Reformulation and asymptotic reductions of interfacial waves

Mark J. Ablowitz

Department of Applied Mathematics

University of Colorado, Boulder

Outline

- Introduction and background
- Nonlocal spectral formulation-classic water waves
  - Dirichlet-Neumann series from the nonlocal spectral formulation
  - Linearization, integral relations
  - Asymptotic reductions: Shallow water–long waves: Benney-Luke (BL), Kadomtsev-Petviashvili (KP), Deep water: NLS equations
- Interfacial waves:
  - One free surface-rigid top and bottom
  - Two free surfaces-rigid bottom
  - Asymptotic reductions: Long waves: ILW-BL, ILW-KP, Coupled NLS
- Lump solutions

Background

- Classic water waves (WW), long history-articles/books
- In 2-dmensions– complex analysis is very useful; provides simplification; not in 3-d.
- In 2006 Ablowitz, Fokas, Musslimani (AFM) developed a nonlocal spectral formulation of WW. Eliminates vertical coordinate.
- Variables: wave height, η, velocity potential on free surface $q = \phi(x, \eta)$;
- Variables: η, q used by Zakharov ('68) in Hamiltonian formulation of WW; found NLS reduction in deep water; Craig & Sulem ('93) derive Dirichlet-Neumann series in terms of η, q. Craig et al study various problems in WW and interfacial waves via DN operator methods
  MJA and Haut ('08): waves with 1 and 2 free interfaces: connect to DN operators/series, asymptotic reductions.

Water Waves

Classical equations: Define the domain D by
$D = \{-\infty < x < \infty, \ -h < y < \eta(x,t), \ t > 0; \ x = (x_1, x_2)\}$ The water wave equations satisfy the following system for $\phi(x, y, t)$ and $\eta(x,t)$:

$$\Delta\phi = 0 \ \text{ in } \ D$$

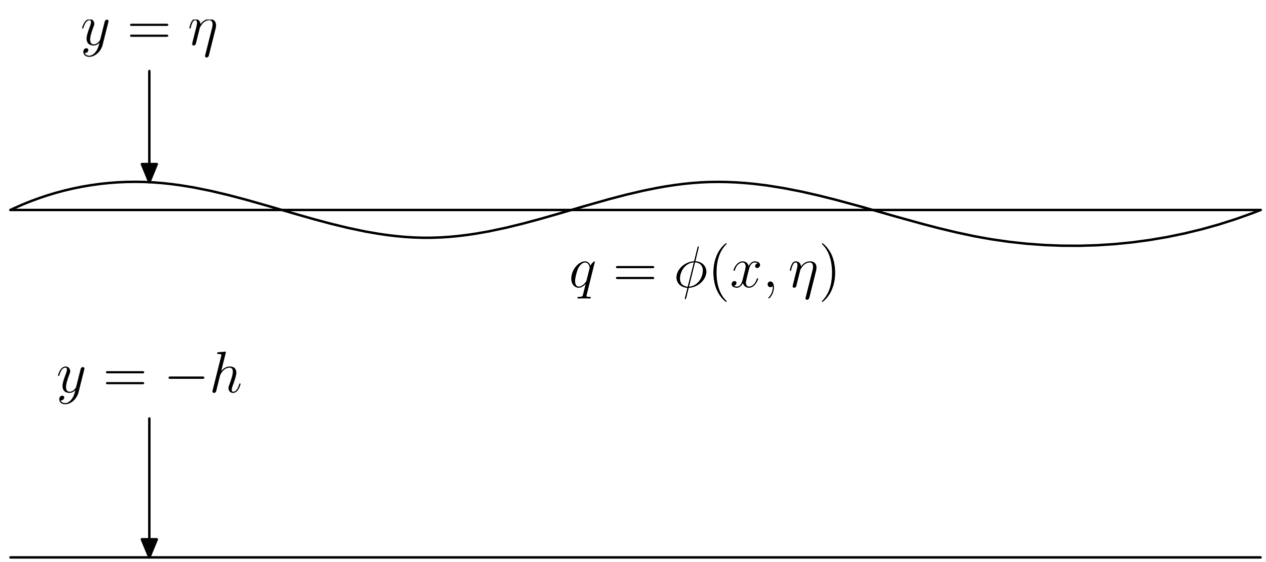
$$\phi_y = 0 \ \text{ on } \ y = -h$$

$$\eta_t + \nabla\phi \cdot \nabla\eta = \phi_y \ \text{ on } \ y = \eta$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) \ \text{ on } \ y = \eta,$$

where g: gravity, σ: surface tension. AFM, JFM, 2006, reformulation. Nonlocal eq. on a *fixed* domain.

Water Waves: figure



Classic water wave configuration

Water Waves-Nonlocal System

AFM find 2 eq., 2 unk: $\eta, q = \phi(x, \eta)$, rapid decay: 1 nonlocal eq. and 1 PDE; fixed domain!

$$\int dx e^{ik\cdot x}(i\eta_t \cosh[\kappa(\eta + h)] + \frac{k \cdot \nabla q}{\kappa} \sinh\left[\kappa(\eta + h)\right]) = 0 \qquad (I)$$

$$q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla \eta)^2}{2(1 + |\nabla \eta|^2)} = \sigma \nabla \cdot (\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}) \qquad (II)$$

$x = (x_1, x_2),\ k = (k_1, k_2),\ \kappa^2 = k_1^2 + k_2^2,\ q(x,t) = \phi(x, \eta(x,t))$

Infinite depth:

$$\int dx e^{ik\cdot x} e^{k\eta}(i\eta_t + \frac{k \cdot \nabla q}{\kappa}) = 0 \qquad (I')$$

Nonlocal System-Green's Identity

Define an associated potential: $\psi(x, y)$ satisfying

$$\Delta\psi(x, y) = 0, -h < y < \eta(x, t), x \in \mathbf{R}^2, \ \psi_y(x, -h) = 0$$

Use Green's identity:

$$\begin{aligned} 0 &= \int_{D(\eta)} \left((\Delta\psi)\,\phi - \left(\Delta\phi\right)\psi\right) dV \\ &= \int_{\partial D(\eta)} \left(\phi\,(\nabla\psi \cdot \vec{n}) - \psi\left(\nabla\phi \cdot \vec{n}\right)\right) dS \end{aligned}$$

$\vec{n}$ is the unit normal;. Eval. using BC's and eq. of motion:

$$\int_{\mathbf{R}^2} \psi(x, \eta)\eta_t \, dx = \int_{\mathbf{R}^2} q\left(\psi_y(x, \eta) - \nabla_x\psi(x, \eta) \cdot \nabla_x\eta\right) dx \quad (W)$$

Nonlocal System-Green's Identity

Write $\psi(x, y)$ in terms of basis func:
$\psi_k(x, y) = e^{ikx}\cosh[\kappa(y + h)]$. Substitute into weak eq (W); integration by parts leads to nonlocal spectral eq.

$$\int dx e^{ik\cdot x}\left(i\eta_t \cosh[\kappa(\eta + h)] + \frac{k \cdot \nabla q}{\kappa} \sinh\left[\kappa(\eta + h)\right]\right) = 0 \qquad (I),$$

Take FT, exp'd cosh, sinh in pwrs of η use pseudo-diff. forms $\Rightarrow$

$$\eta_t = G(\eta)q = \left(\sum_n A_n(\eta)\right)^{-1}\left(\sum_n B_n(\eta)\right) q$$

where $G(\eta)$ is the DN operator; series agrees with Craig et al; recall η_t prop. to normal deriv of ϕ ($\eta_t = \phi_y - \nabla\phi \cdot \nabla\eta$).

WW- Linearized System

Let $\eta, |\nabla q| \to 0$ then eq. (I,II) simplify.

$$\int dx e^{ikx} (i\eta_t \cosh \kappa h + \frac{k \cdot \nabla q}{\kappa} \sinh \kappa h) = 0 \quad (1L)$$

recall $\kappa^2 = k_1^2 + k_2^2$. Use Fourier transform: $\hat{\eta} = \int dx e^{ikx} \eta$

$$i\hat{\eta}_t \cosh \kappa h + \frac{k \cdot \widehat{\nabla q}}{\kappa} \sinh \kappa h = 0 \;\; (1L)$$

$$\widehat{q_t} + (g + \frac{\sigma}{\rho}\kappa^2)\hat{\eta} = 0 \;\; (2L)$$

Then from eq. (1L), (2L)find:

$$\hat{\eta}_{tt} = -(g\kappa + \frac{\sigma}{\rho}\kappa^3) \tanh \kappa h \; \hat{\eta}$$

WW-Nonlocal System-Remarks

- Can find integral relations by taking $k \to 0$. First two (recall: $x \to (x_1, x_2)$)

$$\frac{\partial}{\partial t}\int dx\ \eta(x,t) = 0 \quad (Mass)$$

$$\frac{\partial}{\partial t}\int dx(x_1\eta) = \int dx\ q_{x_1}(\eta + h) \quad (COM)$$

  LHS: COM in x_1 direction -RHS related to x_1 momentum; COM -in x_2-direction and virial identities can also be found.

- May extend formalism to variable depth.
- Can derive KP, Benney-Luke, Boussinesq, NLS systems

WW-Asymptotic Systems

Small ampl., long waves: nondim:
$\epsilon = \frac{a}{h}, \mu = \frac{h}{l_x}, \gamma = \frac{l_x}{l_y}, \epsilon, \mu, \gamma << 1$. Find Benney-Luke (1964) system (nmlz'd surface tension: $\tilde{\sigma}$):

$$q_{tt} - \tilde{\Delta} q + \tilde{\sigma}\mu^2 \tilde{\Delta}^2 q + \varepsilon(\partial_t |\tilde{\nabla} q|^2 + q_t \tilde{\Delta} q) = 0 \ (BL)$$

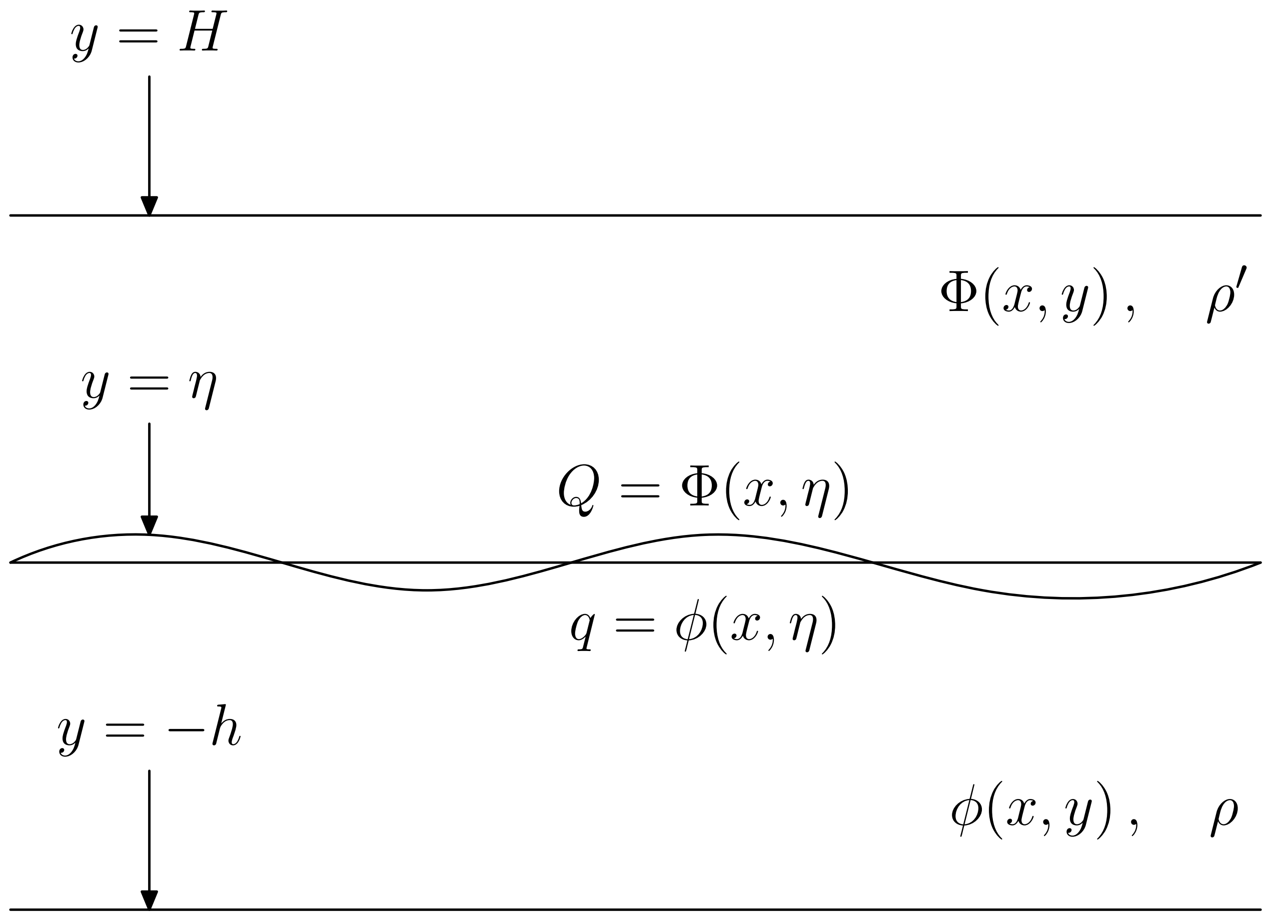
$$\tilde{\Delta} = \partial_{x_1}^2 + \gamma^2 \partial_{x_2}^2 \qquad |\tilde{\nabla} q|^2 = \left(q_{x_1}^2 + \gamma^2 q_{x_2}^2\right).$$

If $\epsilon = \mu^2 = \gamma^2$ then BL yields KP equation. Let:
$\xi = x_1 - t, T = \epsilon t/2, w = q_\xi$:

$$\partial_\xi(w_T - \tilde{\sigma} w_{\xi\xi\xi} + 3(w w_\xi)) + w_{x_2 x_2} = 0$$

Can also find NLS eq.: assume small amplitude, quasi-monchomatic exp'n for η, q *and* use the Riemann-Lesbesq. Lemma; Thus can obtain both shallow and deep water reductions from nonlocal eq.

Interfacial waves–rigid top



One interfacial wave–nonlocal form

$$\int_{\mathbf{R}^2} e^{ikx} \cosh(\kappa(\eta + h))\eta_t \, dx = i \int_{\mathbf{R}^2} e^{ikx} \sinh(\kappa(\eta + h)) \left(\frac{k}{\kappa} \cdot \nabla q \right) dx$$

$$\int_{\mathbf{R}^2} e^{ikx} \cosh(\kappa(\eta - H))\eta_t \, dx = i \int_{\mathbf{R}^2} e^{ikx} \sinh(\kappa(\eta - H)) \left(\frac{k}{\kappa} \cdot \nabla Q \right) dx$$

$$\rho \left(q_t + \frac{1}{2} |\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla \eta)^2}{2\,(1 + |\nabla \eta|^2)} \right) -$$

$$\rho' \left(Q_t + \frac{1}{2} |\nabla Q|^2 + g\eta - \frac{(\eta_t + \nabla Q \cdot \nabla \eta)^2}{2\,(1 + |\nabla \eta|^2)} \right) = \sigma \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right)$$

where quantities defined in prior figure-and earlier.; 3 eq., 3 unkowns η, q, Q: fixed domain!

One interfacial wave–remarks

May derive DN operator, DN series and asymp reductions.
Small ampl., long waves- find ILW-BL eq. (nondim):

$$Q_{tt} - \tilde{c_0}^2 Q_{x_1x_1} - \gamma^2 c_0^2 Q_{x_2x_2} + \epsilon\left(2Q_{x_1}Q_{x_1t} + Q_tQ_{x_1x_1}\right) +$$
$$\mu\left(-\frac{c_0^2}{\alpha\tilde{\rho}}Q_{x_1x_1} + \frac{\tilde{\sigma}}{\tilde{\rho}}Q_{x_1x_1x_1x_1} - \frac{c_0^2}{\tilde{\rho}}i\coth(\alpha D_1)Q_{x_1x_1x_1}\right) = 0\,,$$

where $\mu = \frac{H}{l_x}, \epsilon = \frac{a}{H}, \gamma$ as in std BL, $\tilde{\rho} = \frac{\rho'}{\rho}, \delta = \frac{h}{H}, \alpha = \delta\mu$, $c_0^2 = (1/\tilde{\rho} - 1), \tilde{c_0}^2 = (1/\tilde{\rho} - 1)\left(1 + 1/(\delta\tilde{\rho})\right)$ and psd notation $(D_1 = -i\partial_{x_1})$:

$$\coth(\alpha D_1)Q_{x_1x_1x_1} = \frac{1}{(2\pi)^2 i}\int_{\mathbf{R}^2} e^{ikx}\coth(\alpha k_1)k_1^3\widehat{Q}(k)\,dk\,.$$

One interfacial wave–ILW-KP

Transform: $x = x_1 - \tilde{c}_0 t,\ y = x_2, T = \epsilon t,\ w = Q_x,$ yields ILW-KP

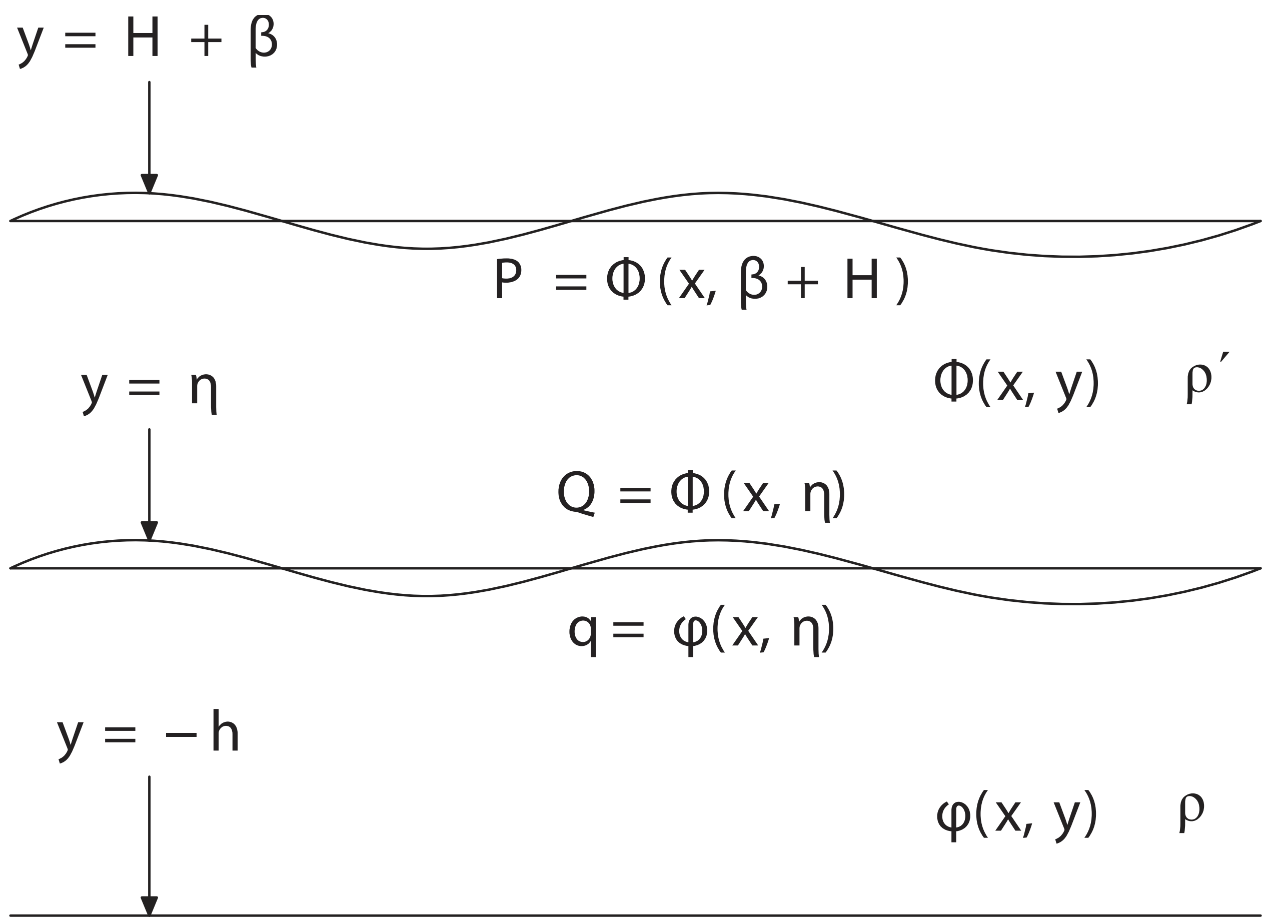
$$2\tilde{c}_0 w_{xt} + \frac{c_0^2}{\tilde{\rho}\alpha} w_{xx} + c_0^2 w_{yy} + \frac{3}{2} c_0 \partial_x^2 \left(w^2\right) - \frac{\tilde{\sigma}}{\tilde{\rho}} w_{xxxx} + \frac{ic_0^2}{\tilde{\rho}} \coth(\alpha D_1) w_{xxx} = 0\,,$$

Recall: $\delta = \frac{h}{H}, \alpha = \delta\mu$; if $\alpha \to 0$ find standard KP eq.; If $\alpha \to \infty$ find BO-KP eq. (cf. MJA, Segur '80)

$$2\tilde{c}_0 w_{xt} + \frac{c_0^2}{\tilde{\rho}\alpha} w_{xx} + c_0^2 w_{yy} + \frac{3}{2} c_0 \partial_x^2 \left(w^2\right) - \frac{\tilde{\sigma}}{\tilde{\rho}} w_{xxxx} - \frac{c_0^2}{\tilde{\rho}} H w_{xxx} = 0\,,$$

where $Hu(x) = P\frac{1}{\pi} \int \frac{u(\xi)}{\xi - x}\, d\xi$

Interfacial waves–two free interfaces



Two-fluid configuration with two free interfaces

Two interfacial waves–nonlocal form

$$\int_{\mathbf{R}^2} e^{ikx}\cosh(\kappa(\eta+h)\eta_t\,dx = i\int_{\mathbf{R}^2} e^{ikx}\frac{\sinh(\kappa(\eta+h)}{\kappa}(k\cdot\nabla)q\}\,dx$$

$$\int_{\mathbf{R}^2} e^{ikx}\sinh(\kappa\beta)\beta_t\,dx - \int_{\mathbf{R}^2} e^{ikx}\sinh(\kappa(\eta-H))\eta_t\,dx =$$

$$-\,i\int_{\mathbf{R}^2} e^{ikx}\frac{\cosh(\kappa(\eta-H))}{\kappa}(k\cdot\nabla)\,Q\,dx + i\int_{\mathbf{R}^2} e^{ikx}\frac{\cosh(\kappa\beta)}{\kappa}(k\cdot\nabla)\,P\,dx$$

$$\int_{\mathbf{R}^2} e^{ikx}\sinh(\kappa(\beta+H))\beta_t\,dx - \int_{\mathbf{R}^2} e^{ikx}\sinh(\kappa\eta)\eta_t\,dx =$$

$$-\,i\int_{\mathbf{R}^2} e^{ikx}\frac{\cosh(\kappa\eta)}{\kappa}(k\cdot\nabla)\,Q\,dx + i\int_{\mathbf{R}^2} e^{ikx}\frac{\cosh(\kappa(\beta+H))}{\kappa}(k\cdot\nabla)\,P\,dx$$

Two interfacial waves–Bernoulli Eq.

$$\rho\left(q_t+\frac{1}{2}|\nabla q|^2+g\eta-\frac{(\eta_t+\nabla q\cdot\nabla\eta)^2}{2\left(1+|\nabla\eta|^2\right)}\right)$$

$$-\rho'\left(Q_t+\frac{1}{2}|\nabla Q|^2+g\eta-\frac{(\eta_t+\nabla Q\cdot\nabla\eta)^2}{2\left(1+|\nabla\eta|^2\right)}\right)=\sigma_1\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)$$

$$P_t+\frac{1}{2}|\nabla P|^2+g\beta-\frac{(\beta_t+\nabla P\cdot\nabla\beta)^2}{2\left(1+|\nabla\beta|^2\right)}=\frac{\sigma_2}{\rho'}\nabla\cdot\left(\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}\right)$$

Thus have 5 eq. in 5 unkowns: η,β,q,Q,P: fixed domain!

Two interfacial waves–Remarks

- Can find integral relations by taking $k \to 0$; e.g.

$$\frac{\partial}{\partial t}\int dx\ (\eta(x,t) - \beta(x,t)) = 0 \quad (Mass)$$

- One interface is a special subcase
- Can find DN operator; connect with Craig et al ’05
- Can find asymptotic reductions; e.g. coupled NLS: small amplitude, quasi-monochomatic limit

Two interfacial waves–Coupled NLS

Let

$$\eta(x, y, t) = e^{i(k_1 x + l_1 y - \omega_1 t)} \eta_{1,0}(X, T) + e^{i(k_2 x - \omega_2 t)} \eta_{0,1}(X, T) + c.c. + O(\epsilon)\,,$$

$$\beta(x, y, t) = e^{i(k_1 x_1 + l_1 y - \omega_1 t)} \beta_{1,0}(X, T) + e^{i(k_2 x - \omega_2 t)} \beta_{0,1}(X, T) + c.c. + O(\epsilon)\,,$$

where $X = \epsilon x$, $T = \epsilon t$. Find coupled NLS

$$i f_\tau + \delta_1 f_{\xi\xi} + \delta_2 |f|^2 f + \delta_3 |g|^2 f = 0\,,$$

$$i g_\tau + i \Delta g_\xi + \delta_4 g_{\xi\xi} + \delta_5 |g|^2 g + \delta_6 |f|^2 g = 0\,,$$

where $\xi = X - C_{g_1} T$, $\tau = \epsilon T$; the coef. $\delta_1, \ldots, \delta_6$ depend on k_1, l_1, k_2, ω_1, ω_2, and fluid parameters, $\delta_1 = \omega_{1,k_1 k_1}/2, \delta_4 = \omega_{2,k_2 k_2}/2, f(\xi, \tau) = \eta_{1,0}(X, T), g(\xi, \tau) = \beta_{0,1}(X, T), C_{g_2} = C_{g_1} + \epsilon \Delta$

Lumps

KP equation in standard form

$$\partial_x(u_t + 6uu_x + u_{xxx}) - 3sgn(\tilde{\sigma})u_{yy} = 0$$

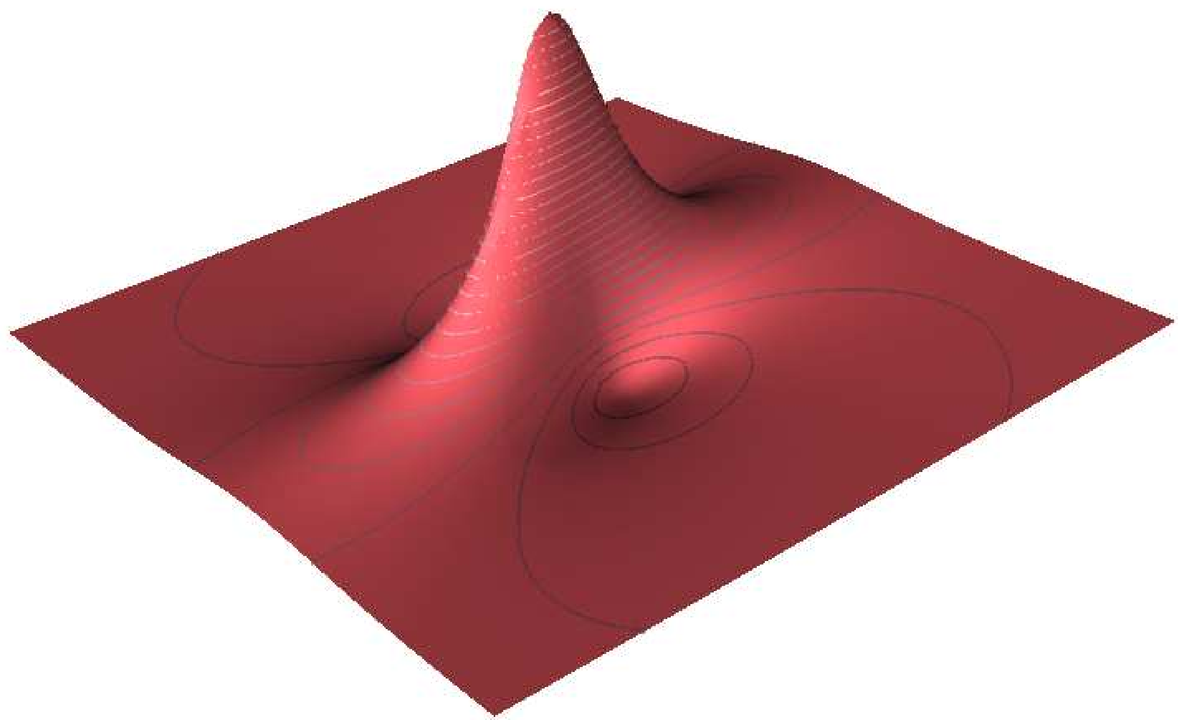
$\tilde{\sigma} > 0$ lumps; $\tilde{\sigma}$ normalized surface tension.
The 1-Lump solution is given by:

$$u = 16\frac{-4(x' - 2k_R y')^2 + 16k_I^2 y'^2 + \frac{1}{k_I^2}}{[4(x' - 2k_R y')^2 + 16k_I^2 y'^2 + \frac{1}{k_I^2}]^2},$$

where $x' = x - c_x t, y' = y - c_y t, c_x = 12(k_R^2 + k_I^2), c_y = 12k_R;$

$$u(0,0) = \frac{4}{3}(c_x - \frac{c_y^2}{12}) > 0$$

Lump Solution of KP



Recall BL eq.

Find Benney-Luke (1964) system ($\tilde{\sigma}$ related to nmlz'd surface tension):

$$q_{tt} - \tilde{\Delta} q + \tilde{\sigma}\mu^2 \tilde{\Delta}^2 q + \varepsilon(\partial_t |\tilde{\nabla} q|^2 + q_t \tilde{\Delta} q) = 0 \ (BL)$$

$(x_1, x_2) \to (x, y),\ \tilde{\Delta} = \partial_x^2 + \gamma^2 \partial_y^2 \qquad |\tilde{\nabla} q|^2 = \left(q_x^2 + \gamma^2 q_y^2\right)$.

When $\epsilon = \mu^2 = \gamma^2$, BL yields KP equation. Let:
$\xi = x - t, T = \epsilon t/2, w = q_\xi$:

$$\partial_\xi (w_T - \tilde{\sigma} w_{\xi\xi\xi} + 3(w w_\xi)) + w_{yy} = 0$$

BL Equation and Lumps

$q = q(x - v_x t, y - v_y t), v_x = 1 - \epsilon c_x; v_y = c_y.$
Below: $c_x = 3, c_y = 0, \tilde{\sigma} = 1/3;$

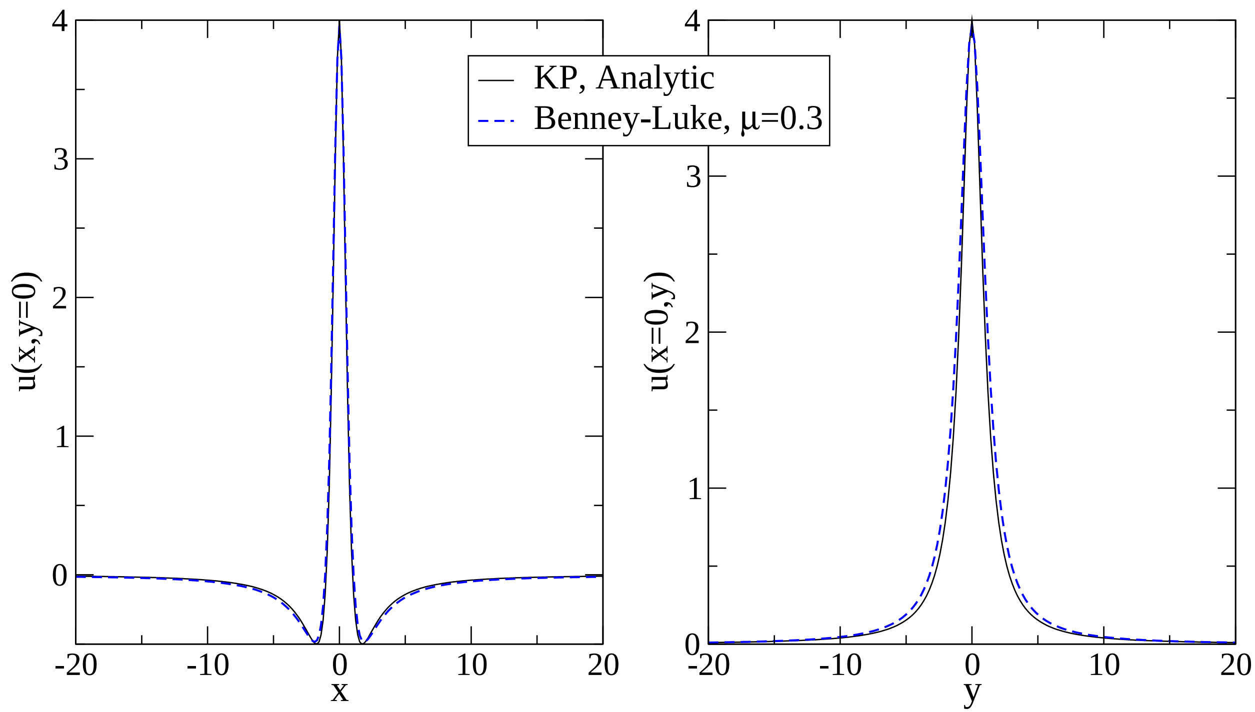


Figure 1: Wave profiles for the KP and BL eq.

BL Eq.and Lumps-con't

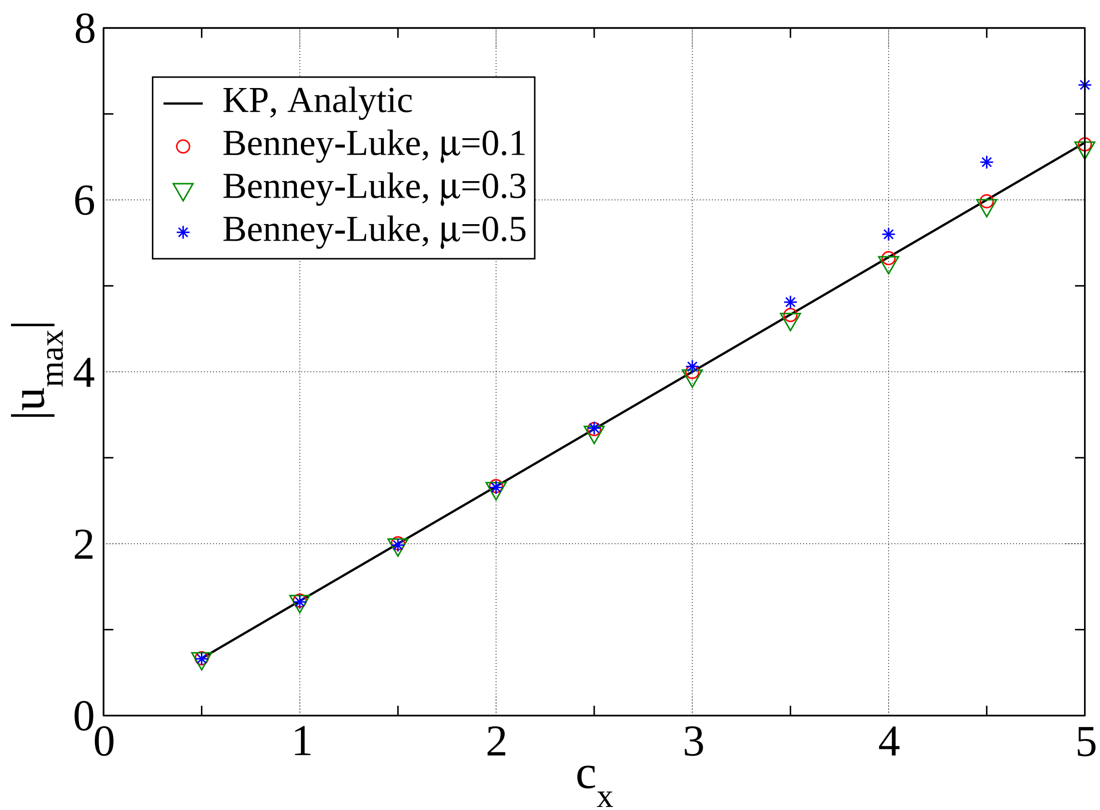


Figure 2: $u(0,0) = u_{max}$ vs. c_x for various values of μ. Fig. shows that KP is a good approx. to the BL equation in this range of parameters.

ILW-KP Eq.

Recall

$$2\tilde{c}_0 w_{xt} + \frac{c_0^2}{\tilde{\rho}\alpha} w_{xx} + c_0^2 w_{yy} + \frac{3}{2} c_0 \partial_x^2 \left(w^2\right) - \frac{\tilde{\sigma}}{\tilde{\rho}} w_{xxxx} + \frac{i c_0^2}{\tilde{\rho}} \coth(\alpha D_1) w_{xxx} = 0\,,$$

where: $\delta = h/H, \;\; \alpha = \delta\mu, \;\; \tilde{\rho} = \rho'/\rho, \;\; D_1 = -i\partial_{x_1}$
and
$c_0^2 = (1/\tilde{\rho} - 1), \tilde{c}_0{}^2 = (1/\tilde{\rho} - 1)\left(1 + 1/(\delta\tilde{\rho})\right)$.

Figures below– typical values: $\tilde{\rho} = 1/2, \tilde{\sigma} = 1$

ILW-KP Lumps

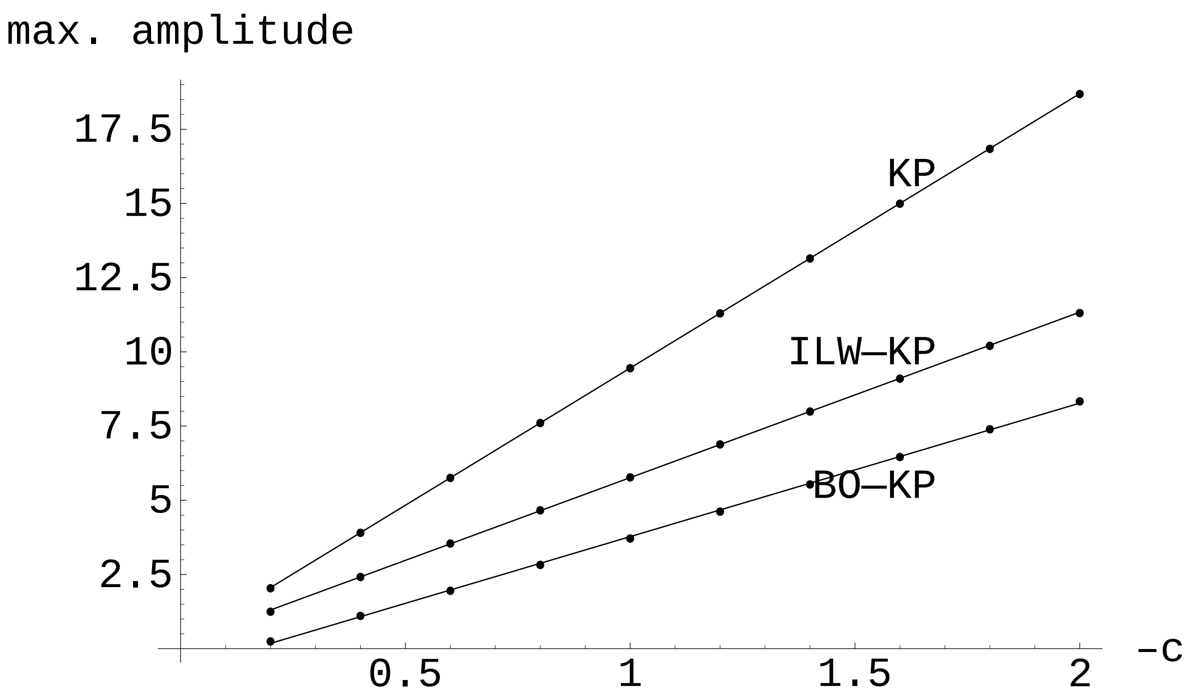


Figure 3: Max. amp. vs -c; x-velocity= c with 0 y-velocity $\delta = 1, \alpha = 1/10$; $\delta = 10; \alpha = 1$, and $\delta = 100; \alpha = 10$, i.e. KP. ILW-KP, and BO-KP resp.

ILW-KP Lumps-con't

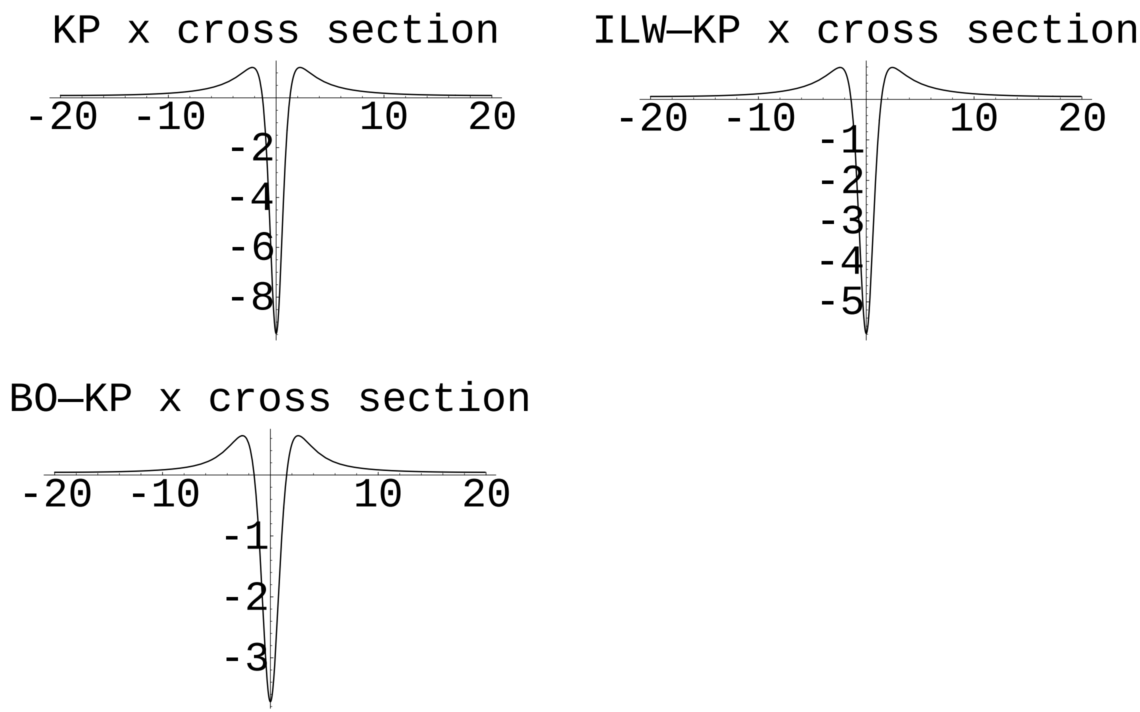


Figure 4: x cross sections $w(x,0)$ $\delta = 1; \alpha = 1/10$, $\delta = 10; \alpha = 1$, and $\delta = 100; \alpha = 10$, i.e. KP. ILW-KP, and BO-KP resp. In each case the x-dir. speed $c = -1$.

ILW-KP Lumps-con't

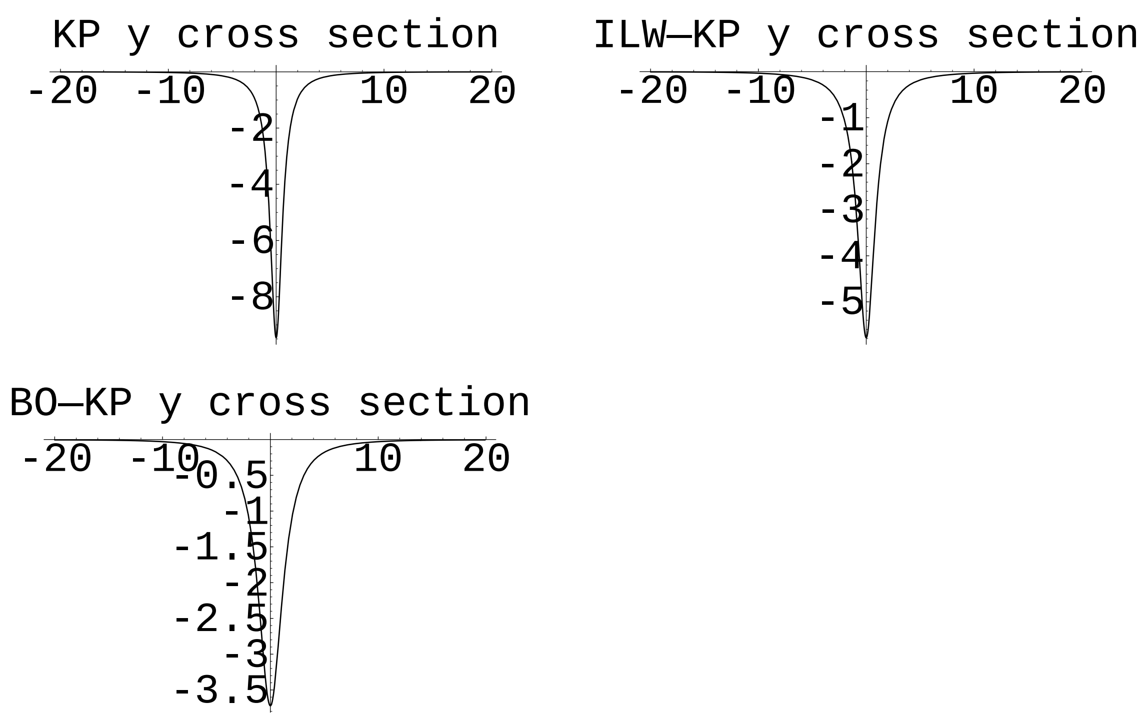


Figure 5: y cross sections $w(0, y)$ $\delta = 1; \alpha = 1/10$, $\delta = 10; \alpha = 1$, and $\delta = 100; \alpha = 10$, i.e. the KP. ILW-KP, and BO-KP regimes, respectively. In each case, the x-dir. speed $c = -1$.

Lumps from nonlocal WW Equations

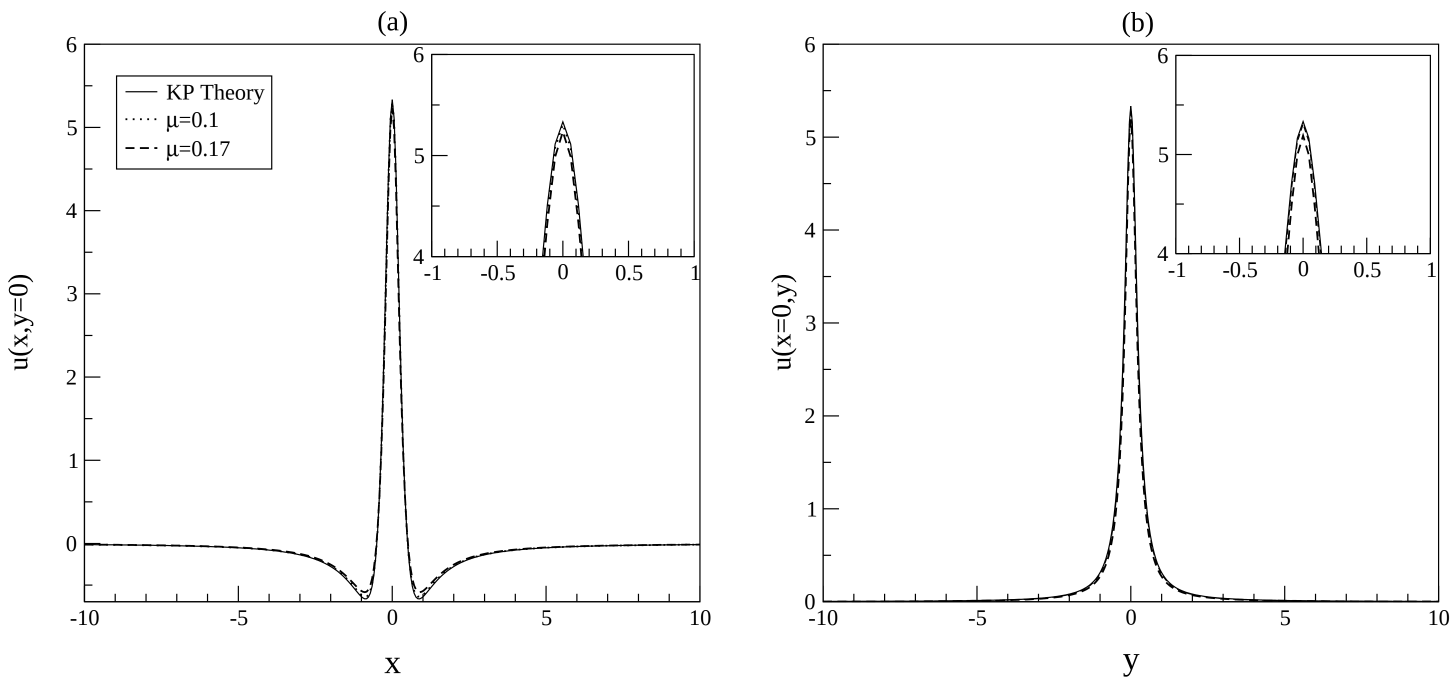


Figure 6: Wave profiles for the full WW equations, $c_x = 4.0, c_y = 0$ Benney-Luke/KP equations are good approx. to the full WW eq. in this range of parameters.

Lumps-WW Equations-figs con't

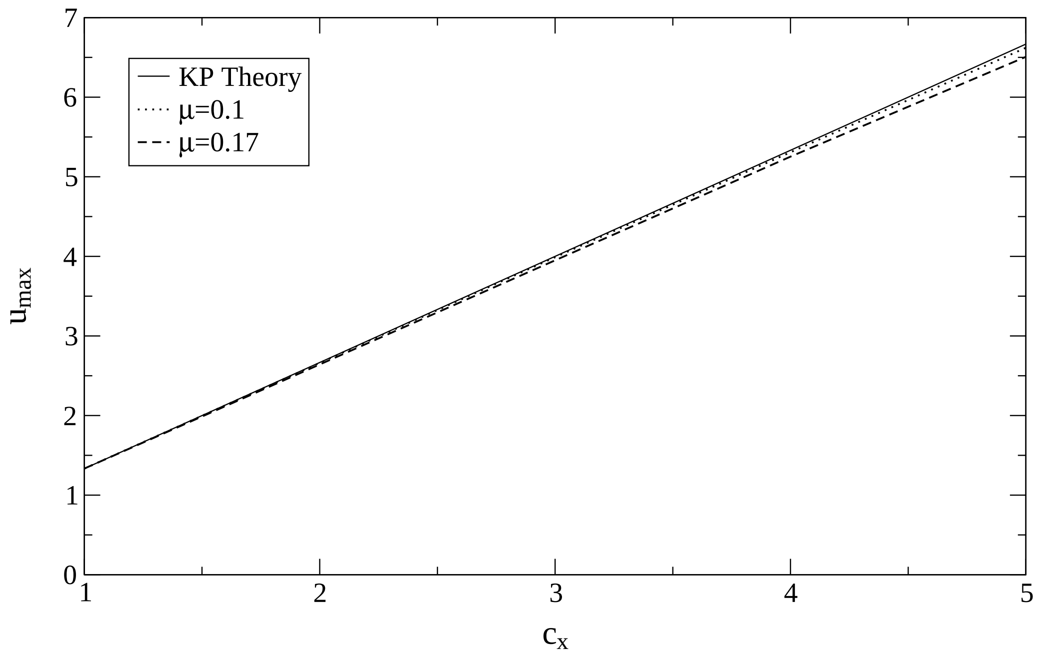


Figure 7: u_{max} vs. c_x the full WW equations for various values of μ. Fig. also shows that the Benney-Luke/KP equations are good approximations to full WW eq. in this range of parameters.

Conclusions

- Derive nonlocal spectral formulations of water waves and interfacial waves on fixed domain
- Find Dirichlet-Neumann operator/series; results agree with those of Craig et al
- Formulations lead to integral relations, asymptotic reductions-long waves, quasi-monochromatic waves
- Can find lump solutions numerically