Soliton resonance and web structure: the Davey-Stewartson equation

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joint work with

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Introduction and motivation

- (2+1)-dimensional integrable systems have a large variety of solutions.
- KPII has a large family of real, nonsingular soliton solutions exhibiting a rich phenomenology, including soliton resonance and web structure.

  These solutions likely to be stable and physically relevant.
- A nontrivial connection exists between soliton solutions of KPII and combinatorial algebraic geometry.
- Soliton resonance and web structure are a generic feature of (2+1)-D integrable systems.

Here: Davey-Stewartson (DS) equation; more precisely, defocusing DSII.

- The underlying mathematical structure of the soliton solutions of DSII is similar to that of KPII.
- At the same time, the physical behavior of the solutions is richer than that of KPII, and includes V-shape solitons and soliton reconnection.

Outline

1. The Kadomtsev-Petviashvili equation

    - KPII, its Wronskian solutions and its line solitons
    - Ordinary soliton solutions and fully resonant solutions
    - Asymptotics line solitons and index pairs
    - Elastic 2-soliton solutions
    - Elastic N-soliton solutions

2. Soliton resonance in other (2+1)-dimensional integrable systems

3. The Davey-Stewartson equation

    - DSII, its Wronskian solutions and its line solitons
    - Restricted soliton solutions
    - V-shape solitons
    - Soliton reconnection
    - Unrestricted solutions, general case

The Kadomtsev-Petviashvili equation and its Wronskian solutions

- Kadomtsev-Petviashvili (KP) equation: $\sigma = i$: KPI; $\sigma = 1$: KPII

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + \sigma^2\frac{\partial^2 u}{\partial y^2} = 0\,.$$

- Bilinear form: $u(x,y,t) = 2\dfrac{\partial^2}{\partial x^2}\log\tau(x,y,t)\,.$

Then the tau-function $\tau(x,y,t)$ satisfies Hirota's bilinear equation:

$$(-4D_xD_t + D_x^4 + 3\sigma^2 D_y^2)\,\tau\cdot\tau = 0\,.$$

Hirota derivatives: $D_x^m f\cdot g = (\partial_x - \partial_{x'})^m f(x,y,t)g(x',y,t)|_{x'=x}$ etc.

- Wronskian solutions:

$$\tau(x,y,t) = \mathrm{Wr}(f_1,\ldots,f_N) = \begin{vmatrix} f_1^{(0)} & \cdots & f_N^{(0)} \\ \vdots & \ddots & \vdots \\ f_1^{(N-1)} & \cdots & f_N^{(N-1)} \end{vmatrix},$$

with

$$\frac{\partial f_n}{\partial y} = \sigma\frac{\partial^2 f_n}{\partial x^2}\,, \qquad \frac{\partial f_n}{\partial t} = \frac{\partial^3 f_n}{\partial x^3}\,.$$

Ordinary line solitons and fully resonant solutions

- Line soliton: take $N = 1$ and $f = e^{\theta_1} + e^{\theta_2}$, with $\theta_m = k_m x + k_m^2 y + k_m^3 t + \theta_{m,0}$ and $k_1, k_2 \in \mathbb{R}$.

$$u = \tfrac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2[\tfrac{1}{2}(\theta_2 - \theta_1)]\,,$$

  travelling wave solution, localized in the xy-plane along the line $\theta_1 = \theta_2$.
  - soliton direction: $c = -dx/dy = k_1 + k_2$. (“velocity” in the xy-plane)
  - soliton amplitude: $a = k_2 - k_1$.

- Ordinary N-soliton solutions: $f_n = e^{\theta_{2n-1}} + e^{\theta_{2n}}\,, \quad n = 1, \dots, N\,.$
  (Pattern in the xy-plane: intersection of N lines, plus small phase shifts.)

- Different choice of eigenfunctions: $f_n = \partial^{n-1} f / \partial x^{n-1}\,, \quad n = 1, \dots, N$.
  (These τ_N's also generate solutions of the Toda lattice hierarchy.)

- Take $f = e^{\theta_1} + \cdots + e^{\theta_M}$, with θ_m as before and $k_1 < \cdots < k_M$.
  Also a combination of exponential terms, like ordinary soliton solutions. Here however each eigenfunction f_n contains all exponentials.

General properties of fully resonant solutions

- Theorem: τ_N produces an (N_-,N_+)-soliton solution:

  $N_- := M-N$ line solitons as $y \to -\infty$, identified by $[n,n+N]$, $n=1,\ldots,M-N$.

  $N_+ := N$ line solitons as $y \to \infty$, identified by $[n,n+M-N]$, $n=1,\ldots,N$.

  The amplitudes and directions of these solitons are completely determined by these index pairs.

- The solution describes a fully resonant interaction of line solitons:

  - the whole interaction is a collection of fundamental resonances;
  - at each resonant vertex Miles' resonance condition is satisfied;
  - the interaction creates a web-like structure with $(N_+-1)(N_--1)$ holes.

- When $M=2N$, N-soliton solutions are produced.

  However these are not the same as ordinary N-soliton solutions.

  - the amplitudes and directions of the solitons are different,
  - the interaction process is different (Y-junctions vs X-junctions; each vertex of an ordinary solution becomes four vertices and a hole).

[GB & Y Kodama, 2003]

Fully resonant solutions: examples

Top:

$N = 1, \;\; M = 5 \;\; \Rightarrow$

(4,1)-soliton solution.

$(k_1, \ldots, k_5) = (-1, -\frac{1}{4}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2})$

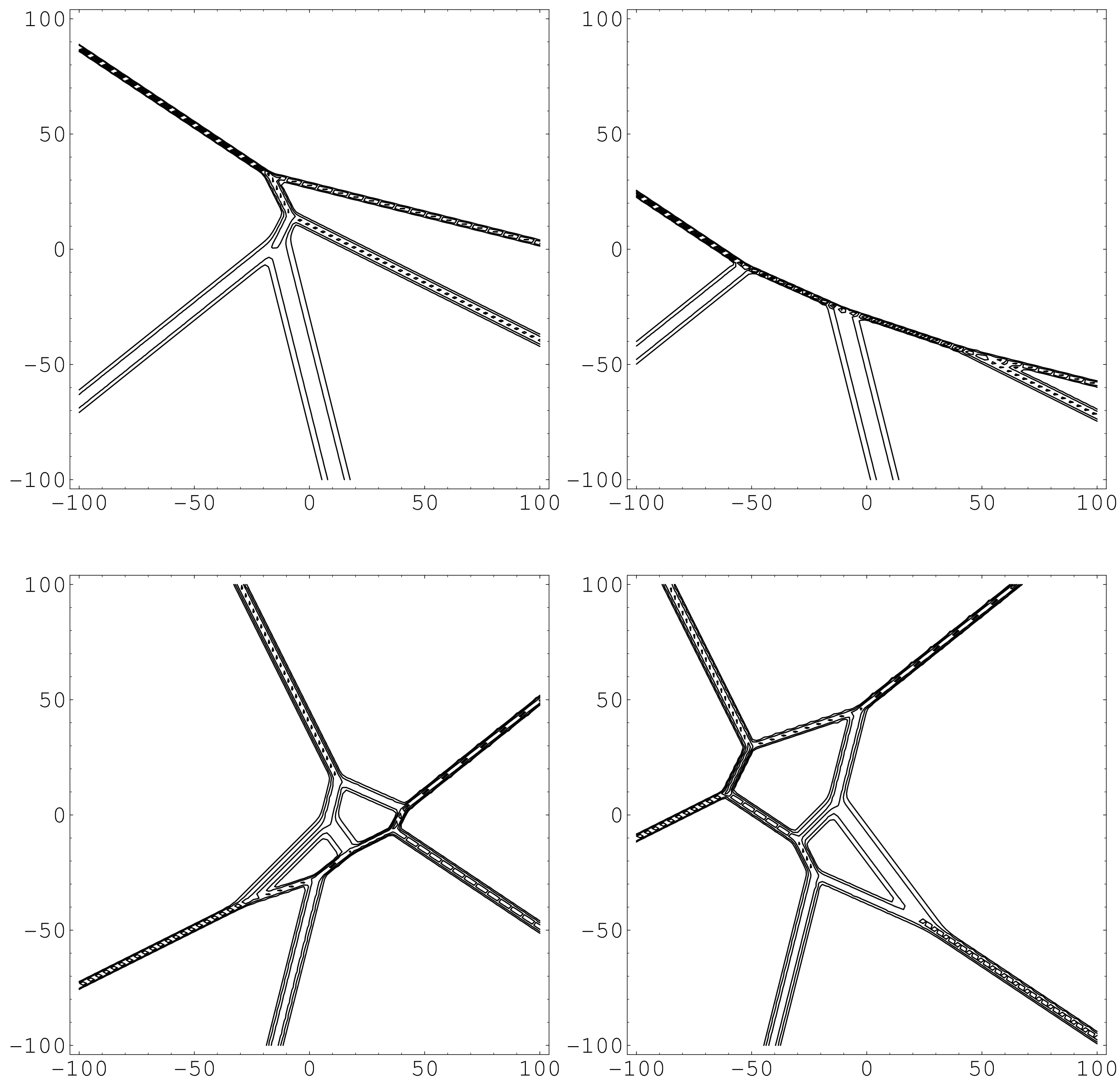
Bottom:

$N = 3, \;\; M = 5 \;\; \Rightarrow$

(3,2)-soliton solution.

$(k_1, \ldots, k_5) = (-2, -1, -\frac{1}{4}, \frac{1}{2}, \frac{3}{2})$

Both solutions are shown
at two different values of time.



General soliton solutions of KPII

- General soliton solutions:

$$f_n(x,y,t) = \sum_{m=1}^{M} a_{n,m}\, e^{\theta_m(x,y,t)}\,,$$

$\theta_m(x,y,t) = k_m x + k_m^2 y + k_m^3 t + \theta_{m,0} =$ exponential "phases".

$N =$ number of eigenfunctions

$M =$ number of exponential phases

$A = (a_{n,m}) = N \times M$ coefficient matrix,

$k_1, \ldots, k_M =$ phase parameters.

- Lemma:

$$\tau_{N,M}(x,y,t) = \sum_{1 \le m_1 < m_2 < \cdots < m_N \le M} V_{m_1,\ldots,m_N}\ A_{m_1,\ldots,m_N}\ e^{\theta_{m_1,\ldots,m_N}}\,.$$

$A_{m_1,\ldots,m_N} =\ N \times N$ minor obtained from columns $m_1, \ldots, m_N$ of A,

$\theta_{m_1,\ldots,m_N} = \theta_{m_1} + \cdots + \theta_{m_N} =$ phase combination,

$V_{m_1,\ldots,m_N} = \prod_{1 \le s < s' \le N} (k_{m_s} - k_{m_{s'}}) =$ van der Monde determinant.

Asymptotics line soliton of KPII

- Note:
  - each phase combination combination is sum of N distinct phases,
  - the only (x,y,t)-dependence of $\tau(x,y,t)$ comes from $\theta_1,\ldots,\theta_M$,
  - if all minors of A are nonnegative, $u(x,y,t)$ is nonsingular and positive.

- Theorem: [GB & Chakravarty, 2006]
  - $u(x,y,t)$ is exponentially localized those lines in the xy-plane where a balance exists between two phase comb's with $N-1$ common phases.
  - Along each of these lines, the solution is (up to exp'lly small terms) a line soliton produced by the two phases being exchanged.

- Each line soliton is identified by an index pair $[i,j]$. WLOG, take $i<j$.
  (θ_i&θ_j are the phases being exchanged in the dominant phase combinations)

- Def: we call asymptotic line solitons those that extends out to infinity as $y\to\infty$ or as $y\to-\infty$.
  In particular, we call incoming/outgoing solitons those as $y\to\mp\infty$.

Incoming and outgoing line solitons of KPII

- Theorem: [GB & Chakravarty, 2006]
  any irreducible, nonnegative coefficient matrix A generates:
  - $N_+ = N$ outgoing line solitons identified by $[i_n^+, j_n^+]$ with $i_n^+ < j_n^+$ and where $i_1^+, \ldots, i_N^+$ label the N pivot columns of A
  - $N_- = M-N$ incoming line solitons identified by $[i_n^-, j_n^-]$ with $i_n^- < j_n^-$ and where $j_1^-, \ldots, j_N^-$ label the $M-N$ non-pivot columns of A.

  The index pairs are uniquely identified by rank conditions on the minors.

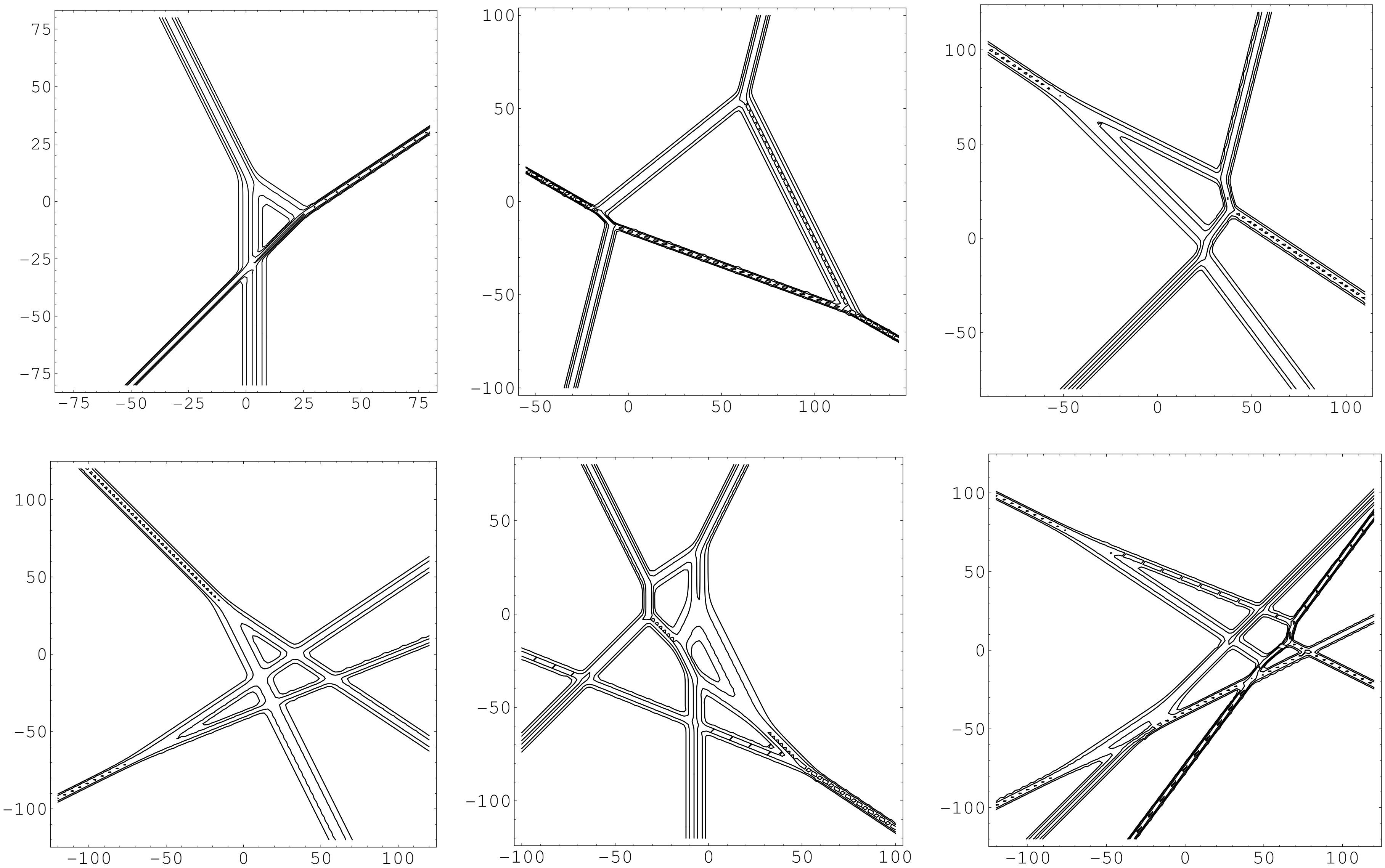
  (We say A is irreducible if rank$(A) = N$ and, when in RREF, every column is nonzero & every row has at least 2 nonzero elements.)

- Corollary: N-soliton solutions are obtained when $M = 2N$.

- Def: we call elastic N-soliton solutions those for which the amplitudes and directions of the incoming and outgoing solitons are the same.

- The map $(i_1^+, \ldots, i_N^+, j_1^-, \ldots, j_{M-N}^-) \mapsto (j_1^+, \ldots, j_N^+, i_1^-, \ldots, i_{M-N}^-)$ identifies a permutation of $1, \ldots, M$. [Chakravarty & Kodama, 2007]

Gallery of solutions solutions of KPII



(None of these is a traveling wave solution. Ordinary soliton solutions with $N > 2$ aren't either.)

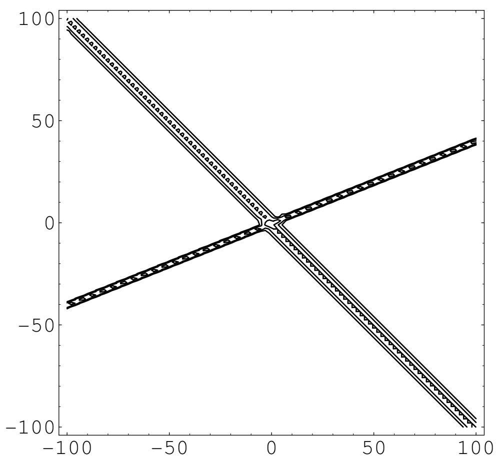
Elastic 2-soliton solutions

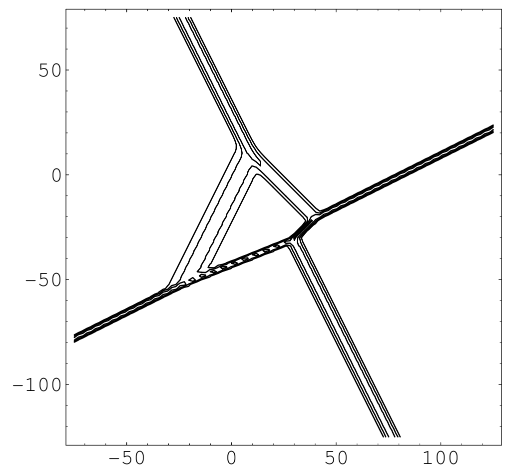
There are three types of elastic 2-soliton solutions: [Kodama, 2004]

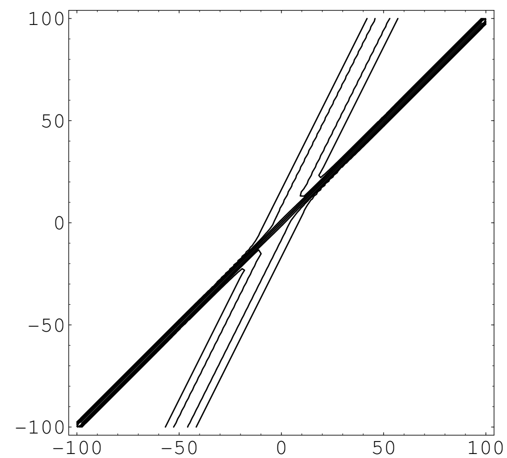
Ordinary	Resonant	Asymmetric
$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & a_{23} & a_{24} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

Solitons:

[1,2], [3,4]	[1,3], [2,4]	[1,4], [2,3]







Lemma: [GB, 2007]

Iff $c_2 - c_1 > a_1 + a_2$ an ordinary 2-soliton solution exists.

Iff $|a_1 - a_2| < |c_2 - c_1| < a_1 + a_2$, a resonant 2-soliton solution exists.

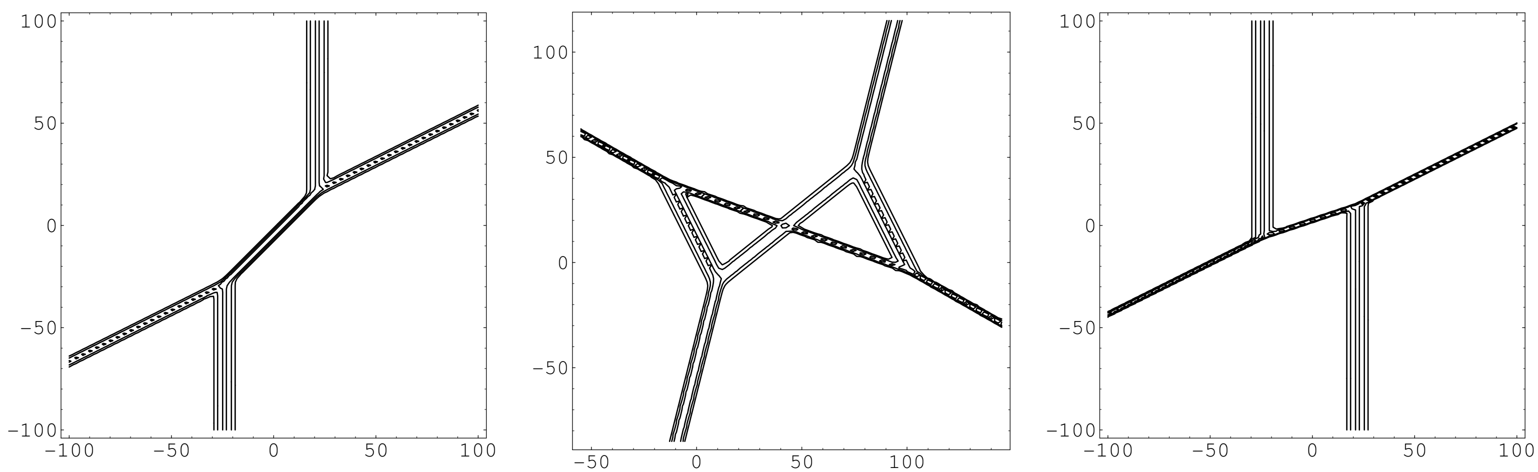
Iff $|c_2 - c_1| < |a_1 - a_2|$, an asymmetric 2-solution exists.

Interaction phase shifts of elastic 2-soliton solutions

The interactions are different for each type: [GB, 2007]

- ordinary solutions: $\delta x_{\mathrm{ord}} = \delta x_s$, and can take any positive value,
- asymmetric solutions: $\delta x_{\mathrm{asym}} = \delta x_s$, and can take any negative value,
- resonant soln's: $\delta x_{\mathrm{res}} = \delta x_s + \delta x_a$; both terms can take any real value;

$$\delta x_s = \log \left| \frac{(c_2 - c_1)^2 - (a_2 - a_1)^2}{(c_2 - c_1)^2 - (a_2 + a_1)^2} \right|, \qquad \delta x_a = \log \left(\frac{a_{23}}{a_{24}} - 1 \right).$$



ordinary resonant asymmetric

A nontrivial contribution to the phase shift exists for resonant solutions.

Elastic N-soliton solutions

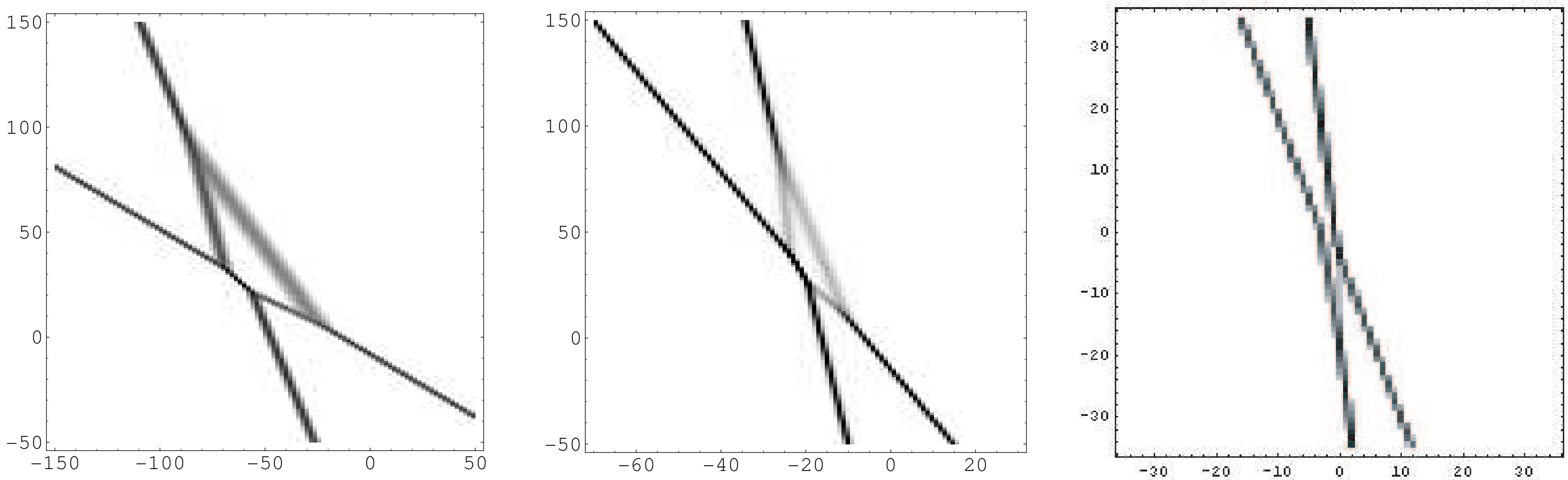
- Lemma: an elastic N-soliton solution is possible only when the pairs $[i_n, j_n]_{n=1}^N$ are disjoint. [GB & Chakravarty, 2006]
- Lemma: A generates an elastic solution iff its zero minors are dual:
$$A_{m_1,\dots,m_N} = 0 \iff A_{\bar{m}_1,\dots,\bar{m}_N} = 0,$$
with $\{m_1,\dots,m_N\} \cup \{\bar{m}_1,\dots,m_N\} = \{1,\dots,2N\}$. [Kodama, 2004]
- Theorem: $\exists$ an elastic solution for any disjoint set of pairs $[i_n, j_n]_{n=1}^N$.
Refinement of Schubert cell decomposition of $\mathrm{Gr}_{N,M}^{\mathrm{tnn}}$. (cf. Postnikov, 2006)
Explicit construction: [Kodama, 2004; GB & Chakravarty, 2006]
Exploit linear algebra constraints derived from soliton asymptotics.
- Corollary: $(2N-1)!!$ types of elastic N-soliton solutions are possible.
(Total number of ways of arrange $2N$ integers in pairs.)
Most of them are partially resonant.
- Many other combinatorial properties can be obtained.
[Kodama, 2004; GB & Chakravarty, 2007; Chakravarty & Kodama, 2007,2008]

Resonance and web structure in other soliton systems

Resonance and web structure are generic for (2+1)-D integrable systems.

cKP and dKP also have resonant solutions with web structure
[S Isojima, R Willox & J Satsuma, 2002 & 2003; Y Kodama & K-i Maruno, 2006]

Moreover, fully resonant solutions also exist in discrete soliton systems:



Left: resonant 2-soliton solution of the 2D Toda lattice;
center: resonant 2-soliton solution of the fully discrete 2D Toda lattice;
right: resonant 2-soliton solution of the ultra-discrete 2D Toda lattice
[K-i Maruno & GB, 2004]

The Davey-Stewartson equation and its bilinear forms

- Davey-Stewartson (DS) system: (subscripts x, y, t denote partial derivs)
  ($\sigma = \mp 1$: DSI/II; $\nu = \mp 1$: focusing/defocusing)

$$i\frac{\partial q}{\partial t} + \frac{1}{2}\sigma\frac{\partial^2 q}{\partial x^2} - \frac{1}{2}\frac{\partial^2 q}{\partial y^2} + 2\sigma q Q + 4\sigma\nu|q|^2 q = 0\,,$$
$$\frac{\partial^2 Q}{\partial x^2} + \sigma\frac{\partial^2 Q}{\partial y^2} = -4\nu\frac{\partial^2}{\partial x^2}(|q|^2)\,.$$

- Real bilinear form: ($\nu = 1$, defocusing)

$$q = e^{4it}\,G/F\,, \qquad Q = (\log F)_{xx}\,.$$

  Then

$$(2iD_t + \sigma D_x^2 - D_y^2)\,G\cdot F = 0\,, \qquad (\sigma D_x^2 + D_y^2)\,F\cdot F + 8GG^* - 8F^2 = 0\,,$$

- Complex bilinear form: perform the change of variables

$$x_1 = \sqrt{\sigma}\,x + y\,, \quad x_{-1} = \sigma\,(-\sqrt{\sigma}\,x + y)\,, \quad x_{\pm 2} = \mp it\,.$$

  Then

$$(D_{x_{\pm 2}} - D_{x_{\pm 1}}^2)\,G\cdot F = 0\,, \qquad \sigma D_{x_1} D_{x_{-1}} F\cdot F + 2GG^* - 2F^2 = 0\,.$$

Wronskian solutions of Hirota's equations

- Can write solutions of Hirota's equations as [Freeman 1984; Ohta 1989]

$$F = C\,\tau_N^{(s)}, \qquad G = C\,\tau_N^{(s+1)}, \qquad \bar{G} = C^*\,\tau_N^{(s-1)},$$

where $s \in \mathbb{Z}$ and $C \in \mathbb{C}$ are arbitrary constants,

$$\tau_N^{(n)} = \mathrm{Wr}_{x_1}(f_1^{(n)}, f_2^{(n)}, \cdots, f_N^{(n)}) = \det\begin{pmatrix} f_1^{(n)} & \cdots & f_1^{(n+N-1)} \\ \vdots & \ddots & \vdots \\ f_N^{(n)} & \cdots & f_N^{(n+N-1)} \end{pmatrix},$$

and $f_1, \ldots, f_N$ solve

$$\frac{\partial f^{(j)}}{\partial x_{\pm 1}} = f^{(j\pm 1)}, \qquad \frac{\partial f^{(j)}}{\partial x_{\pm 2}} = f^{(j\pm 2)},$$

with $f^{(0)} = f$.

- But, to obtain solutions of DS we need $F \in \mathbb{R}$ and $\bar{G} = G^*$.

  This imposes a restriction on the admissible sets of functions $f_1, \ldots, f_N$ as well as the constants s and C.

Wronskian solutions of defocusing DSII

- Lemma: [GB & K-i Maruno, 2006]
  To get solutions of DSII, take $s = -(N-1)/2$ & $C = (2i)^{-N(N-1)/2}$ with

$$f_n = \sum_{m=1}^{M} a_{n,m} e^{\theta_m} \,,$$

where $\theta_m = \theta_{m,0} + \sum_{j=-2}^{2} p_m^j x_j$ and $p_m = e^{i\varphi_m}$, with $a_{n,m} \in \mathbb{R}$ and $\varphi_m \in \mathbb{R}$.

- $A = (a_{n,m}) =$ **real** $N \times M$ real coefficient matrix,
  $\varphi_1, \ldots, \varphi_M =$ **real** phase parameters.

  [An equivalent way to get the same F, G & G^* is to set $s = 0$ & $C = 1$ and multiply each exponential term in f_n by $e^{-i(N-1)\varphi_m/2}$.]

- In terms of the physical variables:

$$\theta_m(x,y,t) = 2\left[x \sin\varphi_m + y \cos\varphi_m - t \sin(2\varphi_m)\right] + \theta_{0,m} \,.$$

- WLOG we can assume $\varphi_1, \ldots, \varphi_M$ are s.t. $-\pi \leq \varphi_1 < \cdots < \varphi_M < \pi$.

Tau-function of DSII via the Binet-Cauchy theorem

- The result is a direct consequence of:

  Lemma: [GB & K-i Maruno, 2006]

$$\tau_N^{(n)} = (2i)^{N(N-1)/2} \sum_{1\le m_1<\cdots<m_N\le M} \Delta_{m_1,\dots,m_N} A_{m_1,\dots,m_N} \times e^{\theta_{m_1,\dots,m_N}+i[n+(N-1)/2]\varphi_{m_1,\dots,m_N}}\,,$$

  where

  $\theta_{m_1,\dots,m_N} = \theta_{m_1} + \cdots + \theta_{m_N} =$ phase combination,

  $\varphi_{m_1,\dots,m_N} = \varphi_{m_1} + \cdots + \varphi_{m_N}$,

  $\Delta_{m_1,\dots,m_N} = \prod_{1\le j<j'\le N} \sin\left[\frac{1}{2}(\varphi_{m_{j'}} - \varphi_{m_j})\right]$ (replaces Van der Monde determinant)

  $A_{m_1,\dots,m_N} = N \times N$ minor of A obtained from columns $m_1,\dots,m_N$.

  (Proof: use Binet-Cauchy theorem)

- Can now verify the reality of F and the conjugacy of G and $\bar{G}$.

- Also, $\Delta_{m_1,\dots,m_N} > 0$ (since $-\pi \le \varphi_1 < \cdots < \varphi_M < \pi$) $\Rightarrow$ nonsingular solutions.

- The tau-function has a similar expression to KP, but here the θ's and the soliton direction are **not** increasing functions of the φ's.

Line solitons of defocusing DSII

- $N = 1$ and $M = 2$:

$$Q(x,y,t) = (\sin\varphi_1 - \sin\varphi_2)^2 \operatorname{sech}^2\left[\tfrac{1}{2}(\theta_1 - \theta_2)\right],$$

$$|q(x,y,t)|^2 = \frac{1}{2}\operatorname{sech}^2[\tfrac{1}{2}(\theta_1 - \theta_2)]\left\{\cosh(\theta_1 - \theta_2) + \cos(\varphi_1 - \varphi_2)\right\}.$$

  $Q =$ **bright** soliton component (intensity peak over zero background)

  $q =$ **dark** soliton component (intensity dip over unit background)

- Both are traveling wave solutions localized along the line $\theta_1 = \theta_2$.

- Soliton direction: $c = \tan\left[\tfrac{1}{2}(\varphi_i + \varphi_j)\right].$

- Soliton amplitude:

$$\max Q = (\sin\varphi_i - \sin\varphi_j)^2,$$

$$1 - \min|q|^2 = \sin^2[\tfrac{1}{2}(\varphi_i - \varphi_j)].$$

- $\max Q = 0$ whenever $\varphi_i - \varphi_j = \pm\pi$

  $\Rightarrow$ all horizontal solitons disappear from the bright component.

  (In contrast, $1 - \min|q|^2 \neq 0$.)

Restricted soliton solutions

- When $-\pi/2 \leq \varphi_1, \ldots, \varphi_M < \pi/2$, the corresponding solutions are s.t.:

  - the soliton directions $c_{j,j'}$ are increasing functions of $\varphi_j + \varphi_{j'}$.
    (Horizontal solitons are not included in this range.)

  - we can divide asymptotic line solitons into incoming and outgoing.
    (incoming/outgoing = extending out to infinity as $y \to \mp\infty$, as for KP.)

  - we can apply to this class of solutions all the tools developed for KP.

- Thus, we have the same results as for KP. In particular:

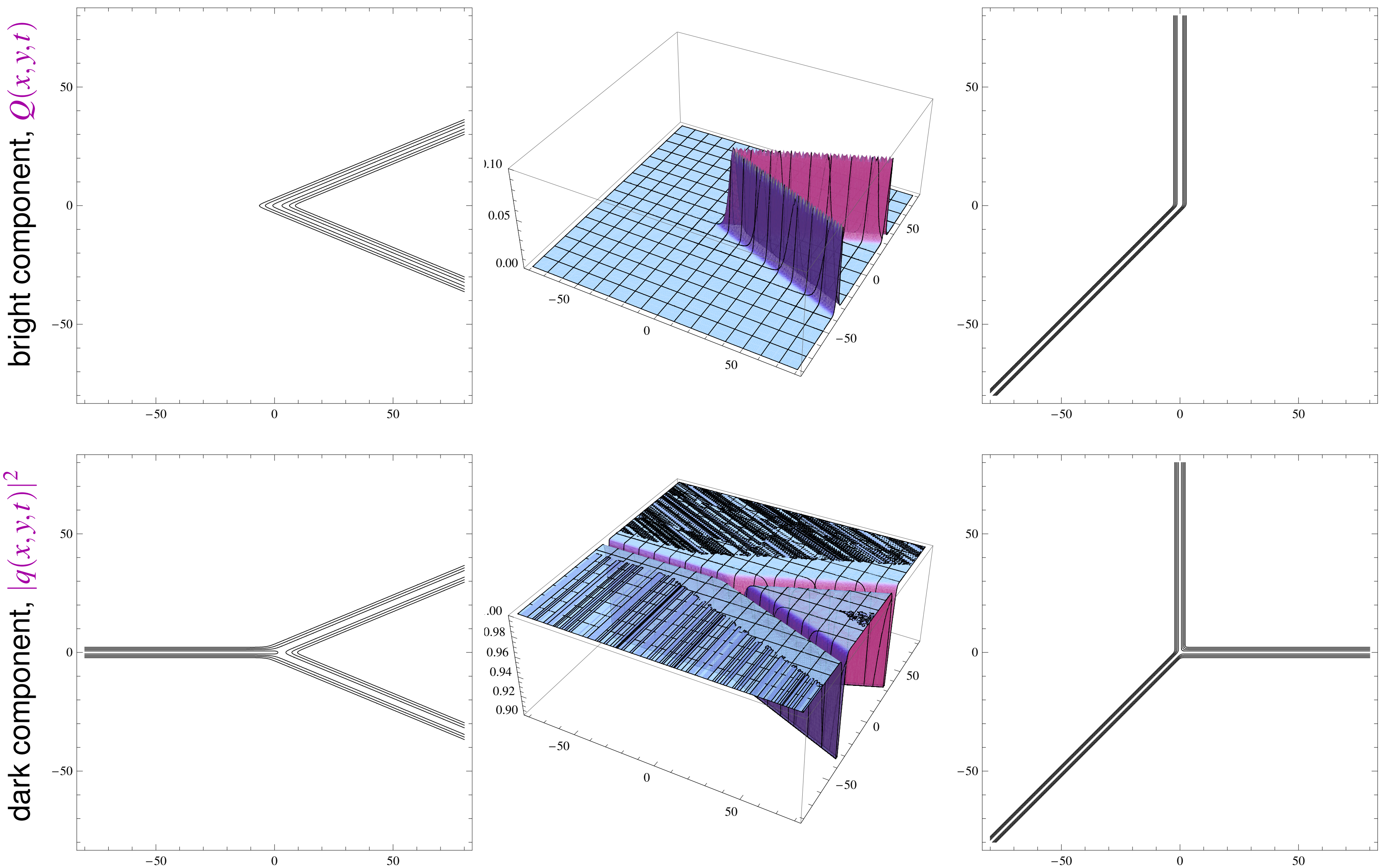
  Any irreducible, nonnegative coeff matrix generates a solution of DSII with:
  - N outgoing solitons identified by $[i_n^+, j_n^+]$, $i_n^+ < j_n^+$,
  - $M-N$ incoming solitons identified by $[i_n^-, j_n^-]$, $i_n^- < j_n^-$,

  with $i_1^+, \ldots, i_N^+$ the N pivot columns and $j_1^-, \ldots, j_N^-$ the $M-N$ non-pivot columns.

- When $\varphi_1, \ldots, \varphi_M$ are in the full range, however, the distinction between incoming and outgoing solitons loses its significance, and new kinds of behavior appear.

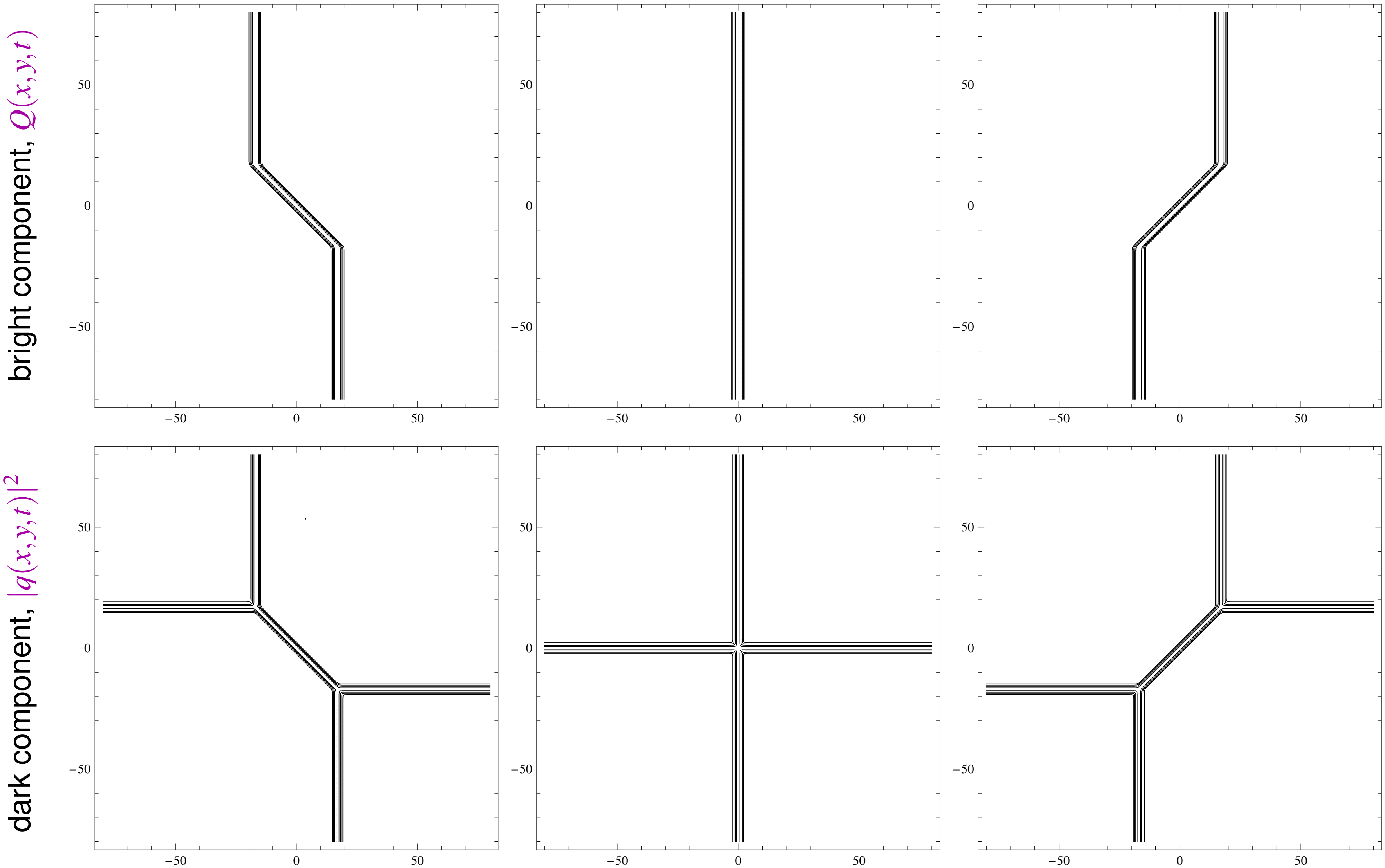
Unrestricted solutions, $N = 1$ & $M = 3$: V-shape solitons

Left&center: $(\varphi_1, \varphi_2, \varphi_3) = (\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4})$. Right: $(\varphi_1, \varphi_2, \varphi_3) = (-\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4})$.



Unrestricted solutions, $N = 1$ & $M = 4$: soliton reconnection

$(\varphi_1, \ldots, \varphi_4) = (-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4})$. Left/center/right: $t = -12, 0, 12$. [cf. Nishinari et al., 1993]



Can have elastic multi-soliton solutions with $N = 1$. But phase shifts are not time-independent!

Unrestricted solutions, general case

- Scalar case, $N = 1$: [GB & K-i Maruno, 2008]
  $\exists\; M-1$ asymptotic line solitons identified the index pairs $[n, n+1]$,
  + 1 asymptotic line soliton identified by the pair $[M, 1]$.

- Def: a pair $[j, i]$ with $j > i$ labels a soliton produced by θ_j and $\theta_i + 2\pi$.
  [It is therefore localized at $\alpha = \frac{1}{2}(\theta_i + \theta_j) + \pi$ instead of $\alpha = \frac{1}{2}(\theta_i + \theta_j)$].

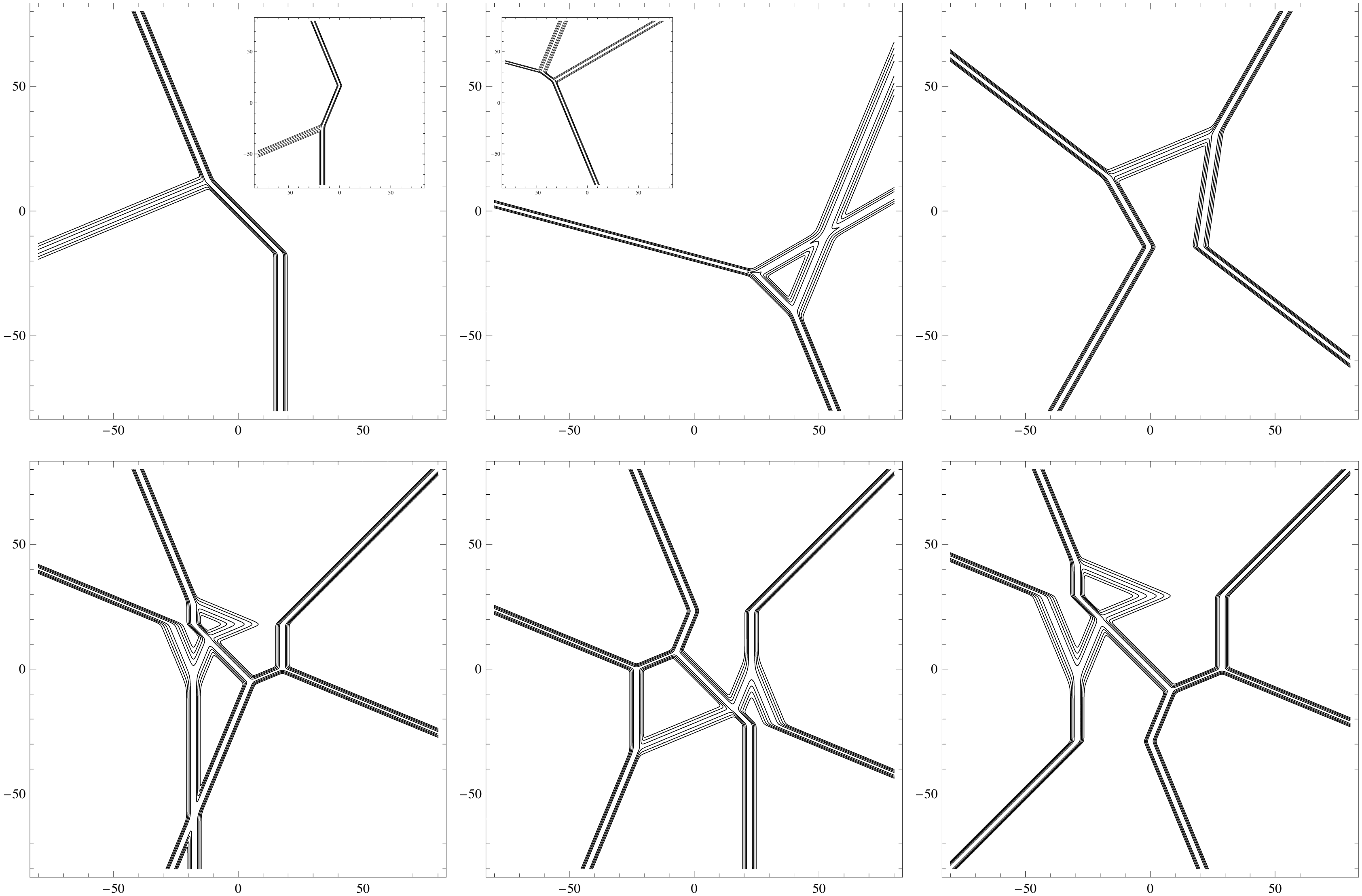
- General case, $N > 1$: [GB & K-i Maruno, 2008]
  - $M - N$ asymptotic solitons identified by $[i_n^-, j_n^-]$, with $i_n^- < j_n^-$ and where $j_1^-, \ldots, j_{M-N}^-$ label the non-pivot columns of A.
  - N asymptotic solitons identified by $[j_n^+, i_n^+]$, with $j_n^+ > i_n^+$ and where $i_1^+, \ldots, i_N^+$ label the pivot columns of A.

  (Use a generalization of the methods of asymptotic analysis developed for KP.)
  But now any soliton can be upstairs/downstairs depending on $\varphi_1, \ldots, \varphi_M$.

- Any solution of DSII also identifies a permutation of $1, \ldots, M$:
  $(i_1^+, \ldots, i_N^+, j_1^-, \ldots, j_{M-N}^-) \mapsto (j_1^+, \ldots, j_N^+, i_1^-, \ldots, i_{M-N}^-)$.

Gallery of solitons solutions of defocusing DSII

(Can define elastic solutions and classify them into non-/partially/fully resonant. And of course the y-independent solutions yield the dark solitons of NLS.)



Summary

- The solitonic sector of (2+1)-dimensional soliton equations is very rich.
- For KPII, any nonnegative irreducible $N \times M$ matrix produces:
  $M - N$ asymptotic line solitons as $y \to -\infty$ (one for each non-pivot),
  N asymptotic line solitons as $y \to \infty$ (one for each pivot).
- $(2N-1)!!$ types of elastic N-soliton solutions of KPII are possible, characterized by their physical properties.
- For DS, a restricted class exists in 1-to-1 correspondence with KP.
- Unrestricted solutions: more general phenomena; horizontal solitons, V-shape, soliton reconnection...
- Can classify even these more general solutions.
- There is a nontrivial connection between integrable systems and combinatorial algebraic geometry.

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