Reciprocal transformations for a Class of Weakly Nonlinear Hydrodynamic-Type Systems

Maciej Błaszak and Artur Sergyeyev

A. Mickiewicz University Poznan-POLAND

Gallipoli 2008

Introduction

Introduction

Consider the system of first order quasi-linear PDEs:

$$\boldsymbol{u}_t = K(\boldsymbol{u})\boldsymbol{u}_x, \tag{1}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n)^T$, K is an $n \times n$ matrix, called *hydrodynamic-type* or *dispersionless* systems.

Introduction

Consider the system of first order quasi-linear PDEs:

$$\boldsymbol{u}_t = K(\boldsymbol{u})\boldsymbol{u}_x, \tag{1}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n)^T$, K is an $n \times n$ matrix, called *hydrodynamic-type* or *dispersionless* systems.

Restriction: *semi-Hamiltonian* in the sense of Tsarev and *weakly nonlinear* (WNSH).

Introduction

Consider the system of first order quasi-linear PDEs:

$$\boldsymbol{u}_t = K(\boldsymbol{u})\boldsymbol{u}_x, \tag{1}$$

where $\boldsymbol{u} = (u_1, \dots, u_n)^T$, K is an $n \times n$ matrix, called *hydrodynamic-type* or *dispersionless* systems.

Restriction: *semi-Hamiltonian* in the sense of Tsarev and *weakly nonlinear* (WNSH).

Contribution (Riemann invariants representation): Ferapontov, Fordy, Pavlov,..

Introduction

Consider the system of first order quasi-linear PDEs:

$$\boldsymbol{u}_t = K(\boldsymbol{u})\boldsymbol{u}_x, \tag{1}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n)^T$, K is an $n \times n$ matrix, called *hydrodynamic-type* or *dispersionless* systems.

Restriction: *semi-Hamiltonian* in the sense of Tsarev and *weakly nonlinear* (WNSH).

Contribution (Riemann invariants representation): Ferapontov, Fordy, Pavlov,..
The general solution in implicit form and linearization: Ferapontov.

Introduction

Consider the system of first order quasi-linear PDEs:

$$\boldsymbol{u}_t = K(\boldsymbol{u})\boldsymbol{u}_x, \tag{1}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n)^T$, K is an $n \times n$ matrix, called *hydrodynamic-type* or *dispersionless* systems.

Restriction: *semi-Hamiltonian* in the sense of Tsarev and *weakly nonlinear* (WNSH).

Contribution (Riemann invariants representation): Ferapontov, Fordy, Pavlov,..
The general solution in implicit form and linearization: Ferapontov.

The construction of WNSH systems in a coordinate-free fashion ?

Introduction

A coordinate-free version of conditions: Pavlov, Svinolupov, Sharipov.

Introduction

A coordinate-free version of conditions: Pavlov, Svinolupov, Sharipov.
Construction of some special subclass (WNSH systems of L-type): Blaszak.

Introduction

A coordinate-free version of conditions: Pavlov, Svinolupov, Sharipov.
Construction of some special subclass (WNSH systems of L-type): Blaszak.

Here we present a construction of large class of WNSH systems in a coordinate-free fashion using the fact that the set of WNSH systems is in one to one correspondence with the set of Killing tensors of Stäckel metrices. Hence:

Introduction

A coordinate-free version of conditions: Pavlov, Svinolupov, Sharipov.
Construction of some special subclass (WNSH systems of L-type): Blaszak.

Here we present a construction of large class of WNSH systems in a coordinate-free fashion using the fact that the set of WNSH systems is in one to one correspondence with the set of Killing tensors of Stäckel metrices. Hence:

WNSH systems $=$ *dispersionless Killing systems.*

The main results are as follows:

Introduction

The main results are as follows:

Starting object: (1,1)-tensor L on $M \in \mathbb{R}^n$, of vanishing Nijenhuis torsion and simple eigenvalues.
Target: an infinite family of d-Killing systems together with their general solutions.

Introduction

The main results are as follows:

Starting object: (1,1)-tensor L on $M \in \mathbb{R}^n$, of vanishing Nijenhuis torsion and simple eigenvalues.
Target: an infinite family of d-Killing systems together with their general solutions.

The construction consists of three steps:

Introduction

The main results are as follows:

Starting object: (1,1)-tensor L on $M \in \mathbb{R}^n$, of vanishing Nijenhuis torsion and simple eigenvalues.
Target: an infinite family of d-Killing systems together with their general solutions.

The construction consists of three steps:

1. From tensor L we construct seed d-Killing systems (*d-Killing systems associated with the L-tensor*) together with infinitely many conservation laws numerated by integer $\beta > 0.$

Introduction

2. Using any r conservation laws $\beta_r,\ 1 \leq r \leq n$ we define r-time reciprocal transformations (RT)and a set of related d-Killing systems.

Introduction

2. Using any r conservation laws $\beta_r,\ 1 \le r \le n$ we define r-time reciprocal transformations (RT)and a set of related d-Killing systems.

3. We show that all constructed systems have common Riemann invariants $\lambda^j,\ j = 1, ..., n,$ which are eigenvalues of the tensor L. In Riemann invariants we derive the general solution for seed systems and the general solution of related d-Killing systems

Dispersionless systems and Killing tensors

Dispersionless systems and Killing tensors

Consider: $M \in \mathbb{R}^n$, $\boldsymbol{u} = (u^1, \ldots, u^n)^T$, (2,0)-metric G, (1,1)-Killing tensors K_i, $i = 1, \ldots, n$ which have a common orthogonal eigenframe of closed one-forms.

Dispersionless systems and Killing tensors

Consider: $M \in \mathbb{R}^n$, $\boldsymbol{u} = (u^1, \ldots, u^n)^T$, (2,0)-metric G, (1,1)-Killing tensors K_i, $i = 1, \ldots, n$ which have a common orthogonal eigenframe of closed one-forms.

Now consider a set of dispersionless Killing system (d-Killing)

$$K_1^{-1}\boldsymbol{u}_{t_1} = K_2^{-1}\boldsymbol{u}_{t_2} = \cdots = K_n^{-1}\boldsymbol{u}_{t_n}. \tag{2}$$

Dispersionless systems and Killing tensors

Consider: $M \in \mathbb{R}^n$, $\boldsymbol{u} = (u^1, \ldots, u^n)^T$, (2,0)-metric G, (1,1)-Killing tensors K_i, $i = 1, \ldots, n$ which have a common orthogonal eigenframe of closed one-forms.

Now consider a set of dispersionless Killing system (d-Killing)

$$K_1^{-1}\boldsymbol{u}_{t_1} = K_2^{-1}\boldsymbol{u}_{t_2} = \cdots = K_n^{-1}\boldsymbol{u}_{t_n}. \tag{2}$$

For any fixed $j \in \{1, \ldots, n\}$ we can rewrite (2) as

$$\boldsymbol{u}_{t_i} = K_i K_j^{-1} \boldsymbol{u}_{t_j}, \quad i = 1, \ldots, n, \quad i \neq j. \tag{3}$$

Dispersionless systems and Killing tensors

Consider: $M \in \mathbb{R}^n$, $\boldsymbol{u} = (u^1, \ldots, u^n)^T$, (2,0)-metric G, (1,1)-Killing tensors K_i, $i = 1, \ldots, n$ which have a common orthogonal eigenframe of closed one-forms.

Now consider a set of dispersionless Killing system (d-Killing)

$$K_1^{-1}\boldsymbol{u}_{t_1} = K_2^{-1}\boldsymbol{u}_{t_2} = \cdots = K_n^{-1}\boldsymbol{u}_{t_n}. \tag{2}$$

For any fixed $j \in \{1, \ldots, n\}$ we can rewrite (2) as

$$\boldsymbol{u}_{t_i} = K_i K_j^{-1}\boldsymbol{u}_{t_j}, \quad i = 1, \ldots, n, \quad i \neq j. \tag{3}$$

The flows (3) with $i = 1, \ldots, n$,, commute for any fixed j.

Seed d-Killing systems

Seed d-Killing systems

Construction of seed systems from L-tensor:

$$K_1 = \mathbb{I}, \quad K_r = \sum_{k=0}^{r-1} \rho_k L^{r-1-k}, \quad r = 2, \dots, n, \tag{4}$$

where

$$\det(\xi \mathbb{I} - L) = \sum_{i=0}^{n} \rho_i \xi^{n-i},$$

and $\mathbb{I}$ is the unit matrix.

Seed d-Killing systems

Construction of seed systems from L-tensor:

$$K_1 = \mathbb{I}, \quad K_r = \sum_{k=0}^{r-1} \rho_k L^{r-1-k}, \quad r = 2, \ldots, n, \tag{4}$$

where

$$\det(\xi \mathbb{I} - L) = \sum_{i=0}^{n} \rho_i \xi^{n-i},$$

and $\mathbb{I}$ is the unit matrix.

(1,1)-tensors K_j are Killing tensors of contravariant metric G (covariant metric $g = G^{-1}$) such that

Seed d-Killing systems

$$\nabla_m L^i_j = \delta^i_m \partial_j(\mathrm{tr}L) + \sum_{k=1}^{n} g_{jm} G^{ik} \partial_k(\mathrm{tr}L), \quad m, i, j = 1, \dots, n.$$

Seed d-Killing systems

$$\nabla_m L^i_j = \delta^i_m \partial_j(\mathrm{tr} L) + \sum_{k=1}^{n} g_{jm} G^{ik} \partial_k(\mathrm{tr} L), \quad m, i, j = 1, \ldots, n.$$

For such a G the tensor L is called *special conformal Killing tensor of trace form* (L-tensor) (Benenti 1992, 2005, Crampin 2003).

Seed d-Killing systems

$$\nabla_m L^i_j = \delta^i_m \partial_j(\mathrm{tr} L) + \sum_{k=1}^{n} g_{jm} G^{ik} \partial_k(\mathrm{tr} L), \quad m, i, j = 1, \dots, n.$$

For such a G the tensor L is called *special conformal Killing tensor of trace form* (L-tensor) (Benenti 1992, 2005, Crampin 2003).

If G fulfils above condition then also LG does.

Seed d-Killing systems

$$\nabla_m L^i_j = \delta^i_m \partial_j(\mathrm{tr}L) + \sum_{k=1}^{n} g_{jm} G^{ik} \partial_k(\mathrm{tr}L), \quad m, i, j = 1, \ldots, n.$$

For such a G the tensor L is called *special conformal Killing tensor of trace form* (L-tensor) (Benenti 1992, 2005, Crampin 2003).

If G fulfils above condition then also LG does.

For example, the most general form of L-tensor for pseudo-Euclidean metric G in Cartesian coordinates is

Seed d-Killing systems

$$\nabla_m L^i_j = \delta^i_m \partial_j (\mathrm{tr} L) + \sum_{k=1}^{n} g_{jm} G^{ik} \partial_k (\mathrm{tr} L), \quad m, i, j = 1, \ldots, n.$$

For such a G the tensor L is called *special conformal Killing tensor of trace form* (L-tensor) (Benenti 1992, 2005, Crampin 2003).

If G fulfils above condition then also LG does.

For example, the most general form of L-tensor for pseudo-Euclidean metric G in Cartesian coordinates is

$$L^{ij} = m x^i x^j + \beta^i x^j + \beta^j x^i + \gamma^{ij}$$

Seed d-Killing systems

$$\nabla_m L^i_j = \delta^i_m \partial_j(\mathrm{tr}L) + \sum_{k=1}^{n} g_{jm} G^{ik} \partial_k(\mathrm{tr}L), \quad m,i,j = 1,\ldots,n.$$

For such a G the tensor L is called *special conformal Killing tensor of trace form* (L-tensor) (Benenti 1992, 2005, Crampin 2003).

If G fulfils above condition then also LG does.

For example, the most general form of L-tensor for pseudo-Euclidean metric G in Cartesian coordinates is

$$L^{ij} = m x^i x^j + \beta^i x^j + \beta^j x^i + \gamma^{ij}$$

where m, β^i and $\gamma^{ij} = \gamma^{ji}$ are arbitrary constants.

Seed d-Killing systems

For the seed systems we have an infinite set of conservation laws (Blaszak 1999, 2003):

$$D_{t_i}(V_j^{(k)}) = D_{t_j}(V_i^{(k)}), \quad i, j = 1, \ldots, n, \quad i \neq j, \quad k \in \mathbb{Z}. \tag{5}$$

Seed d-Killing systems

For the seed systems we have an infinite set of conservation laws (Blaszak 1999, 2003):

$$D_{t_i}(V_j^{(k)}) = D_{t_j}(V_i^{(k)}), \quad i,j = 1,\ldots,n, \quad i \neq j, \quad k \in \mathbb{Z}. \tag{5}$$

where

$$V_r^{(k)} = V_{r+1}^{(k-1)} - \rho_r V_1^{(k-1)}, k \in \mathbb{Z},$$

with the initial condition

$$V_r^{(0)} = -\delta_r^n, \qquad r = 1,\ldots,n.$$

New d-Killing systems via reciprocal map

New d-Killing systems via reciprocal map

Construction:

seed sys. $\Rightarrow$ *conservation laws* $\Rightarrow$ *RT* $\Rightarrow$ *new d-Killing sys.*

New d-Killing systems via reciprocal map

Construction:

seed sys. $\Rightarrow$ *conservation laws* $\Rightarrow$ *RT* $\Rightarrow$ *new d-Killing sys.*

The k-time RT in question:

$$d\tilde{t}_{s_i} = -\sum_{j=1}^{n} V_j^{(\beta_i)} dt_j, \quad i = 1, \dots, k, \qquad (6)$$
$$\tilde{t}_m = t_m, \quad m = 1, 2, \dots, n, \quad m \neq s_i.$$

New d-Killing systems via reciprocal map

Construction:

seed sys. $\Rightarrow$ *conservation laws* $\Rightarrow$ *RT* $\Rightarrow$ *new d-Killing sys.*

The k-time RT in question:

$$\begin{aligned} d\tilde{t}_{s_i} &= -\sum_{j=1}^{n} V_j^{(\beta_i)} dt_j, \quad i = 1, \dots, k, \\ \tilde{t}_m &= t_m, \quad m = 1, 2, \dots, n, \quad m \neq s_i. \end{aligned} \tag{6}$$

The numbers s_i, $i = 1, \dots, k$, are a k-tuple of distinct integers from the set $\{1, \dots, n\}$, and β_j are positive integers:

$$\beta_1 > \beta_2 > \cdots > \beta_k > n - 1.$$

New d-Killing systems via reciprocal map

The inverse transformation:

$$dt_{s_i} = -\sum_{j=1}^{n} \tilde{V}_j^{(n-s_i)} d\tilde{t}_j, \quad i = 1, \ldots, k,$$

$$t_m = \tilde{t}_m, \quad m = 1, 2, \ldots, n, \quad m \neq s_i.$$

New d-Killing systems via reciprocal map

The inverse transformation:

$$dt_{s_i} = -\sum_{j=1}^{n} \tilde{V}_j^{(n-s_i)} d\tilde{t}_j, \quad i = 1, \dots, k,$$
$$t_m = \tilde{t}_m, \quad m = 1, 2, \dots, n, \quad m \neq s_i.$$

Here $\tilde{V}_j^{(m)}$ are functions constructed as follows:

New d-Killing systems via reciprocal map

The inverse transformation:

$$dt_{s_i} = -\sum_{j=1}^{n} \tilde{V}_j^{(n-s_i)} d\tilde{t}_j, \quad i = 1, \dots, k,$$
$$t_m = \tilde{t}_m, \quad m = 1, 2, \dots, n, \quad m \neq s_i.$$

Here $\tilde{V}_j^{(m)}$ are functions constructed as follows:

$$0 = V_j^{(m)} + \sum_{p=1}^{k} \tilde{V}_{s_p}^{(m)} V_j^{(\beta_p)}, \; j = s_1, \dots, s_k,$$

$$\tilde{V}_j^{(m)} = V_j^{(m)} + \sum_{p=1}^{k} \tilde{V}_{s_p}^{(m)} V_j^{(\beta_p)}, \; j \neq s_1, \dots, s_k.$$

New d-Killing systems via reciprocal map

The RT sends the set of the seed systems into:

$$\tilde{K}_1^{-1}\boldsymbol{u}_{\tilde{t}_1} = \tilde{K}_2^{-1}\boldsymbol{u}_{\tilde{t}_2} = \cdots = \tilde{K}_n^{-1}\boldsymbol{u}_{\tilde{t}_n}, \tag{7}$$

$$\Updownarrow$$

$$\boldsymbol{u}_{\tilde{t}_i} = \tilde{K}_i\tilde{K}_j^{-1}\boldsymbol{u}_{\tilde{t}_j}, \quad i = 1, \ldots, n, \quad i \neq j$$

New d-Killing systems via reciprocal map

The RT sends the set of the seed systems into:

$$\tilde{K}_1^{-1}\boldsymbol{u}_{\tilde{t}_1} = \tilde{K}_2^{-1}\boldsymbol{u}_{\tilde{t}_2} = \cdots = \tilde{K}_n^{-1}\boldsymbol{u}_{\tilde{t}_n}, \tag{7}$$

$$\Updownarrow$$

$$\boldsymbol{u}_{\tilde{t}_i} = \tilde{K}_i \tilde{K}_j^{-1}\boldsymbol{u}_{\tilde{t}_j}, \quad i = 1, \ldots, n, \quad i \neq j$$

for any fixed $j \in \{1, \ldots, n\}$, where

New d-Killing systems via reciprocal map

The RT sends the set of the seed systems into:

$$\tilde{K}_1^{-1}\boldsymbol{u}_{\tilde{t}_1} = \tilde{K}_2^{-1}\boldsymbol{u}_{\tilde{t}_2} = \cdots = \tilde{K}_n^{-1}\boldsymbol{u}_{\tilde{t}_n}, \tag{7}$$

$$\Updownarrow$$

$$\boldsymbol{u}_{\tilde{t}_i} = \tilde{K}_i\tilde{K}_j^{-1}\boldsymbol{u}_{\tilde{t}_j}, \quad i = 1,\ldots,n, \quad i \neq j$$

for any fixed $j \in \{1,\ldots,n\}$, where

$$\tilde{K}_{s_i} = -\sum_{j=1}^{k}\tilde{V}_{s_i}^{(n-s_j)}K_{s_j}, \quad i = 1,\ldots,k,$$

$$\tilde{K}_m = K_m - \sum_{j=1}^{k}\tilde{V}_m^{(n-s_j)}K_{s_j}, \quad m = 1,2,\ldots,n, \quad m \neq s_i$$

New d-Killing systems via reciprocal map

It is readily verified that new d-Killing systems possesses infinite set of conservation laws:

$$D_{\tilde{t}_i}(\tilde{V}_j^{(m)}) = D_{\tilde{t}_j}(\tilde{V}_i^{(m)}), \quad i,j = 1,\ldots,n, \quad i \neq j.$$

General solution from separation relations

General solution from separation relations

Now we relate d-Killing systems to separable systems.

General solution from separation relations

Now we relate d-Killing systems to separable systems.

Metric G with n Killing tensors: *Stäckel metric.*

General solution from separation relations

Now we relate d-Killing systems to separable systems.

Metric G with n Killing tensors: *Stäckel metric.*

Consider a class of separable systems given by the *separation relations* of the form

$$\sum_{j=1}^{n} \Phi_i^j(\lambda^i) H_j = f_i(\lambda^i)\mu_i^2 + \gamma_i(\lambda^i), \quad i = 1, \ldots, n, \tag{8}$$

General solution from separation relations

Now we relate d-Killing systems to separable systems.

Metric G with n Killing tensors: *Stäckel metric.*

Consider a class of separable systems given by the *separation relations* of the form

$$\sum_{j=1}^{n} \Phi_i^j(\lambda^i) H_j = f_i(\lambda^i)\mu_i^2 + \gamma_i(\lambda^i), \quad i = 1, \ldots, n, \tag{8}$$

H_j are separable Hamiltonians and $f_i(\xi), \gamma_i(\xi)$ are arbitrary functions of a single variable.

General solution from separation relations

Solving (8) with respect to Hamiltonians $H_i = H_i(\boldsymbol{\lambda}, \mu)$, $i = 1, \ldots, n$, where $\mu = (\mu_1, \ldots, \mu_n)^T$,we get

$$H_i = \mu^T K_i G_f \mu + V_i^{\gamma}(\lambda), \quad i = 1, \ldots, n, \quad K_1 = \mathbb{I},$$

General solution from separation relations

Solving (8) with respect to Hamiltonians $H_i = H_i(\boldsymbol{\lambda}, \mu)$, $i = 1, \ldots, n$, where $\mu = (\mu_1, \ldots, \mu_n)^T$,we get

$$H_i = \mu^T K_i G_f \mu + V_i^{\gamma}(\lambda), \quad i = 1, \ldots, n, \quad K_1 = \mathbb{I},$$

where G_f is a metric tensor, K_i are related Killing tensors and $V_i^{\gamma}(\lambda)$ are separable potentials. Moreover, all H_i Poisson commute.

General solution from separation relations

Solving (8) with respect to Hamiltonians $H_i = H_i(\boldsymbol{\lambda}, \mu)$, $i = 1, \ldots, n$, where $\mu = (\mu_1, \ldots, \mu_n)^T$,we get

$$H_i = \mu^T K_i G_f \mu + V_i^{\gamma}(\lambda), \quad i = 1, \ldots, n, \quad K_1 = \mathbb{I},$$

where G_f is a metric tensor, K_i are related Killing tensors and $V_i^{\gamma}(\lambda)$ are separable potentials. Moreover, all H_i Poisson commute.

From the first half of Hamiltonian equations we have

$$\lambda_{t_k} = \frac{\partial H_i}{\partial \mu} = 2K_k G_f \mu, \qquad k = 1, ..., n.$$

General solution from separation relations

We can rewrite it in the form

$$K_1^{-1}\boldsymbol{\lambda}_{t_1} = \cdots = K_n^{-1}\boldsymbol{\lambda}_{t_n} \Longleftrightarrow \frac{\lambda^i_{t_1}}{v^i_1} = \cdots = \frac{\lambda^i_{t_n}}{v^i_n}, \quad i = 1, \ldots, n, \tag{9}$$

as $K_i(\lambda) = \mathrm{diag}(\nu_i^1(\lambda), ..., \nu_i^n(\lambda))$, $i = 1, \ldots, n$ and

General solution from separation relations

We can rewrite it in the form

$$K_1^{-1}\boldsymbol{\lambda}_{t_1} = \cdots = K_n^{-1}\boldsymbol{\lambda}_{t_n} \Longleftrightarrow \frac{\lambda^i_{t_1}}{v^i_1} = \cdots = \frac{\lambda^i_{t_n}}{v^i_n}, \quad i = 1, \ldots, n, \tag{9}$$

as $K_i(\lambda) = \mathrm{diag}(\nu_i^1(\lambda), ..., \nu_i^n(\lambda))$, $i = 1, \ldots, n$ and

$$v^i_r = (-1)^{r+1}\frac{\det \mathbf{\Phi}^{ir}}{\det \mathbf{\Phi}^{i1}}$$

General solution from separation relations

We can rewrite it in the form

$$K_1^{-1}\boldsymbol{\lambda}_{t_1} = \cdots = K_n^{-1}\boldsymbol{\lambda}_{t_n} \Longleftrightarrow \frac{\lambda^i_{t_1}}{v^i_1} = \cdots = \frac{\lambda^i_{t_n}}{v^i_n}, \quad i = 1, \ldots, n, \tag{9}$$

as $K_i(\lambda) = \mathrm{diag}(\nu_i^1(\lambda), ..., \nu_i^n(\lambda))$, $i = 1, \ldots, n$ and

$$v^i_r = (-1)^{r+1}\frac{\det \boldsymbol{\Phi}^{ir}}{\det \boldsymbol{\Phi}^{i1}}$$

where $\boldsymbol{\Phi} = (\Phi^i_j(\lambda^i))$ is so called *Stäckel matrix* and $\boldsymbol{\Phi}^{ik}$ is the $(n-1)\times(n-1)$ matrix obtained from $\boldsymbol{\Phi}$ by removing its i^{th} row and k^{th} column.

General solution from separation relations

On the other hand, such constructed functions v^i for arbitrary r are general solution (Ferapontov 1991) for the system of equations:

General solution from separation relations

On the other hand, such constructed functions v^i for arbitrary r are general solution (Ferapontov 1991) for the system of equations:

$$\frac{\partial}{\partial\lambda_j}\left(\frac{\partial v^i/\partial\lambda^k}{v^k - v^i}\right) = \frac{\partial}{\partial\lambda_k}\left(\frac{\partial v^i/\partial\lambda^j}{v^j - v^i}\right), \qquad i \neq j \neq k \neq i = 1,\dots,n.$$

$$\partial v^i/\partial\lambda^i = 0, \quad i = 1,\dots,n.$$

General solution from separation relations

On the other hand, such constructed functions v^i for arbitrary r are general solution (Ferapontov 1991) for the system of equations:

$$\frac{\partial}{\partial\lambda_j}\left(\frac{\partial v^i/\partial\lambda^k}{v^k - v^i}\right) = \frac{\partial}{\partial\lambda_k}\left(\frac{\partial v^i/\partial\lambda^j}{v^j - v^i}\right), \qquad i \neq j \neq k \neq i = 1, \ldots, n.$$

$$\partial v^i/\partial\lambda^i = 0, \quad i = 1, \ldots, n.$$

The first equation means *semi-Hamiltonian* property while the second one *weakely nonlinear* property.

General solution from separation relations

Conclusions:

General solution from separation relations

Conclusions:

d-Killing systems = WNSH systems

Riemann invariants = separation coordinates

General solution from separation relations

Conclusions:

d-Killing systems = WNSH systems

Riemann invariants = separation coordinates

The general solution for (9) is given by an Jacobi inversion problem for separation relations (8):

$$\sum_{j=1}^{n} \int^{\lambda^j} \frac{\Phi_j^{n-s}(\xi)}{\varphi_j(\xi)} d\xi = t_s, \quad s = 1, \ldots, n, \tag{10}$$

Conclusions:

d-Killing systems = WNSH systems

Riemann invariants = separation coordinates

The general solution for (9) is given by an Jacobi inversion problem for separation relations (8):

$$\sum_{j=1}^{n} \int^{\lambda^j} \frac{\Phi_j^{n-s}(\xi)}{\varphi_j(\xi)} d\xi = t_s, \quad s = 1, \ldots, n, \tag{10}$$

where $\varphi_i(\xi)$ are arbitrary functions of a single variable

$$\varphi_i(\xi) = [f_i(\xi)(\sum_{j=1}^{n} \Phi_i^j(\xi) a_j - \gamma_i(\xi))]^{1/2}, \quad a_j = const.$$

General solution from separation relations

Now let us identify our systems under consideration.

General solution from separation relations

Now let us identify our systems under consideration. Separation relations for seed systems are:

$$\sum_{j=1}^{n}(\lambda^i)^{n-j}H_j = f_i(\lambda^i)\mu_i^2 + +\gamma_i(\lambda^i), \quad i = 1, \ldots, n,$$

General solution from separation relations

Now let us identify our systems under consideration. Separation relations for seed systems are:

$$\sum_{j=1}^{n} (\lambda^i)^{n-j} H_j = f_i(\lambda^i)\mu_i^2 + +\gamma_i(\lambda^i), \quad i = 1, \ldots, n,$$

while for conservation laws (5):

$$V_j^{(k)}(\lambda) \Longleftrightarrow \gamma_i(\lambda^i) = (\lambda^i)^k.$$

General solution from separation relations

Now let us identify our systems under consideration. Separation relations for seed systems are:

$$\sum_{j=1}^{n}(\lambda^i)^{n-j}H_j = f_i(\lambda^i)\mu_i^2 + +\gamma_i(\lambda^i), \quad i = 1,\ldots,n,$$

while for conservation laws (5):

$$V_j^{(k)}(\lambda) \Longleftrightarrow \gamma_i(\lambda^i) = (\lambda^i)^k.$$

Separation relations for systems (7) after RT (Blaszak, Sergyeyev 2008): for $i = 1,\ldots,n$

$$\sum_{j=1}^{k}(\lambda^i)^{\beta_j}\tilde{H}_{s_j} + \sum_{p=1,p\neq s_1,\ldots,s_k}^{n}(\lambda^i)^{n-p}\tilde{H}_p = f_i(\lambda^i)\mu_i^2 + \gamma_i(\lambda^i).$$

General solution from separation relations

The related Jacobi inversion problems are respectively

$$\sum_{j=1}^{n} \int^{\lambda^j} \frac{\xi^{n-r}}{\varphi_j(\xi)} d\xi = t_r, \quad r = 1, \ldots, n.$$

General solution from separation relations

The related Jacobi inversion problems are respectively

$$\sum_{j=1}^{n} \int^{\lambda^j} \frac{\xi^{n-r}}{\varphi_j(\xi)} d\xi = t_r, \quad r = 1, \dots, n.$$

and

$$\sum_{j=1}^{n} \int^{\lambda^j} \frac{\xi^{\beta_q}}{\varphi_j(\xi)} d\xi = \tilde{t}_{s_q}, \quad q = 1, \dots, k,$$

$$\sum_{j=1}^{n} \int^{\lambda^j} \frac{\xi^{n-i}}{\varphi_j(\xi)} d\xi = \tilde{t}_i, \quad i = 1, \dots, n, \quad i \neq s_q, \quad q = 1, \dots, k.$$

THE END