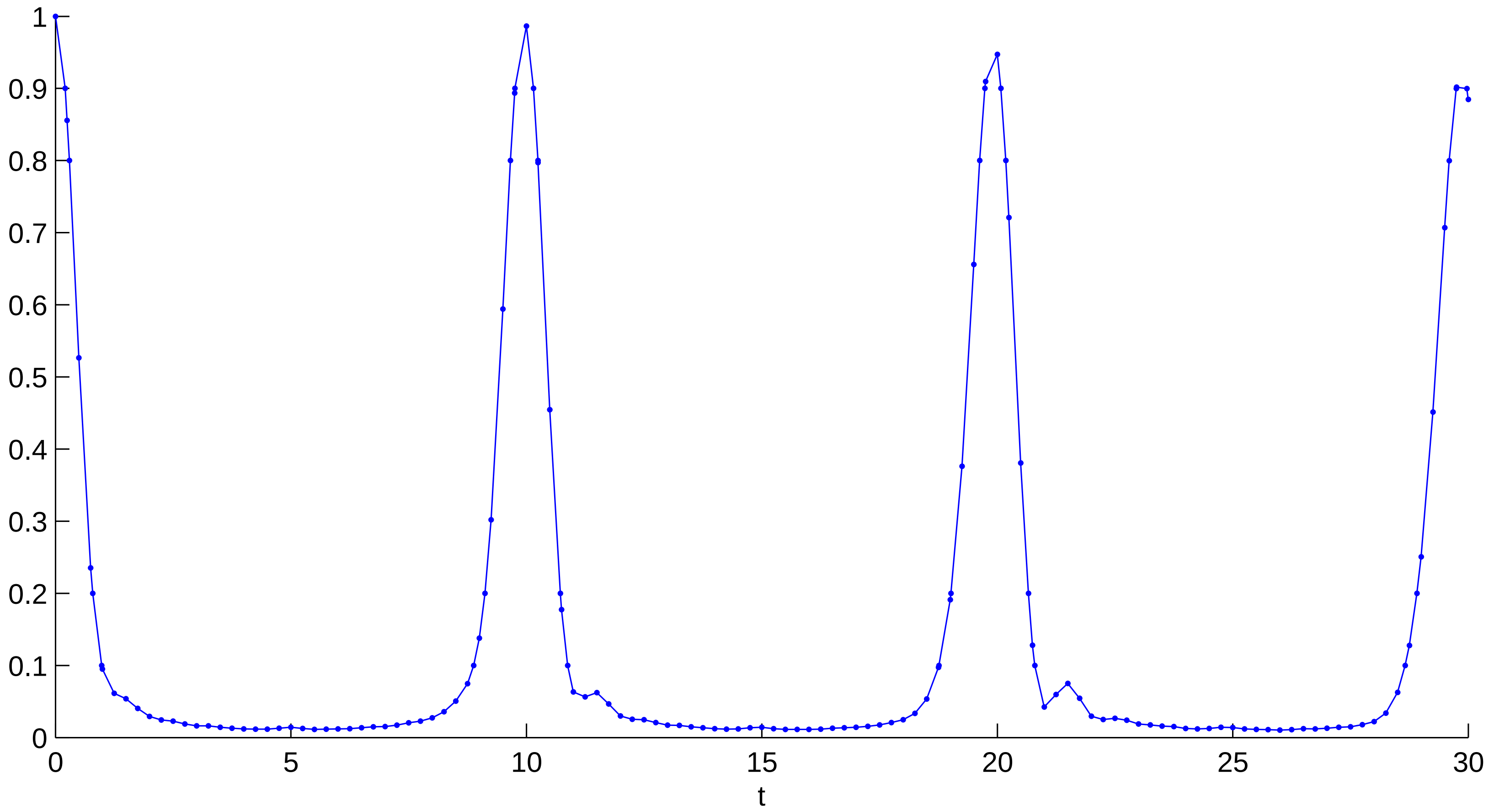
$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = -\left(\mathbf{A}_3 \boldsymbol{u}\right)\left(\mathbf{D}_c \boldsymbol{u}\right) - \delta^2 \mathbf{D}_{3c} \boldsymbol{u}$$

Recurrence



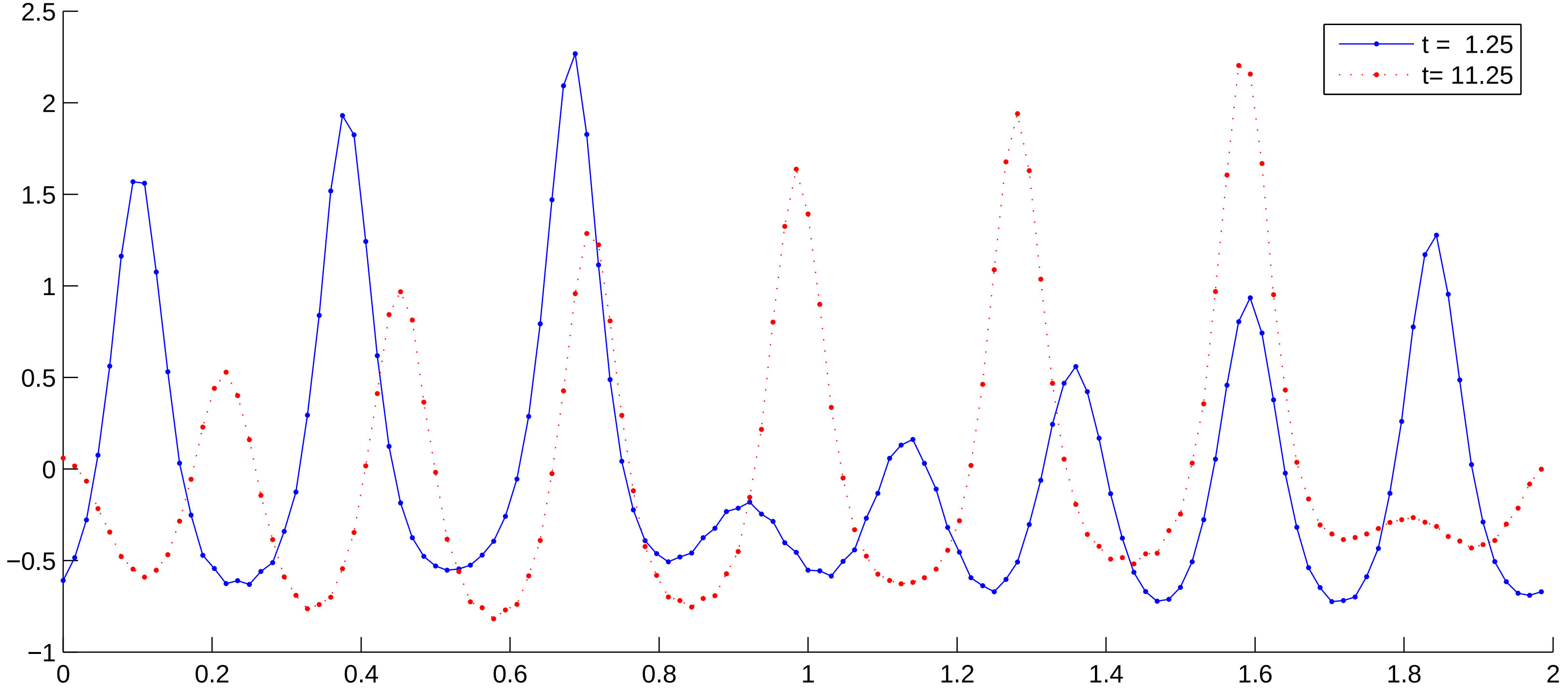
Simulation with $N = 128$

Recurrence



Simulation with $N = 128$

Emergence of Solitons



Simulation with $N = 128$

Emergence of Solitons

KdV solitary waves

$$u(x,t) = 12\delta^2 k^2 \,\mathrm{sech}^2 \left[k(x - 4\delta^2 k^2 t)\right]$$

are the limit of periodic, cnoidal-wave solutions

$$u(x,t) = 12\delta^2 k^2 m \,\mathrm{cn}^2 \left[k(x - 4\delta^2 k^2 (2m-1)t, m)\right]$$

The solitary waves are *solitons*:
they pass through one another and regain their original shape & velocity.

Recurrence

Recurrence has been explained in terms of the solitons.
[Zabusky & Kruskal, 1965; Osborne & Bergamasco, 1986]

Q: How does recurrence depend on the *spatial* discretization?

Hypotheses:

H1: Higher-accuracy discretization gives better recurrence.

H2: Integrable discretization gives better recurrence.

High-accuracy Discretizations

Pseudo-spectral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = -\boldsymbol{u}(\mathbf{D}_f \boldsymbol{u}) - \delta^2 \mathbf{D}_f^3 \boldsymbol{u}$$

Pseudo-spectral (Conservation Form):

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = -\tfrac{1}{2}\mathbf{D}_f \boldsymbol{u}^2 - \delta^2 \mathbf{D}_f^3 \boldsymbol{u}$$

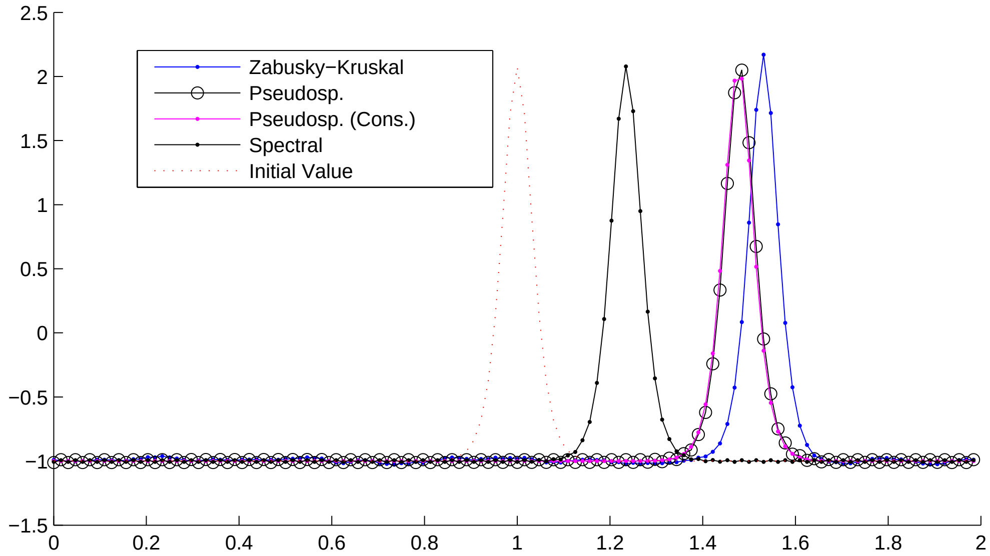
Spectral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = -\mathbf{D}_f \mathbf{F}^{-1}\left[(\mathbf{F}\boldsymbol{u}) * (\mathbf{F}\boldsymbol{u})\right] - \delta^2 \mathbf{D}_f^3 \boldsymbol{u}$$

$\mathbf{D}_f = \mathbf{F}^{-1}\mathbf{\Omega}\mathbf{F}$: Fourier differentiation matrix

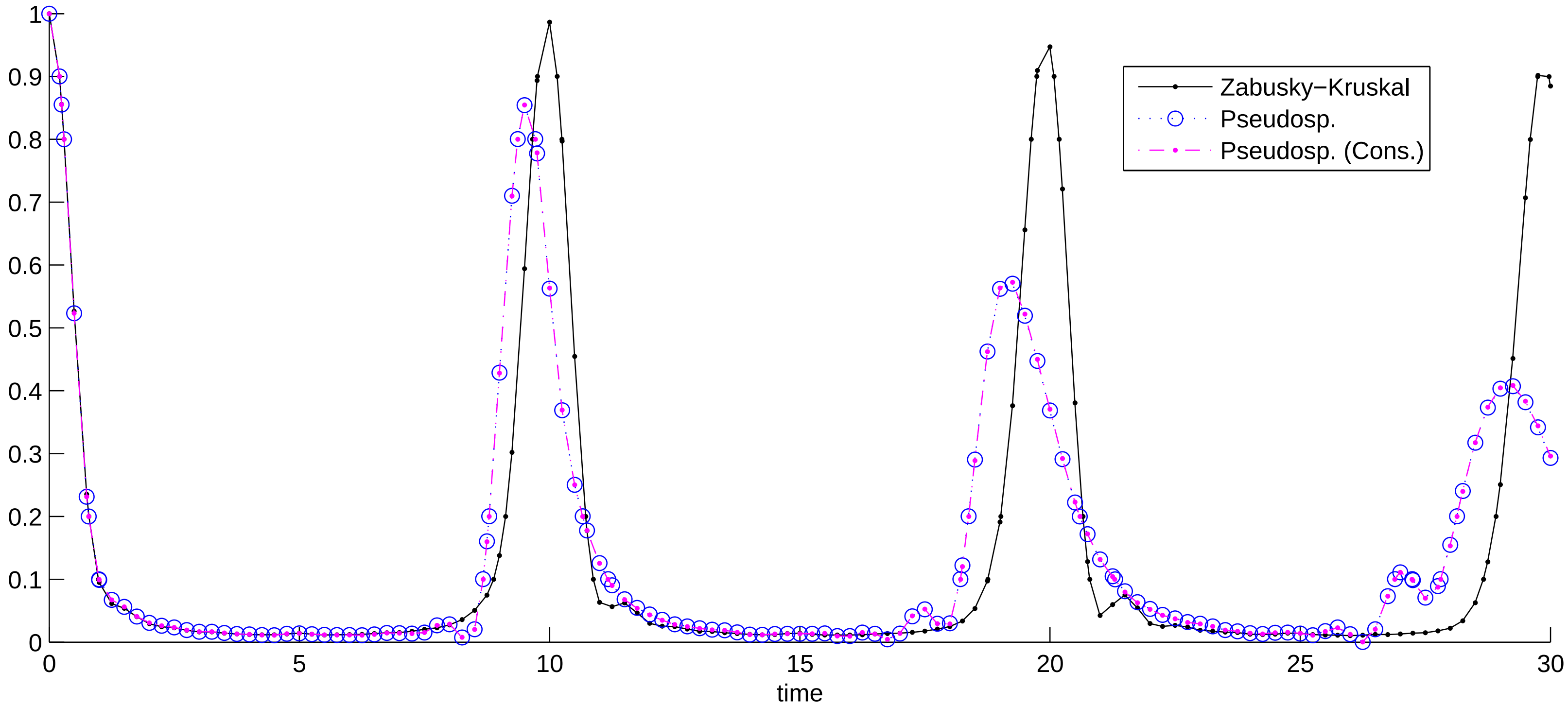
$\mathbf{F}$: discrete Fourier transform matrix

Solitons



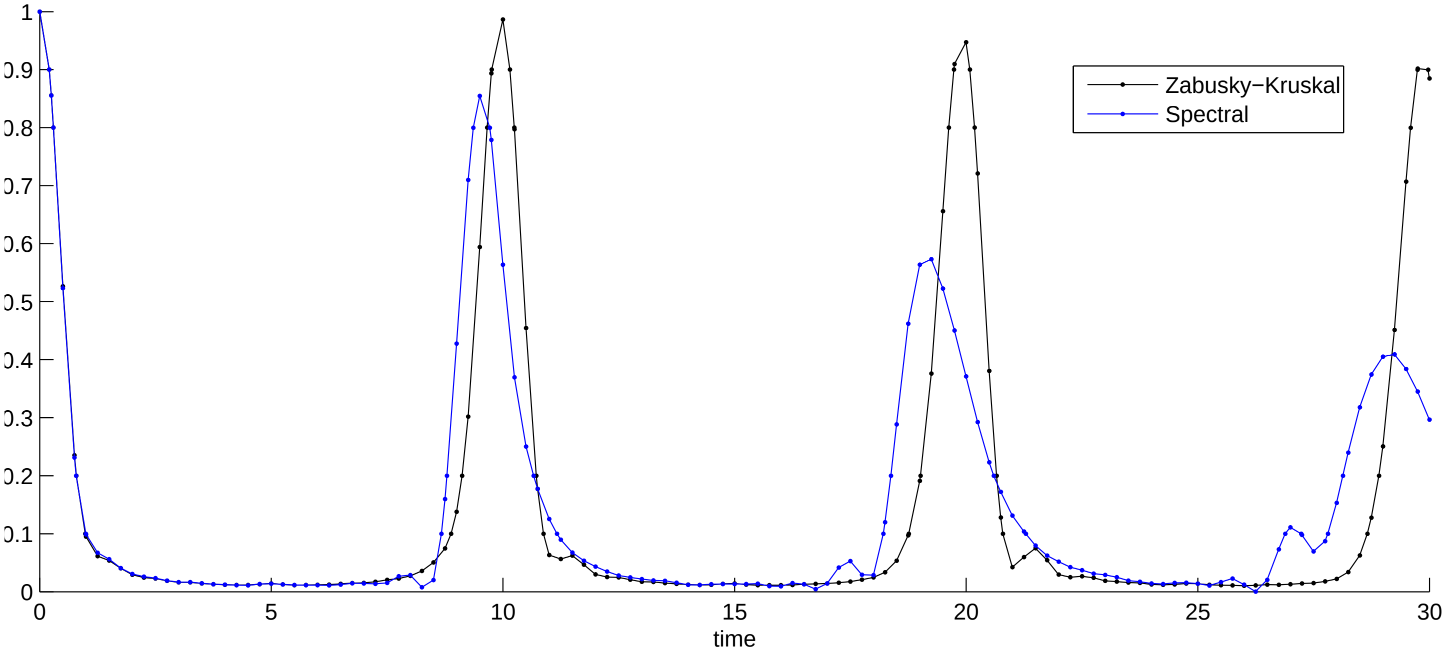
Simulation with $N = 128$ plotted at $t = 9.75$

Recurrence



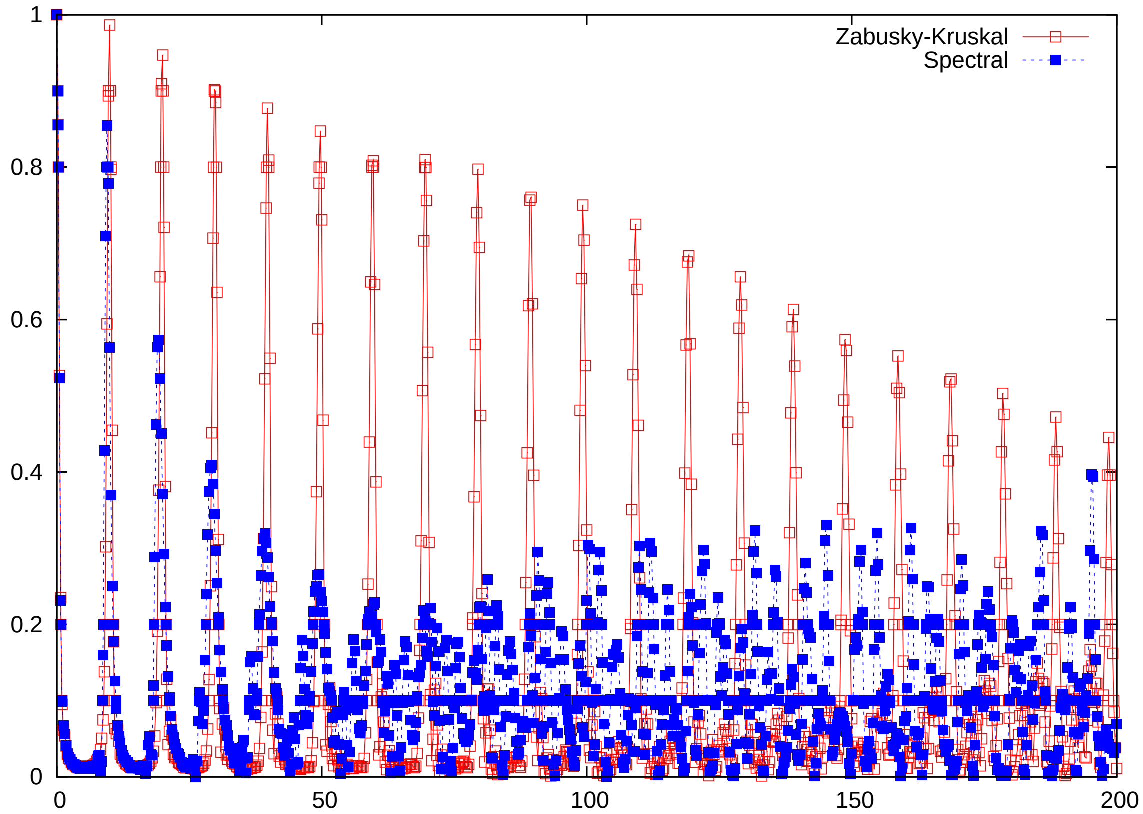
Simulation with $N = 128$

Recurrence



Simulation with $N = 128$

Recurrence



Simulation with $N = 128$

Recurrence

Surprise(?): Recurrence is *worse* in pseudospectral and spectral schemes.

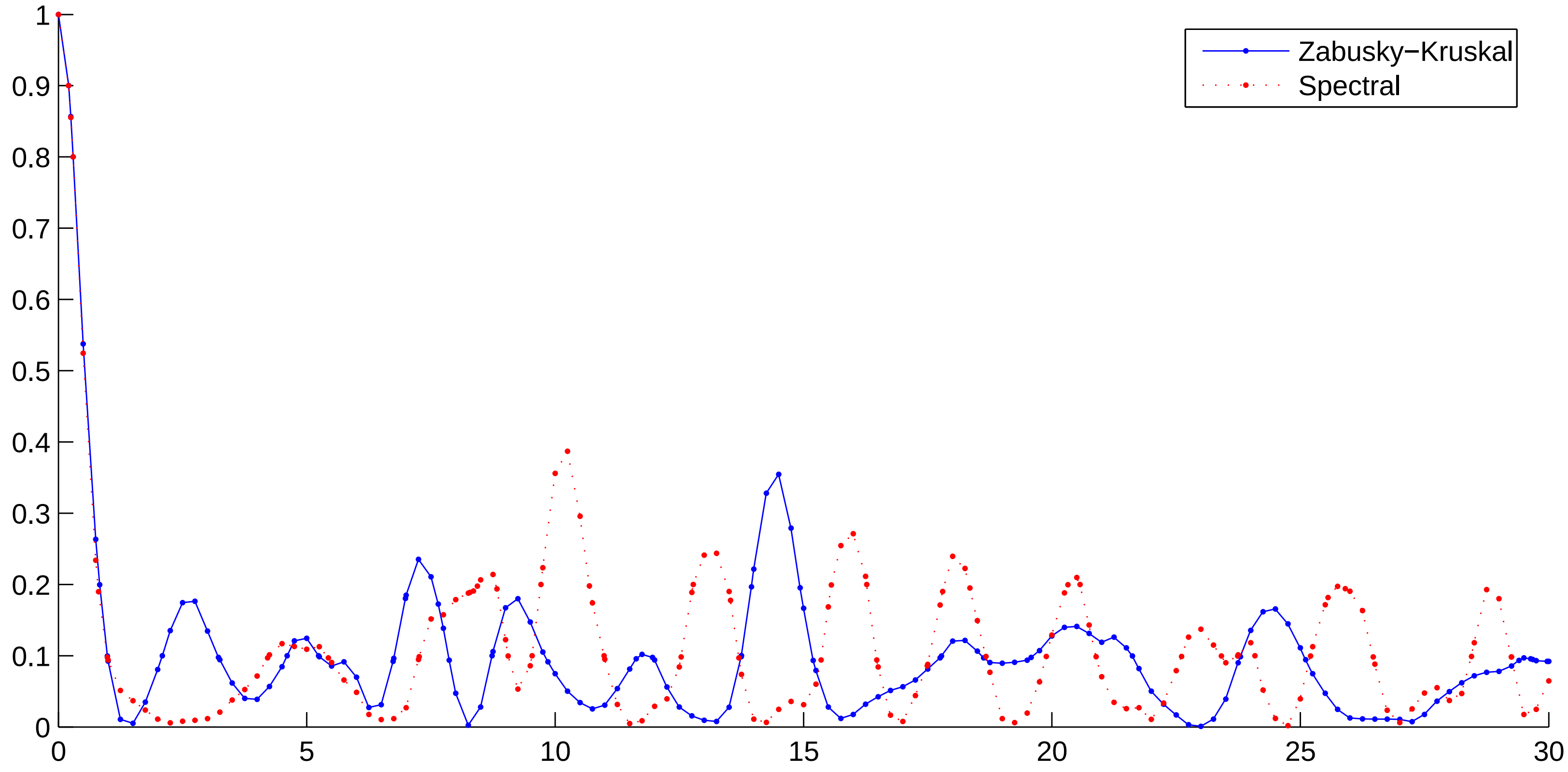
Higher accuracy does *not* yield better recurrence.

Q: What about a "rougher" grid (e.g., $N = 64$)?

A1: Zabusky-Kruskal and Spectral discretizations don't show recurrence.

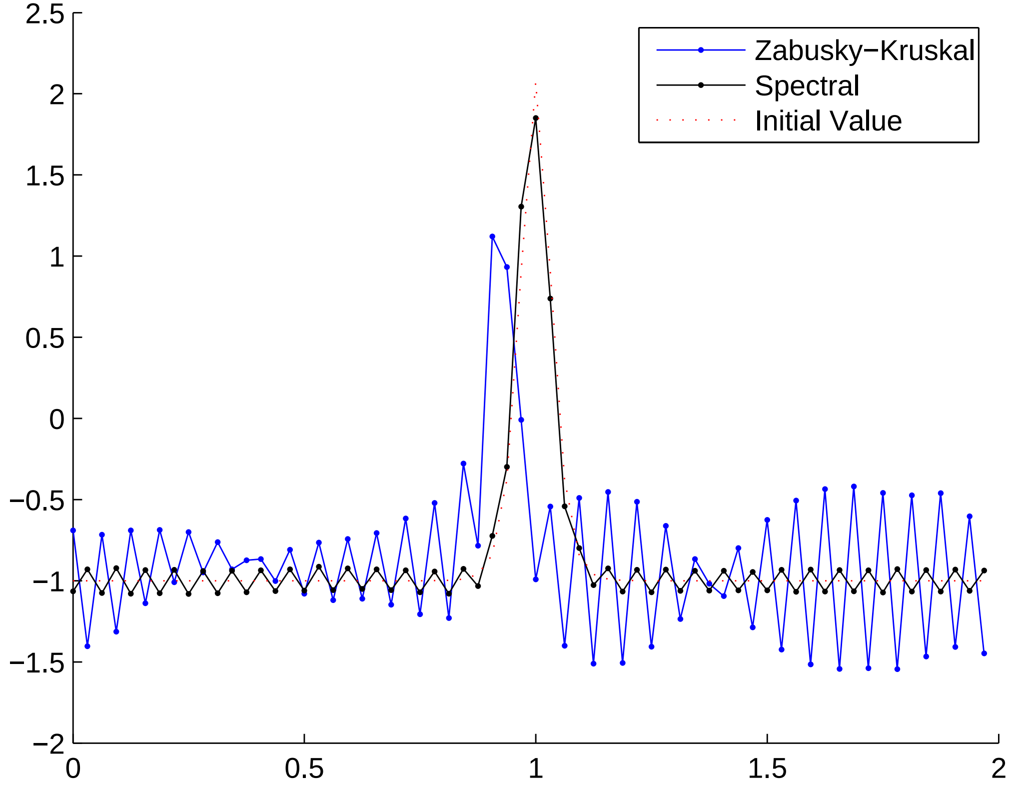
A2: Pseudospectral discretizations manifest a nonlinear instability.

No Recurrence



Simulation with $N = 64$

Solitons(?)



Simulation with $N = 64$ plotted at $t = .75$

Nonlinear Instability

In pseudospectral discretizations there is rapid uncontrolled growth of the solution for “rough” grids.

The nonlinear terms induce aliasing.

Preservation of $\sum_n u_n^2$ in other discretizations precludes the instability.

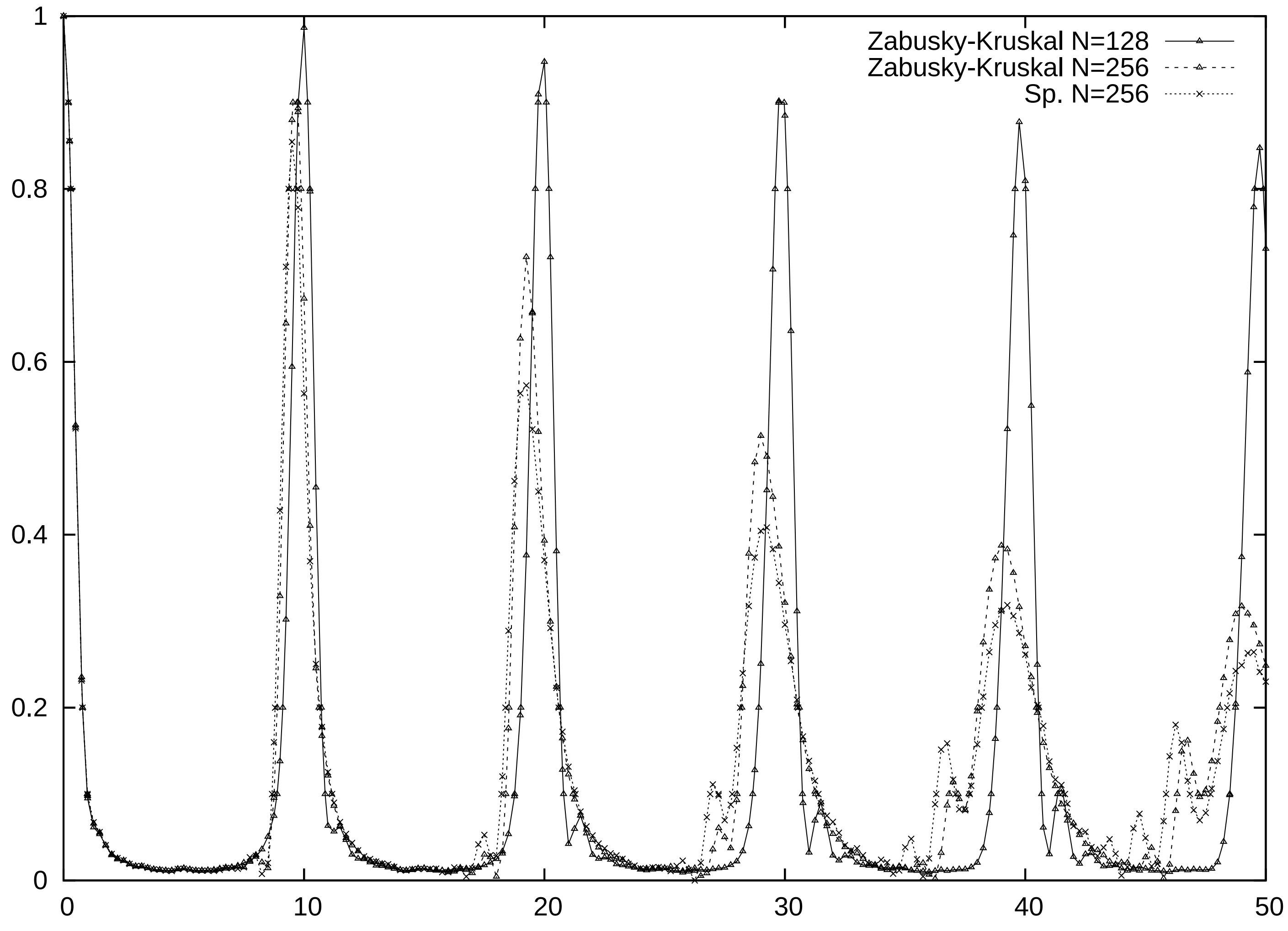
A similar instability exists in:

simple finite-difference discretizations of KdV,

discretizations of viscous Burger’s equation [Maritz & Schoombie],

discretizations inviscid Burger’s equation. [Majda & Timofeyev, 2002]

Finer Grid



Zabusky-Kruskal N=128
Zabusky-Kruskal N=256
Sp. N=256
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50

Integrable Discretization of KdV

KdV can be associated with the Zakharov-Shabat Scattering problem in the form

$$\boldsymbol{\psi}_x = \begin{pmatrix} ik & u \\ -1 & -ik \end{pmatrix} \boldsymbol{\psi}$$

Forward difference $\Rightarrow$ discrete (Ablowitz-Ladik) scattering problem:

$$\boldsymbol{\psi}_{n+1} = \begin{pmatrix} z & U_n \\ \alpha & z^{-1} \end{pmatrix} \boldsymbol{\psi}_n = \mathbf{S}_n \boldsymbol{\psi}_n$$

where

$$U_n = h u_n, \qquad \alpha = -h, \qquad z = e^{ikh}$$

Integrable Discretization of KdV

Discrete Compatibility Condition:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{S}_n = \mathbf{T}_{n+1}\mathbf{S}_n - \mathbf{S}_n\mathbf{T}_n$$

where

$$\boldsymbol{\psi}_{n+1} = \mathbf{S}_n\boldsymbol{\psi}_n \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{\psi}_n = \mathbf{T}_n\boldsymbol{\psi}$$

Compatibility condition is equivalent to:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}U_n = (1-\alpha U_n)\,[&-\alpha U_{n-1}(U_{n-2}-U_n) - \alpha U_{n+1}(U_n - U_{n+2}) \\ &- \alpha(U_{n-1}+2U_n+U_{n+1})(U_{n-1}-U_{n+1}) \\ &+U_{n-2}-2U_{n-1}+2U_{n+1}-U_{n+2}]\end{aligned}$$

Integrable Discretization of KdV

Rescale:

$$U_n \to \tfrac{h}{6} u_n, \qquad \alpha \to -\tfrac{h}{\delta^2} \qquad \tau \to \tfrac{3\delta^2}{h^4}$$

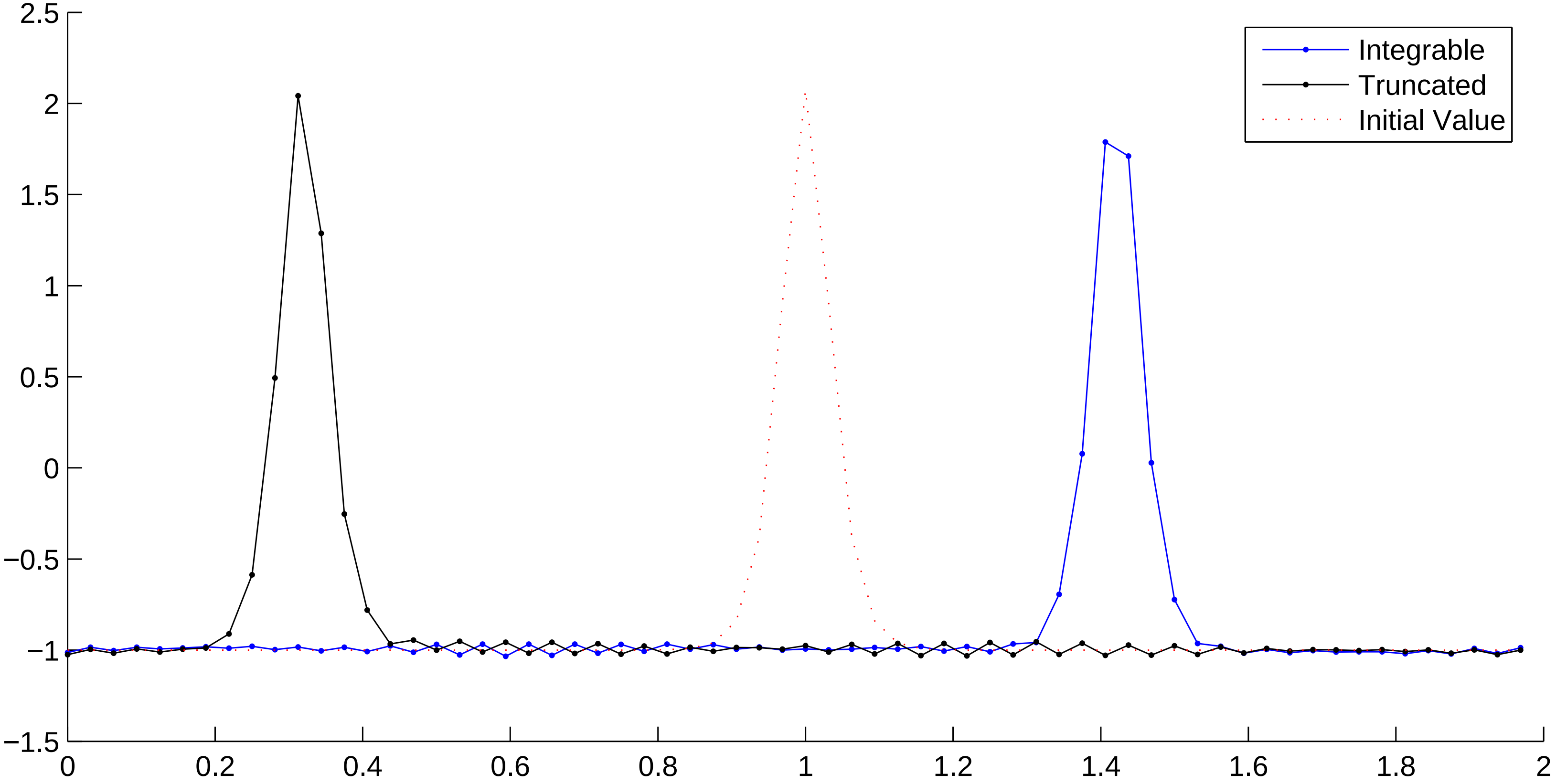
IDKdV:

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n = \left(1 + \frac{h^2 u_n}{6\delta^2}\right) \left[\frac{u_{n-1}(u_{n-2} - u_n)}{12h} + \frac{u_{n+1}(u_n - u_{n+2})}{12h} \right.$$
$$\left. + \frac{(u_{n-1} + 2u_n + u_{n+1})(u_{n-1} - u_{n+1})}{12h} + \delta^2 \frac{u_{n-2} - 2u_{n-1} + 2u_{n+1} - u_{n+2}}{2h^3} \right]$$

Truncated (non-integrable):

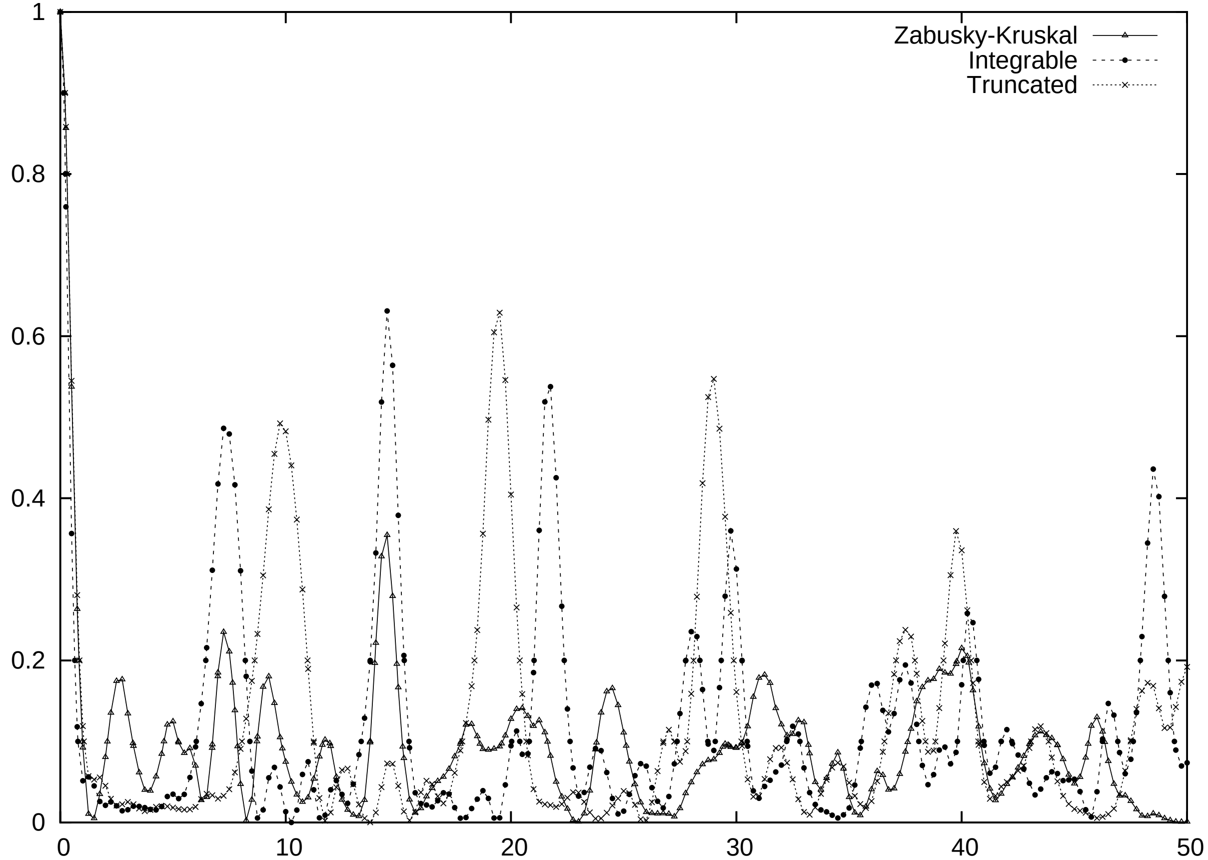
$$\frac{\mathrm{d}}{\mathrm{d}t} u_n = \frac{u_{n-1}(u_{n-2} - u_n)}{12h} + \frac{u_{n+1}(u_n - u_{n+2})}{12h}$$
$$+ \frac{(u_{n-1} + 2u_n + u_{n+1})(u_{n-1} - u_{n+1})}{12h} + \delta^2 \frac{u_{n-2} - 2u_{n-1} + 2u_{n+1} - u_{n-2}}{2h^3}$$

Solitons



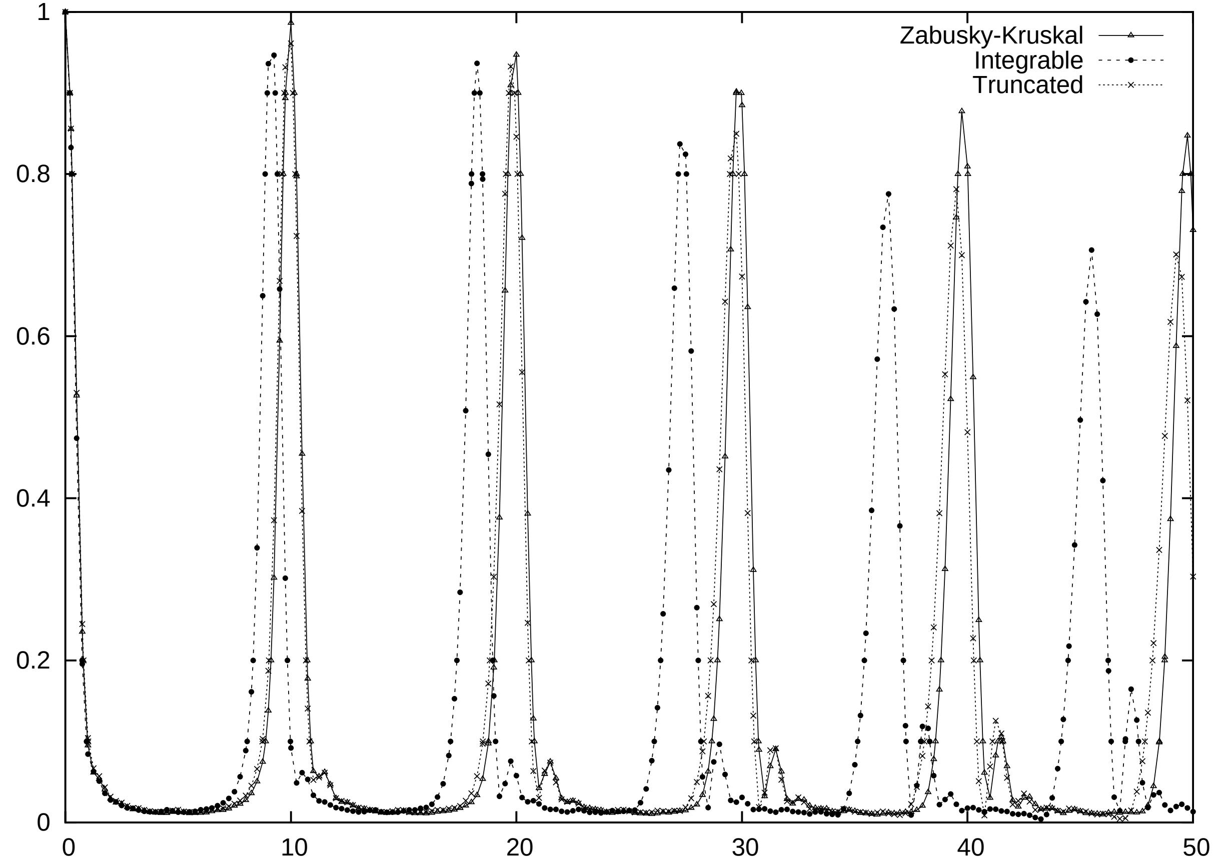
Simulation with $N = 64$ plotted at $t = 39.75$

Recurrence



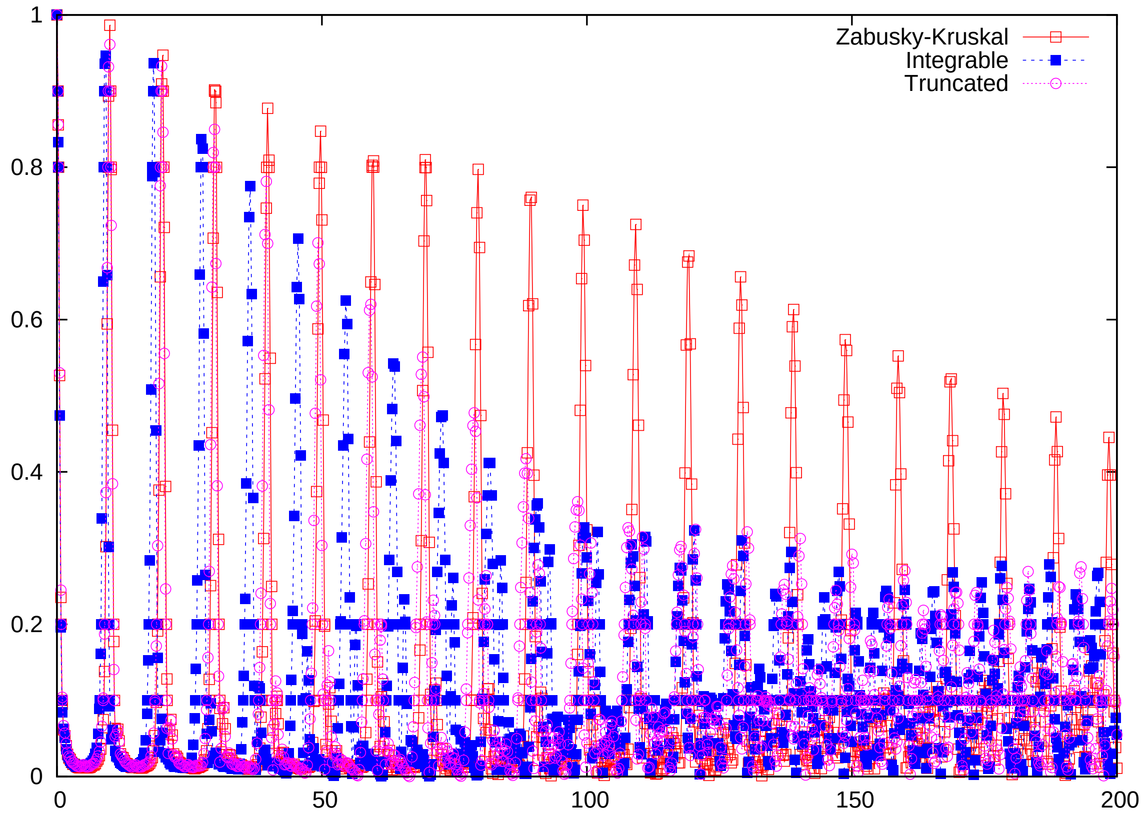
Simulation with $N = 64$ ($\frac{h^2}{6\delta^2} = .336$)

Recurrence



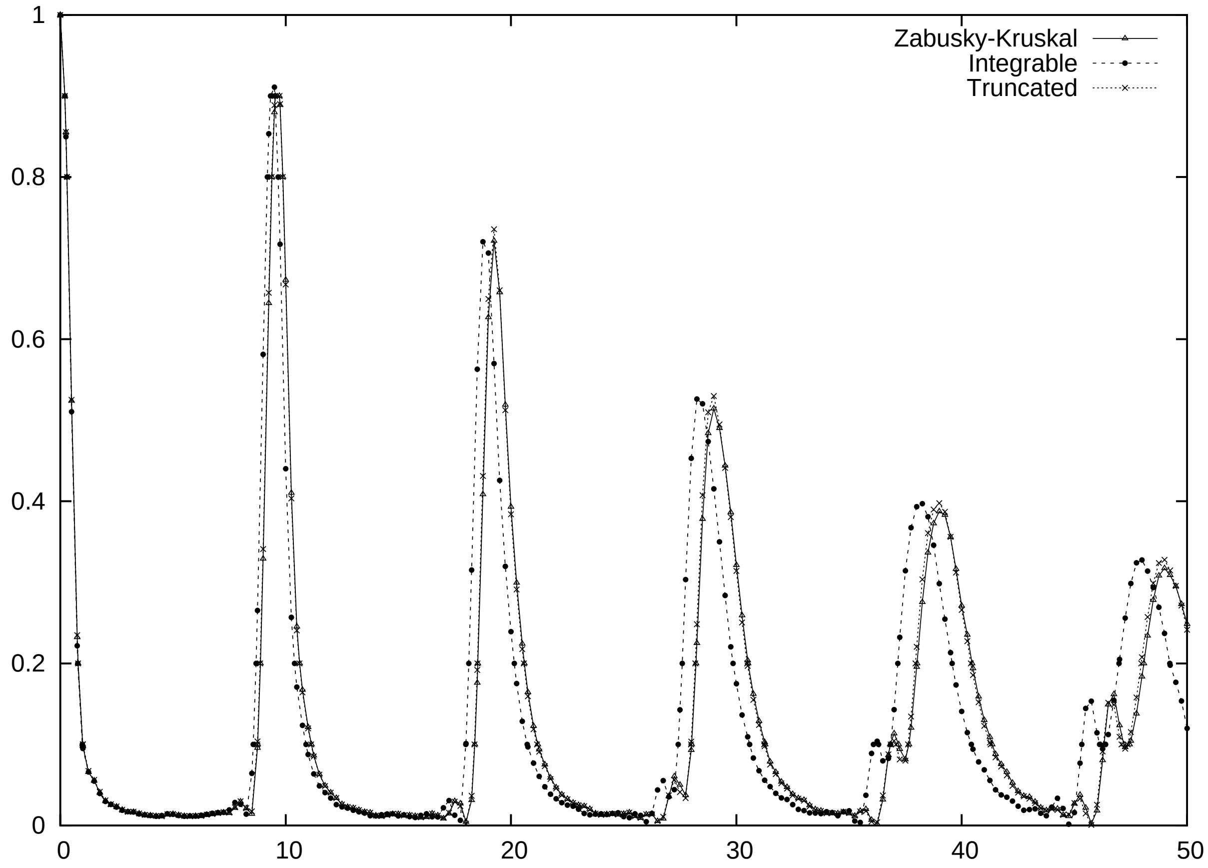
Simulation with $N = 128$ $(\frac{h^2}{6\delta^2} = .0841)$

Recurrence



Simulation with $N = 128$

Recurrence



Simulation with $N = 256$ ($\frac{h^2}{6\delta^2} = .0210$)

Observations

Presence of solitons in discretization not sufficient to capture recurrence.

Recurrence is *worse* in pseudospectral and spectral discretizations than carefully-chosen (Zabusky-Kruskal) finite-differences.

Recurrence not always strengthened by decrease the grid size.

Integrable discretization does not capture recurrence better than Zabusky-Kruskal discretization.

(Related?) Phenomena:

- uncontrolled nonlinear instability
- (modulated) grid-scale oscillations