Temporally-periodic solitons of the parametrically driven damped nonlinear Schrödinger equation

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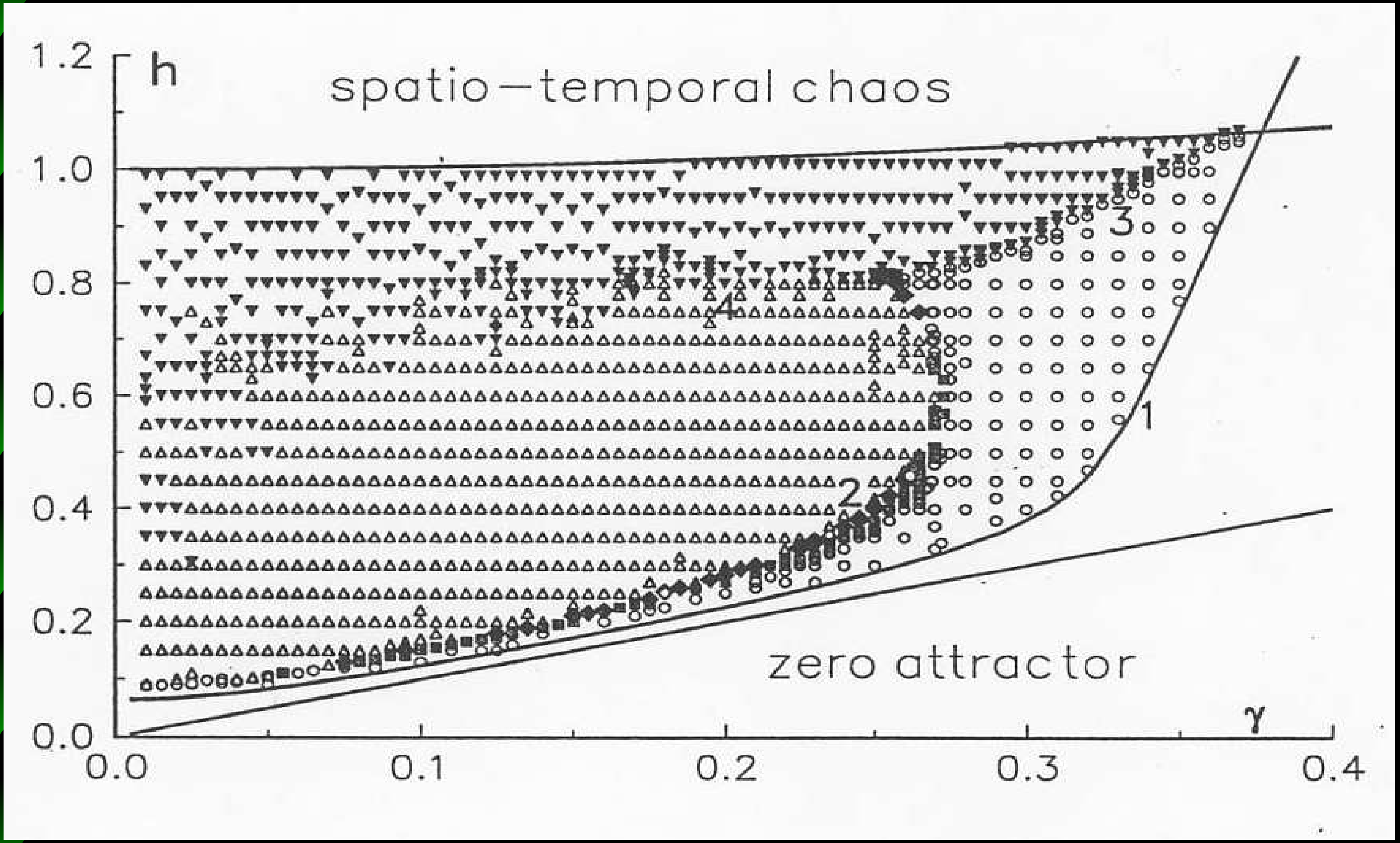
Motivation

We consider the parametrically driven, damped NLS

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi$$



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Aims

Analysing the boundary value problem instead of the initial-boundary value problem provides insights into:

- Period doubling of single time-periodic solitons and time-periodic bound states
- Direct transiton from periodic soliton to spatiotemporal chaos
- Relation between single solitons and bound states

Parametrically driven damped NLS

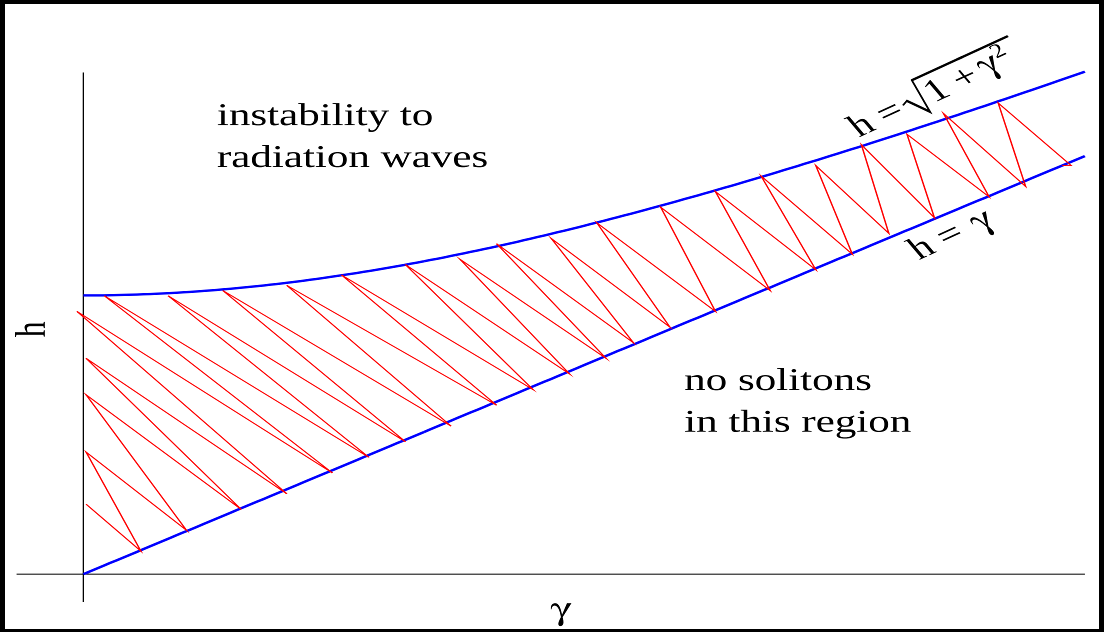
$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi,$$

Explicit soliton solution

$$\psi^{\pm} = A_{\pm} e^{-i\theta_{\pm}} \operatorname{sech}(A_{\pm} x), \quad \text{where}$$

$$\begin{aligned} A_+ &= \sqrt{1 + \sqrt{h^2 - \gamma^2}}, \quad \theta_+ = \frac{1}{2} \arcsin\left(\frac{\gamma}{h}\right), \\ A_- &= \sqrt{1 - \sqrt{h^2 - \gamma^2}}, \quad \theta_- = \frac{\pi}{2} - \theta_+, \end{aligned}$$

Existence region



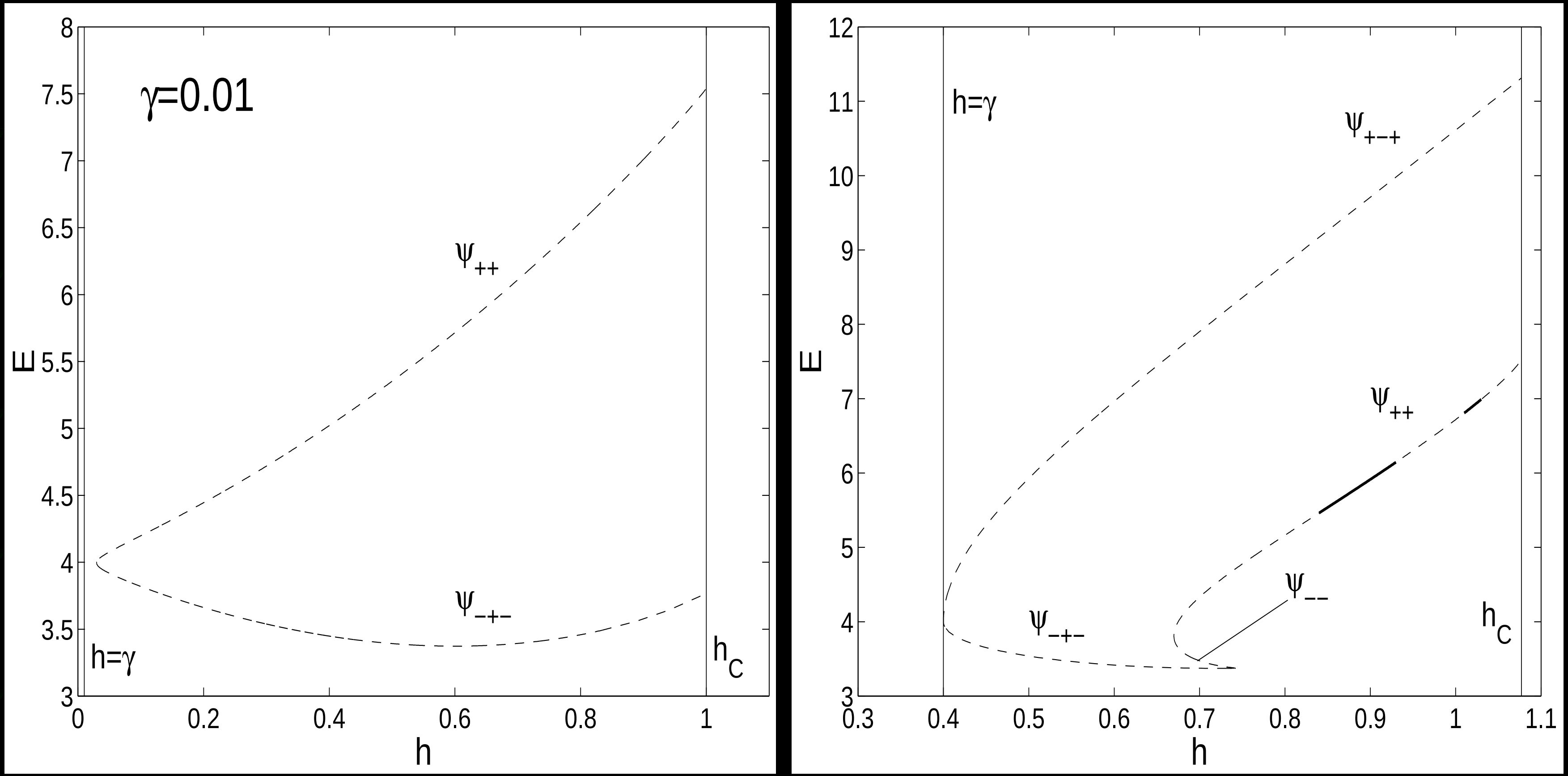
For $h < \gamma$ ***all*** initial conditions decay to zero:

$$\dot{N} + 2\gamma N = ih\int(\psi^2 - \psi^{*2})dx, \quad \text{where} \quad N = \int|\psi|^2 dx.$$

Since $\dot{N} \leq 2(h-\gamma)N,$ we have $N(t) \to 0$ as $t \to \infty.$

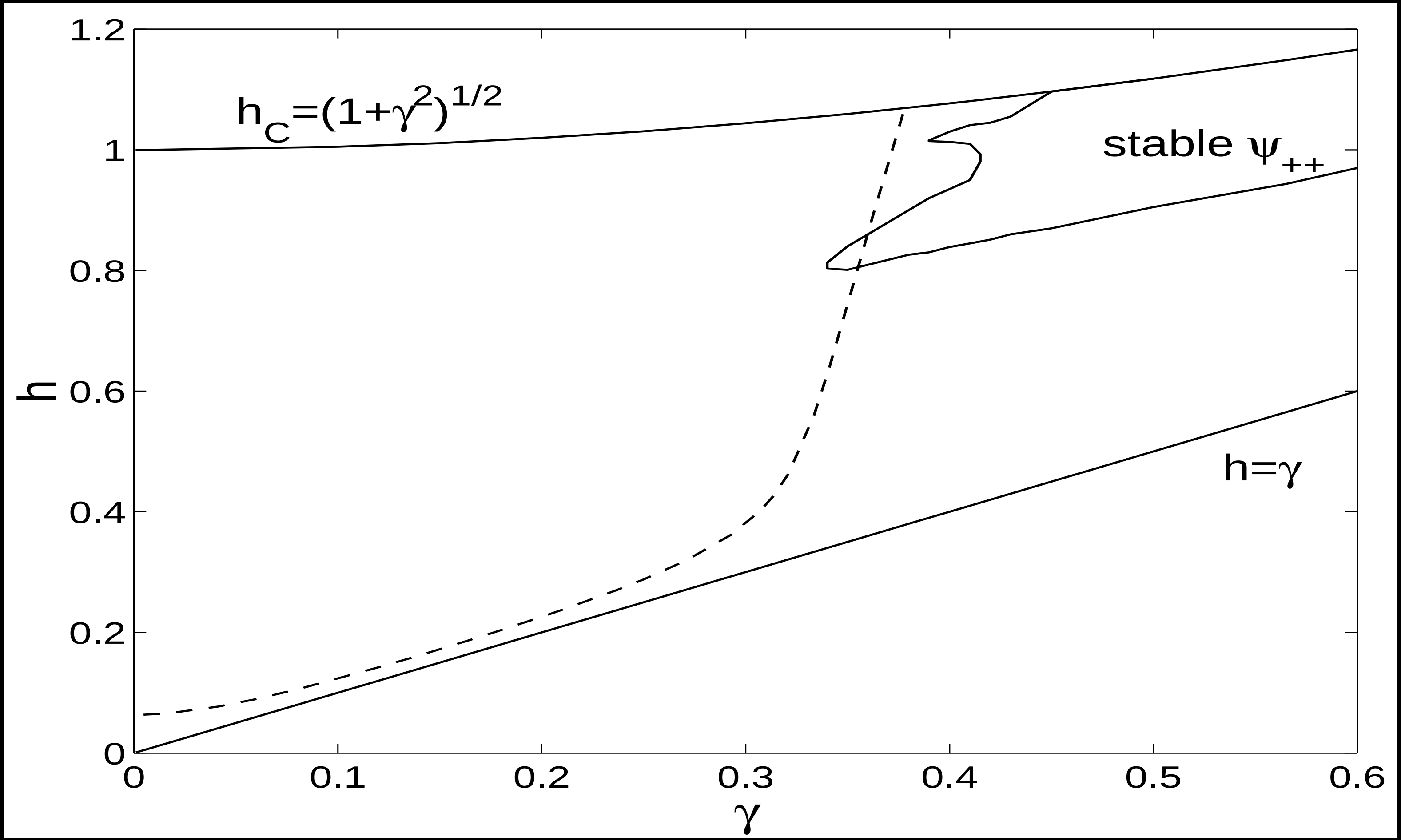
Bound states

$$E = \frac{1}{T}\int_0^T dt \int_{-\infty}^{\infty} \mathcal{E}(x,t)dx,\ \mathcal{E}(x,t) = |\psi_x|^2 + |\psi|^2 - |\psi|^4 + h\mathrm{Re}(\psi^2)$$



Energy of the stationary multisoliton solutions obtained by continuation in h for the fixed $\gamma = 0.01$ and $\gamma = 0.4$. Solid curves show stable and the dashed ones unstable solutions.

Stability of stationary solitons and bound states



The existence and stability chart of the stationary single soliton and two-soliton complexes.

Stability of time-periodic solitons

Periodic solution $\psi_0(x,t) = \mathcal{R}(x,t) + i\mathcal{I}(x,t)$

perturbed by $\psi(x,t) = \psi_0(x,t) + u(x,t) + iv(x,t)$

gives $J\mathbf{w}_t = \mathcal{H}\mathbf{w}$, with $\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$\mathcal{H} = \begin{pmatrix} -\partial_x^2 + 1 + h - 6\mathcal{R}^2 - 2\mathcal{I}^2 & -4\mathcal{R}\mathcal{I} + \gamma \\ -4\mathcal{R}\mathcal{I} - \gamma & -\partial_x^2 + 1 - h - 2\mathcal{R}^2 - 6\mathcal{I}^2 \end{pmatrix}$$

Solution with initial condition $\mathbf{w}(x,0)$ can be written as

$$\mathbf{w}(x,t) = \mathcal{M}_t\mathbf{w}(x,0)$$

If the period of $\psi_0(x,t)$ is T:

$$\mathcal{M}_T\mathbf{w}(x) = \mu\mathbf{w}(x)$$

Floquet multipliers

Consider the eigenvalue problem

$$\mathcal{M}_T \mathbf{w}(x) = \mu \mathbf{w}(x)$$

Eigenvalues μ are Floquet multipliers and λ, where $\mu = e^{\lambda T}$, are Floquet exponents.
For each λ there is a solution $\mathbf{w}(x,t)$ such that

$$\mathbf{w}(x,t) = e^{\lambda t}\mathbf{p}(x,t),$$

where $\mathbf{p}(x,t)$ is a periodic function: $\mathbf{p}(x,t+T) = \mathbf{p}(x,t)$ for all t.

Numerical stability analysis

Expand

$$\mathbf{w}(x,t) = \sum_{n=-N}^{N} \mathbf{w}_n e^{-iq_n x} = \sum_{n=-N}^{N} \begin{pmatrix} u_n \\ v_n \end{pmatrix} e^{-iq_n x}$$

Here $\mathbf{w}_n = \mathbf{w}_n(t), \quad q_n = \pi n / L$. Substituting into linearised problem gives

$$J\dot{\mathbf{w}}_m = \sum_{n=-N}^{N} H_{mn}(t)\mathbf{w}_n,$$

with $\quad H_{mn}(t) = \frac{1}{2L}\int_{-L}^{L} e^{i(q_m - q_n)x} \times$

$$\begin{pmatrix} q_n^2 + 1 + h - 6\mathfrak{R}^2 - 2\mathfrak{I}^2 & -4\mathfrak{R}\mathfrak{I} + \gamma \\ -4\mathfrak{R}\mathfrak{I} - \gamma & q_n^2 + 1 - h - 2\mathfrak{R}^2 - 6\mathfrak{I}^2 \end{pmatrix} dx.$$

Solution of the form

$$u_n(t) = e^{\lambda t} f_n(t), \quad v_n(t) = e^{\lambda t} g_n(t),$$

where $f_n(t+T) = f_n(t)$, $g_n(t+T) = g_n(t)$ for all t.

Form $\quad \vec{w} = (u_{-N}, v_{-N}, u_{-N+1}, v_{-N+1}, ..., u_N, v_N)$.

The problem reduces to $\vec{w}_t = \hat{H}\vec{w}$, where $\hat{H} = J^{-1} H_{mn}$.
The principal fundamental matrix:

$$M_t = \left(\vec{w}^{(1)}_{-N}, \vec{w}^{(2)}_{-N}, ..., \vec{w}^{(1)}_{N}, \vec{w}^{(2)}_{N} \right).$$

Here $\vec{w}^{(1,2)}_{\alpha}(t)$ (with $\alpha = -N, ..., N$) is the solution with the ICs

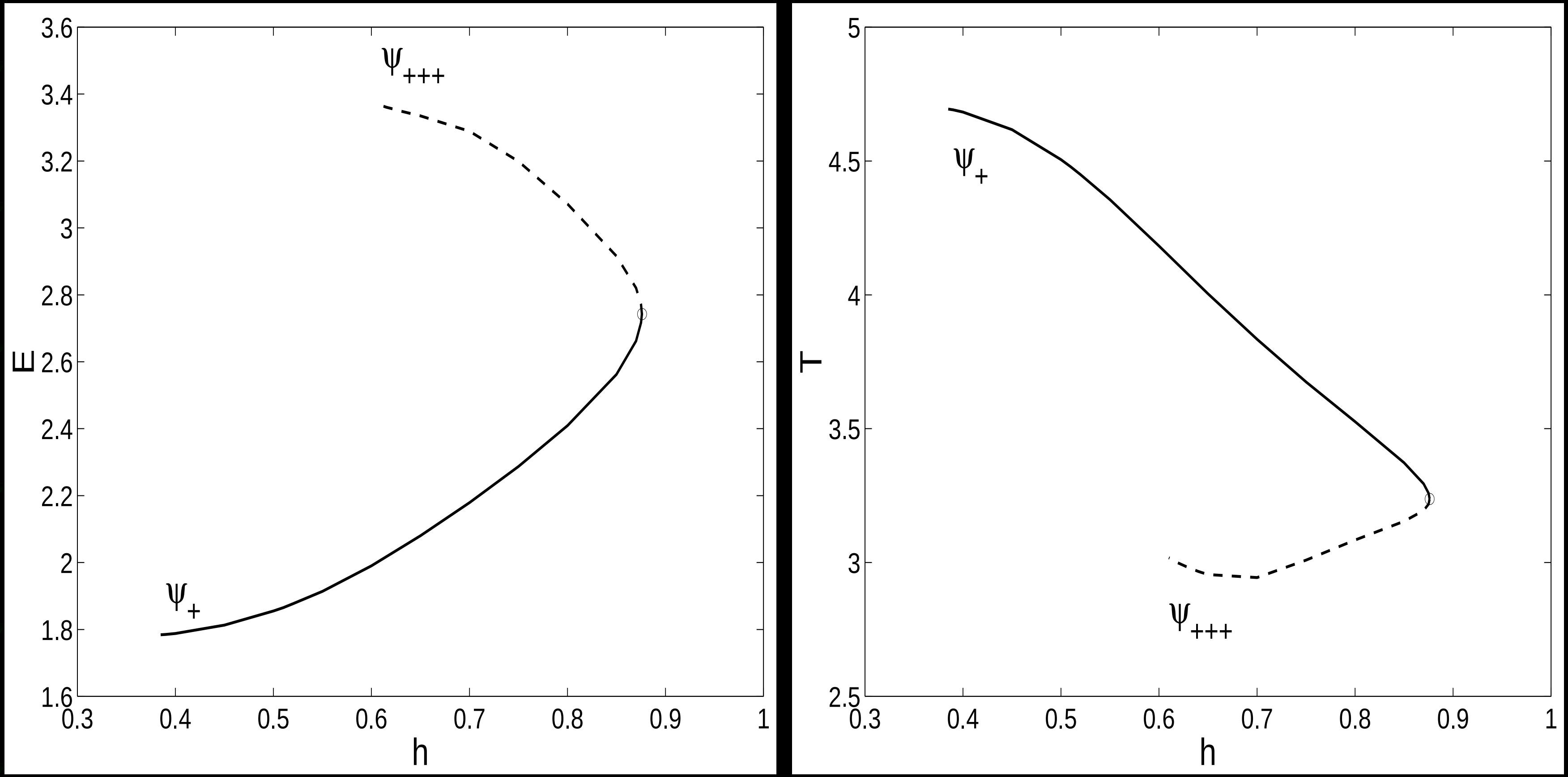
$$u_n(0) = \delta_{n\alpha}, \quad v_n(0) = 0 \quad (n = -N, ..., N),$$

$$u_n(0) = 0, \quad v_n(0) = \delta_{n\alpha} \quad (n = -N, ..., N).$$

Floquet multipliers are eigenvalues of the monodromy matrix with $t = T$

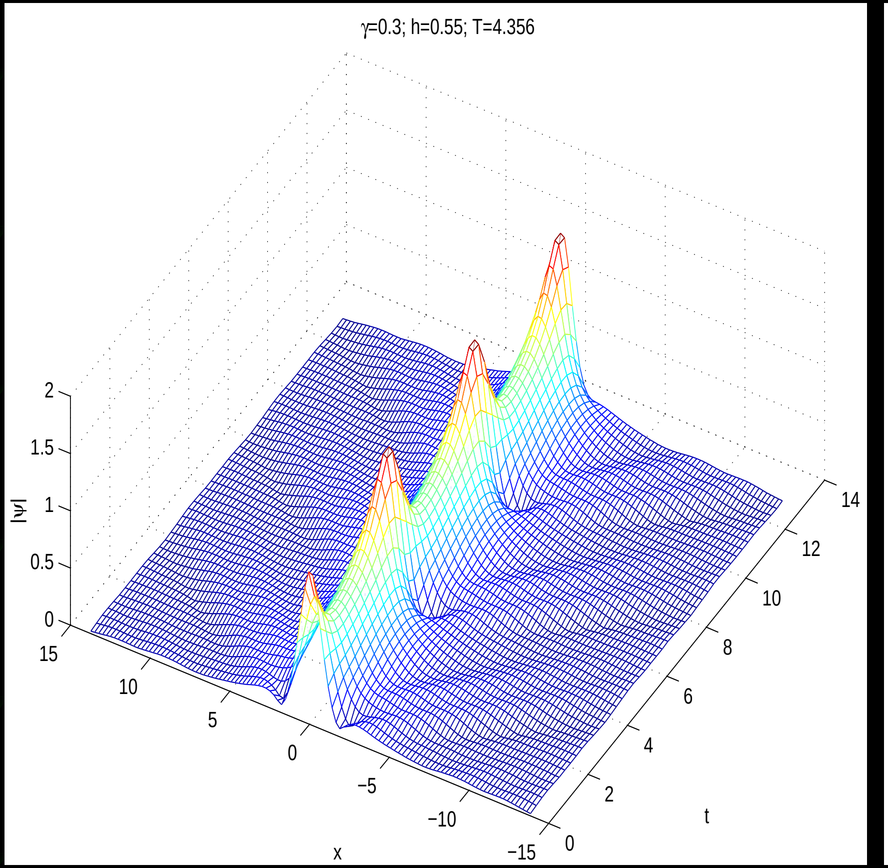
$$M_T \vec{w} = \mu \vec{w}.$$

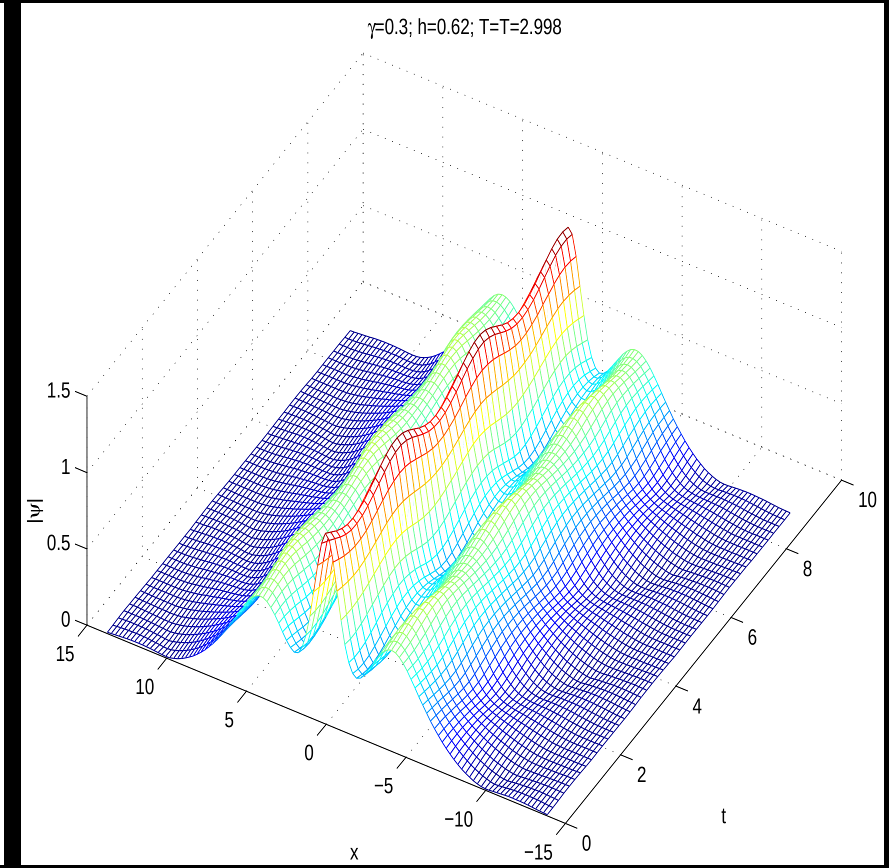
No period doubling for moderate dampings γ



The energy and the period of the periodic solution for $\gamma = 0.3$. The solid curve shows the stable and the dashed one unstable branch.

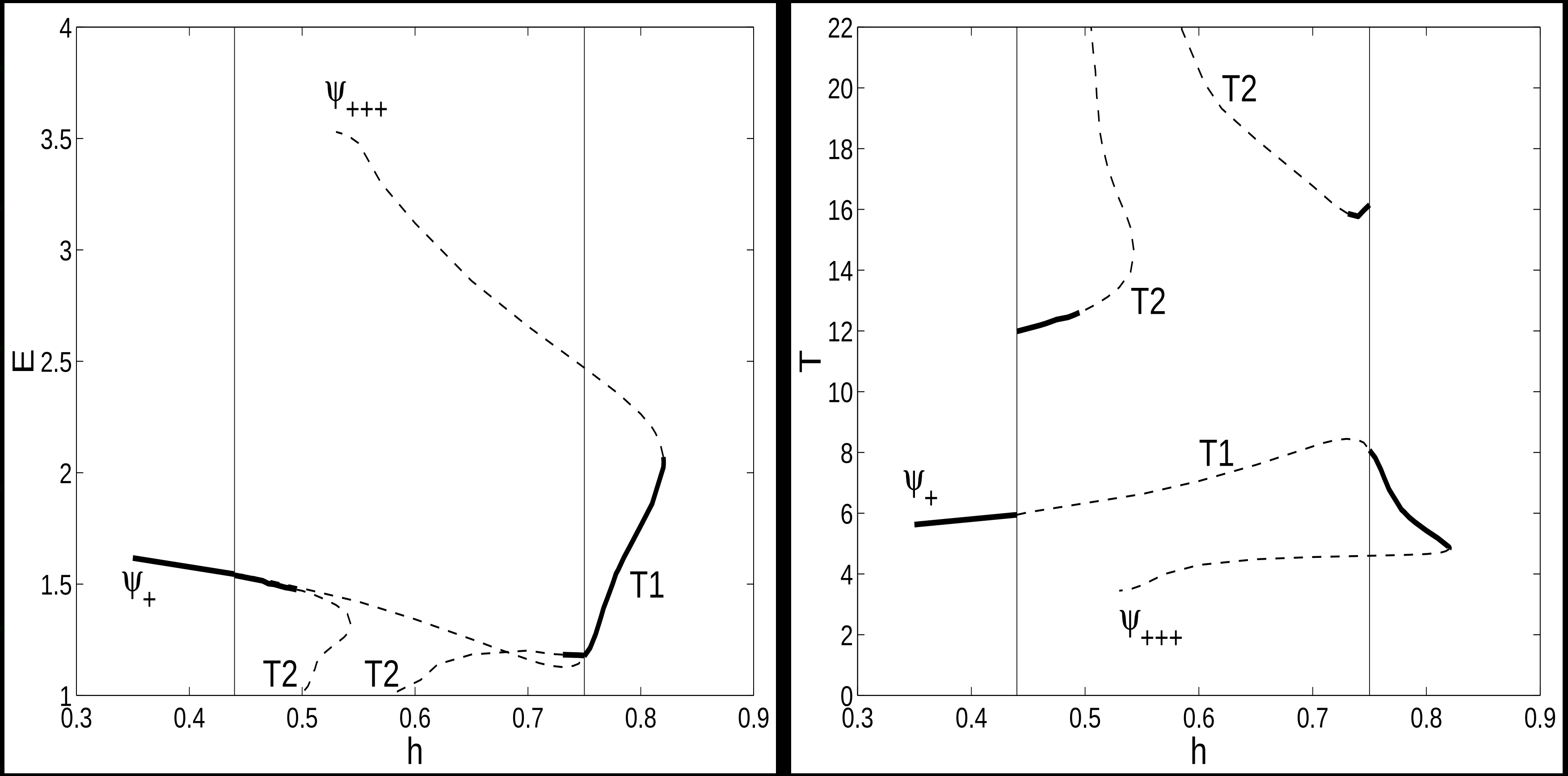
Time evolution for $\gamma = 0.3$





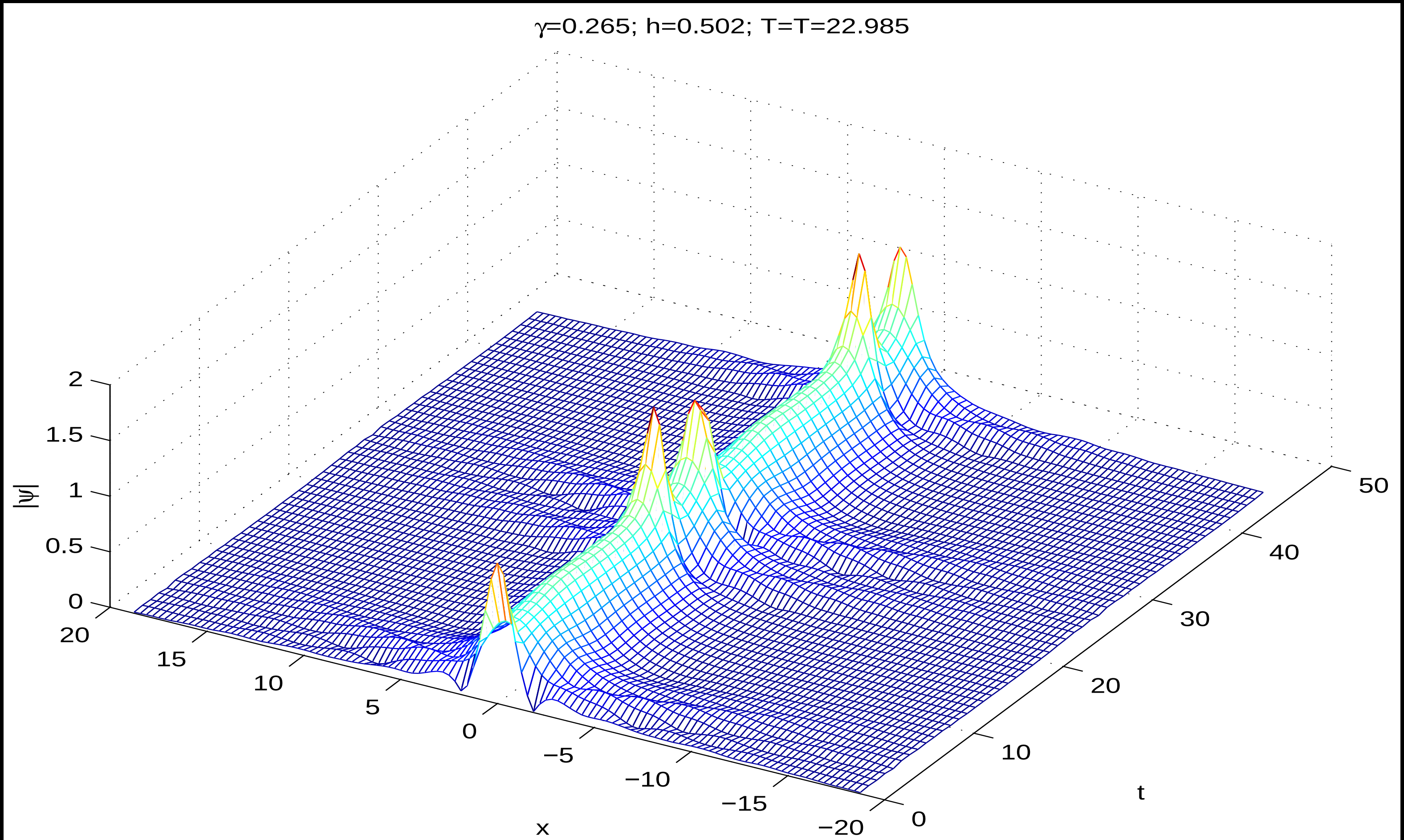
The absolute value of the periodic solution with $\gamma = 0.3$ for $h = 0.55$, $T = 4.356$ and $h = 0.62$, $T = 2.998$. In each case several periods of oscillation are shown.

Period doubling for small dampings γ



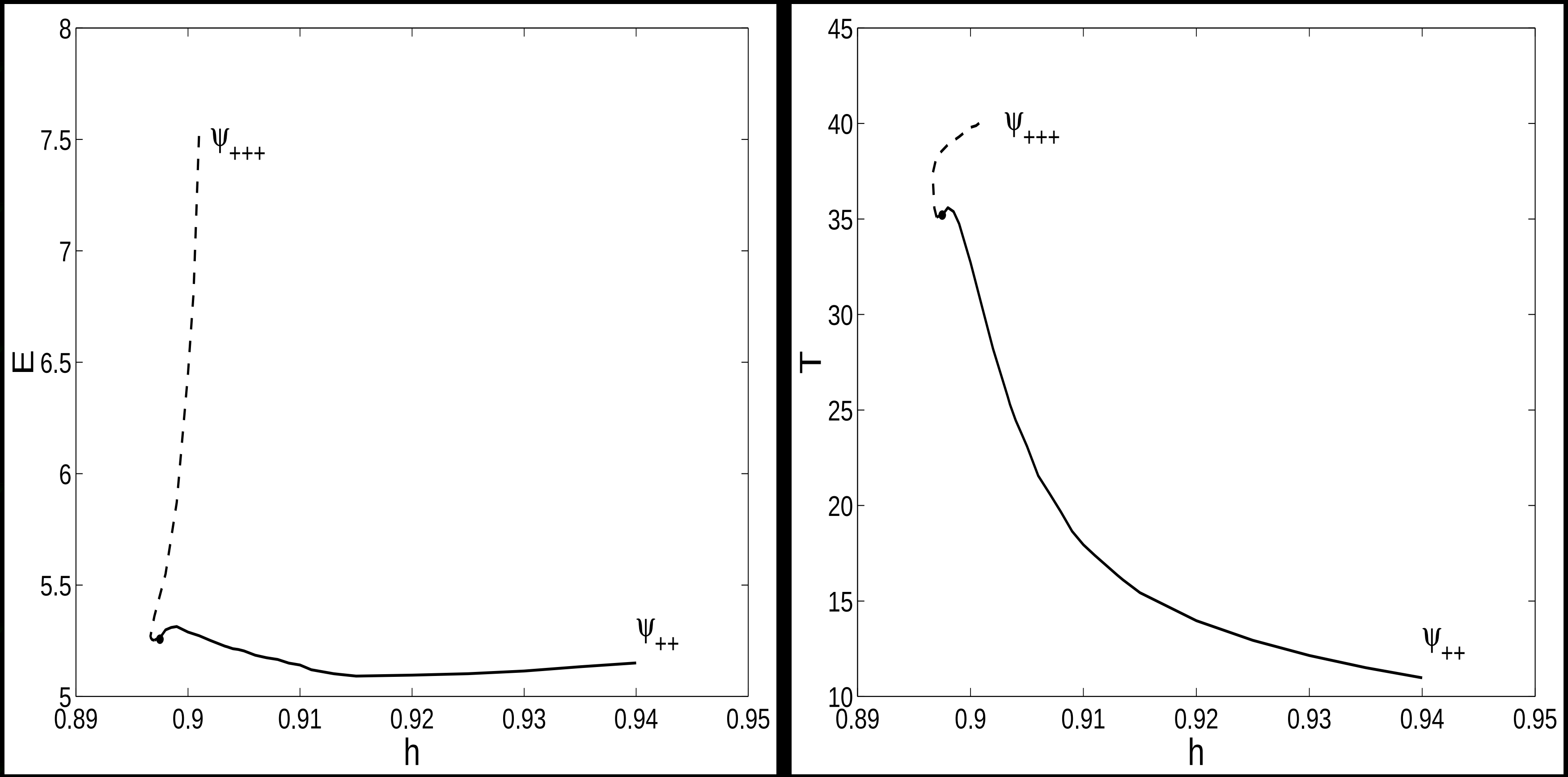
The energy and the period of the periodic solution for $\gamma = 0.265$. Solid curves show the stable and the dashed ones unstable branches.

Teme evolution for $\gamma = 0.265$



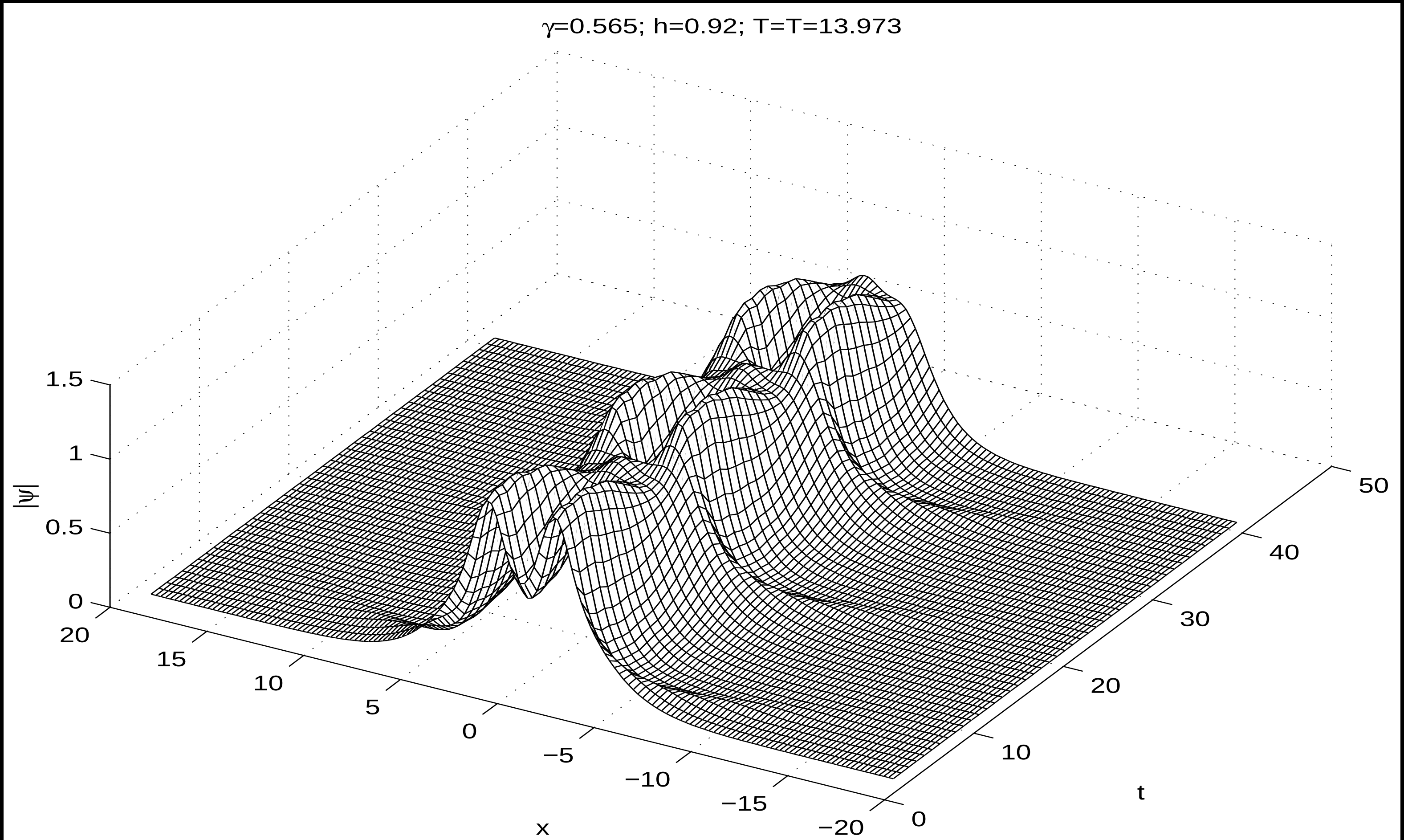
The absolute value of the double-periodic solution for $h = 0.502$, $T \approx 23$. A rapid oscillation is followed by a long quasistationary epoch.

Strong damping γ



The energy and the period of the periodic solution for $\gamma = 0.565$. The solid curve shows the stable and the dashed one unstable branch. The stability boundary is marked by a black blob.

Time evolution of periodic bound state



Clearly visible is the two-soliton spatial structure of the solution. Here $h = 0.92$, $T \approx 14$.

Conclusions

- Small damping: Period doubling confirmed; character of the bifurcation elucidated; accurate bifurcation value determined
- Large damping: Direct transition from temporally periodic soliton to spatiotemporal chaos explained; accurate value for the onset of the spatiotemporal chaos determined
- Bifurcations of bound states of solitons described