On a class of multidimensional integrable hierarchies and their reductions

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Outline

Introduction

Consider a pair for principal chiral field equations

$$\partial_{\tau_1}\psi = \frac{1}{\lambda - 1}U\psi, \qquad \partial_{\tau_2}\psi = \frac{1}{\lambda + 1}V\psi.$$

From compatibility condition for linear problems one gets

$$\frac{\partial_{\tau_2}U}{\lambda - 1} - \frac{\partial_{\tau_1}V}{\lambda + 1} + \left[\frac{U}{\lambda - 1}, \frac{V}{\lambda + 1}\right] = 0,$$

and as a result the equations

$$\partial_{\tau_2}U = \frac{1}{2}[U, V], \qquad \partial_{\tau_1}V = \frac{1}{2}[U, V].$$

For U, V belonging to matrix Lie algebra - standard $(1+1)$ dimensional principal chiral field equations.

It is also possible to consider U, V belonging to infinite-dimensional Lie algebra of vector fields, then we get multidimensional equations for coefficients of vector field. Consider

$$U = \sum_{i=1}^{N} u_i \frac{\partial}{x_i},$$

$$V = \sum_{i=1}^{N} v_i \frac{\partial}{x_i}.$$

Then equations

$$\partial_{\tau_2} U = \frac{1}{2}[U, V], \qquad \partial_{\tau_1} V = \frac{1}{2}[U, V]$$

give $(2+N)$-dimensional closed system of equations for the functions u_i, v_i.

Considering two-dimensional (variables x, y) Hamiltonian vector fields (Hamiltonians H_1 for U, H_2 for V), it is possible to introduce potential Θ,

$$H_1 = \partial_{\tau_1}\Theta, \quad H_2 = \partial_{\tau_2}\Theta.$$

For Θ we obtain Husain equation (1994) in cone variables

$$\partial_{\tau_1}\partial_{\tau_2}\Theta + \frac{1}{2}\left\{\partial_{\tau_1}\Theta, \partial_{\tau_2}\Theta\right\}_{(x,y)} = 0.$$

Dunajski equation

A canonical Plebański form of null-Kähler metrics (signature (2,2))

$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - \Theta_{xx}\mathrm{d}z^2 - \Theta_{yy}\mathrm{d}w^2 + 2\Theta_{xy}\mathrm{d}w\mathrm{d}z. \tag{1}$$

The conformal anti-self-duality (ASD) condition leads to Dunajski equation

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f, \tag{2}$$

$$\Box f = f_{xw} + f_{yz} + \Theta_{yy}f_{xx} + \Theta_{xx}f_{yy} - 2\Theta_{xy}f_{xy} = 0. \tag{3}$$

Equations (2,3) represent a compatibility condition for the linear system $L_0\Psi = L_1\Psi = 0$, where $\Psi = \Psi(w, z, x, y, \lambda)$ and

$$
\begin{aligned}
L_0 &= (\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x) - \lambda\partial_y + f_y\partial_\lambda, \\
L_1 &= (\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x) + \lambda\partial_x - f_x\partial_\lambda.
\end{aligned}
\tag{4}
$$

The case $f = 0$ corresponds to metrics of the form (1) satisfying Einstein equations, and Dunajski equation (2), (3) reduces to Plebański second heavenly equation.

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Dressing scheme

Nonlinear vector Riemann problem

$$\mathbf{S}_+ = \mathbf{F}(\mathbf{S}_-). \tag{5}$$

We consider three-component problem (5)

$$
\begin{aligned}
S_+^0 &= F^0(S_-^0, S_-^1, S_-^2), \\
S_+^1 &= F^1(S_-^0, S_-^1, S_-^2), \\
S_+^2 &= F^2(S_-^0, S_-^1, S_-^2)
\end{aligned}
\tag{6}
$$

for the functions $S^0 \to \lambda + O(\frac{1}{\lambda})$, $S^1 \to -\lambda z + x + O(\frac{1}{\lambda})$, $S^2 \to \lambda w + y + O(\frac{1}{\lambda})$, $\lambda \to \infty$, where x, y, w, z are the variables of Dunajski equation ('times').

Let us consider a linearized problem

$$\delta S_+^i = \sum_j F_{\cdot j}^i \delta S_-^j.$$

The linear space of solutions of this problem is spanned by the functions
$\mathbf{S}_x$, $\mathbf{S}_y$, $\mathbf{S}_\lambda$, which can be multiplied by λ^n and arbitrary function of times.
Expanding the functions $\mathbf{S}_z$, $\mathbf{S}_w$ into the basis, we obtain linear equations

$$((\partial_w + u_y \partial_y + v_y \partial_x) - \lambda \partial_y + f_y \partial_\lambda)\mathbf{S} = 0,$$
$$((\partial_z - u_x \partial_y - v_x \partial_x) + \lambda \partial_x - f_y \partial_\lambda)\mathbf{S} = 0, \qquad (7)$$

u, v, f can be expressed through the coefficients of expansion of S^0, S^1, S^2 at $\lambda = \infty$

$$u = S_1^2 - wS_1^0, \quad v = S_1^1 + zS_1^0, \quad f = S_1^0, \tag{8}$$

$$S^0 = \lambda + \sum_{n=1}^{\infty} \frac{S_n^0}{\lambda^n}, \; S^1 = -z\lambda + x + \sum_{n=1}^{\infty} \frac{S_n^1}{\lambda^n},$$

$$S^2 = w\lambda + y + \sum_{n=1}^{\infty} \frac{S_n^2}{\lambda^n}, \quad \lambda = \infty.$$

To get a Lax pair for Dunajski equation, we should consider the reduction $v_x = -u_y$, then we can introduce a potential Θ,

$$v = \Theta_y, \ u = -\Theta_x. \tag{9}$$

Proposition
Sufficient condition to provide the reduction

$$v_x = -u_y$$

in terms of the Riemann problem (5) is

$$\det F^i_{.j} = 1. \tag{10}$$

Proof Condition (10) implies that

$$\mathrm{d}S_+^0 \wedge \mathrm{d}S_+^1 \wedge \mathrm{d}S_+^2 = \mathrm{d}S_-^0 \wedge \mathrm{d}S_-^1 \wedge \mathrm{d}S_-^2,$$

and thus the form

$$\Omega = \mathrm{d}S^0 \wedge \mathrm{d}S^1 \wedge \mathrm{d}S^2$$

is analytic in the complex plane. Then the determinant of the matrix

$$J = \begin{pmatrix} S_\lambda^0 & S_\lambda^1 & S_\lambda^2 \\ S_x^0 & S_x^1 & S_x^2 \\ S_y^0 & S_y^1 & S_y^2 \end{pmatrix} \tag{11}$$

is also analytic.

Considering behavior of det J at $\lambda = \infty$, we come to the conclusion that

$$\det J = 1.$$

Calculating the coefficient of expansion of det J at $\lambda = \infty$ corresponding to λ^{-1}, we get

$$S_{1x}^1 + z S_{1x}^0 + S_{1y}^2 - w S_{1y}^0 = 0,$$

then, according to (8),

$$v_x = -u_y.$$

$\square$

Constructing solutions. Simple example

$$S_+^1 = S_-^1, \tag{12}$$
$$S_+^2 = S_-^2 \exp(-iF(S_-^2 \cdot S_-^0, S_-^1)), \tag{13}$$
$$S_+^0 = S_-^0 \exp(iF(S_-^2 \cdot S_-^0, S_-^1)), \tag{14}$$

where F is an arbitrary function of two variables.

The function $S^2 \cdot S^0$ is analytic. Then we get an expression

$$\phi = S^2 \cdot S^0 = \lambda^2 w + \lambda y + 2fw + u. \tag{15}$$

Equation (14) now reads

$$S_+^0 = S_-^0 \exp(iF(\phi, -\lambda z + x)).$$

The solution to this equation looks like

$$S^0 = \lambda \exp\left(\frac{1}{2\pi}\int_\gamma \frac{\mathrm{d}\lambda'}{\lambda - \lambda'} F(\phi(\lambda'), -\lambda'z + x)\right)$$

Considering the expansion of this expression in λ, we obtain the equations

$$\frac{1}{2\pi}\int_\gamma \mathrm{d}\lambda F(\phi(\lambda), -\lambda z + x) = 0, \tag{16}$$

$$\frac{1}{2\pi}\int_\gamma \lambda\mathrm{d}\lambda F(\phi(\lambda), -\lambda z + x) = f. \tag{17}$$

u, f are defined as implicit functions. Solution to Dunajski equation is given by $\Theta_x = -u$, $\Theta_y = zf$.

To simplify the example and exclude integration with respect to λ, it is possible to consider function F of the form

$$F(\phi, S^1) = \sum_i \frac{f_i(\phi)}{S^1 - c_i} = \sum_i \frac{f_i(\phi)}{-\lambda z + x - c_i},$$

where f_i are some analytic functions and c_i some constans. Then, performing integration in equations (16), (17) (considering γ as a small circle around infinity) we obtain

$$\sum_i f_i\left(\phi\left(\frac{x - c_i}{z}\right)\right) = 0, \qquad (18)$$

$$\sum_i \frac{x - c_i}{z} f_i\left(\phi\left(\frac{x - c_i}{z}\right)\right) = zf. \qquad (19)$$

We have obtained a solution to Dunajski equation, depending on arbitrary function of two variables, in terms of implicit functions.

Functional dependence on the function of two variables indicates that the solution we have constructed correspons to some $(2+1)$-dimensional reduction of Dunajski equation. It is possible to find the reduced equations explicitely, using the fact that linear equations (4) have analytic solutions ϕ and $-\lambda z + x$. Substituting these solutions to (4) and using (9), we obtain

$$(\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x)(2\frac{w}{z}\Theta_y - \Theta_x) + \frac{y}{z}\Theta_{yy} = 0, \qquad (20)$$

$$(\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x)(2\frac{w}{z}\Theta_y - \Theta_x) - \frac{y}{z}\Theta_{xy} = 0, \qquad (21)$$

HE2 hierarchy

(Takasaki)
We start from two formal Laurent series in λ,

$$
S^1 = \sum_{n=0}^{\infty} t_n^1 \lambda^n + \sum_{n=1}^{\infty} S_n^1(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},
$$

$$
S^2 = \sum_{n=0}^{\infty} t_n^2 \lambda^n + \sum_{n=1}^{\infty} S_n^2(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},
$$

We denote $x = t_0^1$, $y = t_0^2$, introduce the Poisson bracket $\{f, g\} := f_x g_y - f_y g_x$ and the projectors $(\sum_{-\infty}^{\infty} u_n \lambda^n)_+ = \sum_{n=0}^{\infty} u_n \lambda^n$, $(\sum_{-\infty}^{\infty} u_n \lambda^n)_- = \sum_{-\infty}^{n=-1} u_n \lambda^n$.

Heavenly equation hierarchy is defined by the relation

$$(\tilde{d}S^1 \wedge \tilde{d}S^2)_- = 0$$

Differential $\tilde{d}f = \sum_{n=0}^{\infty} \partial_n^1 f \, dt_n^1 + \sum_{n=0}^{\infty} \partial_n^2 f \, dt_n^2$
Equivalent Lax-Sato form

$$\partial_n^1 \mathbf{S} = -\{(\lambda^n S^2)_+, \mathbf{S}\},$$
$$\partial_n^2 \mathbf{S} = \{(\lambda^n S^1)_+, \mathbf{S}\},$$
$$\{S^1, S^2\} = 1.$$

Riemann problem ($+$ area conservation condition):

$$S_+^1 = F^1(\lambda, S_-^1, S_-^2),$$
$$S_+^2 = F^2(\lambda, S_-^1, S_-^2).$$

dKP hierarchy

Two formal Laurent series, usual notations $L(p)$ (Lax function), $M(p)$ (Orlov function), here, respectively, $S^0(\lambda)$, $S^1(\lambda)$:

$$S^0 = \lambda + \sum_{n=1}^{\infty} S_n^0(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n}, \quad \text{(corresponds to } L(p))$$

$$S^1 = \sum_{n=0}^{\infty} t_n^1(S^0)^n + \sum_{n=1}^{\infty} S_n^1(\mathbf{t}^1, \mathbf{t}^2)(S^0)^{-n} \quad \text{(corr. to } M(p))$$

Generating relation (plays a role of Hirota identity)

$$(\mathrm{d}S^0 \wedge \mathrm{d}S^1)_- = 0,$$

differential $\mathrm{d}f = \sum_{n=0}^{\infty} \partial_n^1 f \mathrm{d}t_n^1 + \partial_\lambda f \mathrm{d}\lambda$.

Equivalent Lax-Sato form (here the Poisson bracket is $\{f, g\}_{(\lambda,x)} := f_\lambda g_x - f_x g_\lambda$)

$$\partial_n^1 \mathbf{S} = \frac{1}{n+1} \{(S^0)_+^{n+1}, \mathbf{S}\}_{(\lambda,x)},$$
$$\{S^0, S^1\}_{(\lambda,x)} = 1.$$

Riemann problem (+ area conservation condition):

$$S_+^0 = F^1(S_-^0, S_-^1),$$
$$S_+^1 = F^2(S_-^0, S_-^1).$$

Dunajski equation hierarchy and related hierarchies

To define Dunajski equation hierarchy, we consider three formal Laurent series in λ, depending on two infinite sets of additional variables ('times')

$$S^0 = \lambda + \sum_{n=1}^{\infty} S_n^0(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},$$

$$S^1 = \sum_{n=0}^{\infty} t_n^1(S^0)^n + \sum_{n=1}^{\infty} S_n^1(\mathbf{t}^1, \mathbf{t}^2)(S^0)^{-n}$$

$$S^2 = \sum_{n=0}^{\infty} t_n^2(S^0)^n + \sum_{n=1}^{\infty} S_n^2(\mathbf{t}^1, \mathbf{t}^2)(S^0)^{-n},$$

We denote $x = t_0^1$, $y = t_0^2$, $\partial_n^1 = \frac{\partial}{\partial t_n^1}$, $\partial_n^2 = \frac{\partial}{\partial t_n^2}$ and introduce the projectors $(\sum_{-\infty}^{\infty} u_n \lambda^n)_+ = \sum_{n=0}^{\infty} u_n \lambda^n$, $(\sum_{-\infty}^{\infty} u_n \lambda^n)_- = \sum_{-\infty}^{n=-1} u_n \lambda^n$. Dunajski equation hierarchy is defined by the relation

$$(\mathrm{d}S^0 \wedge \mathrm{d}S^1 \wedge \mathrm{d}S^2)_- = 0, \tag{22}$$

where the differential includes both times and a spectral variable,

$$\mathrm{d}f = \sum_{n=0}^{\infty} \partial_n^1 f \mathrm{d}t_n^1 + \sum_{n=0}^{\infty} \partial_n^2 f \mathrm{d}t_n^2 + \partial_\lambda f \mathrm{d}\lambda.$$

Proposition

The relation (22) is equivalent to the set of equations

$$\partial_n^1 \mathbf{S} = \sum_{i=0,1,2} (J_{1i}^{-1}(S^0)^n)_+ \partial_i \mathbf{S}, \tag{23}$$

$$\partial_n^2 \mathbf{S} = \sum_{i=0,1,2} (J_{2i}^{-1}(S^0)^n)_+ \partial_i \mathbf{S}, \tag{24}$$

$$\det J = 1, \tag{25}$$

where

$$J = \begin{pmatrix} S_\lambda^0 & S_\lambda^1 & S_\lambda^2 \\ S_x^0 & S_x^1 & S_x^2 \\ S_y^0 & S_y^1 & S_y^2 \end{pmatrix}, \tag{26}$$

$\partial_0 = \partial_\lambda$, $\partial_1 = \partial_x$, $\partial_2 = \partial_y$.

The proof of (22) $\Rightarrow$ hierarchy (23,24,25) is based on the following statement.

Lemma
Given identity (22), for arbitrary first order operator $\hat{U}$,

$$\hat{U}\mathbf{S} = \left(\sum_i (u_i^1(\lambda, \mathbf{t}^1, \mathbf{t}^2)\partial_i^1 + u_i^2(\lambda, \mathbf{t}^1, \mathbf{t}^2)\partial_i^2) + u^0(\lambda, \mathbf{t}^1, \mathbf{t}^2)\partial_\lambda \right) \mathbf{S}$$

with 'plus' coefficients $((u_i^1)_- = (u_i^2)_- = u_-^0 = 0)$, the condition $(\hat{U}\mathbf{S})_+ = 0$ (for S^1 and S^2 modulo the derivatives of S^0) implies that $\hat{U}\mathbf{S} = 0$.

The statement (23,24,25) $\Rightarrow$ (22) directly follows from the relation

Lemma

$$\begin{vmatrix} \partial_{\tau_0} S^0 & \partial_{\tau_0} S^1 & \partial_{\tau_0} S^2 \\ \partial_{\tau_1} S^0 & \partial_{\tau_1} S^1 & \partial_{\tau_1} S^2 \\ \partial_{\tau_2} S^0 & \partial_{\tau_2} S^1 & \partial_{\tau_2} S^2 \end{vmatrix} = \begin{vmatrix} V_{\tau_0+}^0 & V_{\tau_0+}^1 & V_{\tau_0+}^2 \\ V_{\tau_1+}^0 & V_{\tau_1+}^1 & V_{\tau_1+}^2 \\ V_{\tau_2+}^0 & V_{\tau_2+}^1 & V_{\tau_2+}^2 \end{vmatrix} \qquad (27)$$

where τ_0, τ_1, τ_2 is an arbitrary set of three times of the hierarchy (23,24,25), and $V_{\tau+}^i$ are the coefficients of corresponding vector fields given by the r.h.s. of equations (23,24),

$$\partial_{\tau_j} \mathbf{S} = \sum_{i=0,1,2} V_{\tau_j+}^i \partial_i \mathbf{S}.$$

Proposition

The flows of the DE hierarchy (23,24) commute and the condition
det $J = 1$ (25) is preserved by the dynamics.

Condition det $J = 1$ defines a reduction for the DE hierarchy (23, 24). The
general hierarchy in the unreduced case is given by equations (23, 24), and
the analogue of relation (22) is

$$((\det J)^{-1} \mathrm{d}S^0 \wedge \mathrm{d}S^1 \wedge \mathrm{d}S^2)_- = 0. \qquad (28)$$

Two-component case of relation (28)

$$((\det J)^{-1} \mathrm{d}S^0 \wedge \mathrm{d}S^1)_- = 0$$

and corresponding equations (23, 24) define a hierarchy for the system
introduced by Manakov and Santini (dispersionless KP minus area
conservation)

A special subclass of the hierarchies of the type (28), (22) is singled out by the condition $S^0 = \lambda$. In this case (22) is transformed to the relation

$$(\mathrm{d}\lambda \wedge \mathrm{d}S^1 \wedge \mathrm{d}S^2)_- = 0 \Rightarrow (\tilde{\mathrm{d}}S^1 \wedge \tilde{\mathrm{d}}S^2)_- = 0,$$

where the differential $\tilde{\mathrm{d}}$ takes into account only times (Plebanski second heavenly equation hierarchy). A two-component case of (28) under the condition $S^0 = \lambda$ reduces to

$$((S_x^1)^{-1}\tilde{\mathrm{d}}S^1)_- = 0.$$

Hierarchy, introduced by L. Martinez Alonso and A.B. Shabat

In a more explicit form, Dunajski equation hierarchy (23, 24) can be written as

$$\partial_n^1 \mathbf{S} = + \left((S^0)^n \begin{vmatrix} S_\lambda^0 & S_\lambda^2 \\ S_y^0 & S_y^2 \end{vmatrix} \right)_+ \partial_x \mathbf{S} - \left((S^0)^n \begin{vmatrix} S_\lambda^0 & S_\lambda^2 \\ S_x^0 & S_x^2 \end{vmatrix} \right)_+ \partial_y \mathbf{S} -$$
$$\left((S^0)^n \begin{vmatrix} S_x^0 & S_x^2 \\ S_y^0 & S_y^2 \end{vmatrix} \right)_+ \partial_\lambda \mathbf{S},$$

$$\partial_n^2 \mathbf{S} = - \left((S^0)^n \begin{vmatrix} S_\lambda^0 & S_\lambda^1 \\ S_y^0 & S_y^1 \end{vmatrix} \right)_+ \partial_x \mathbf{S} + \left((S^0)^n \begin{vmatrix} S_\lambda^0 & S_\lambda^1 \\ S_x^0 & S_x^1 \end{vmatrix} \right)_+ \partial_y \mathbf{S} +$$
$$\left((S^0)^n \begin{vmatrix} S_x^0 & S_x^1 \\ S_y^0 & S_y^1 \end{vmatrix} \right)_+ \partial_\lambda \mathbf{S}.$$

It is easy to check that for $S^0 = \lambda$ Dunajski equation hierarchy reduces to first heavenly equation hierarchy, while for $S^2 = y$ it reduces to dispersionless KP hierarchy.

Waterbag-type reduction

$$S^0 = \lambda + \sum_{n=1}^{N} \ln\left(\frac{\lambda - u_n^0}{\lambda - v_n^0}\right),$$

$$S^1 = \sum_{n=0}^{\infty} t_n^1 (S^0)^n + \sum_{n=1}^{N} \ln\left(\frac{\lambda - u_n^1}{\lambda - v_n^1}\right)$$

$$S^2 = \sum_{n=0}^{\infty} t_n^2 (S^0)^n + \sum_{n=1}^{N} \ln\left(\frac{\lambda - u_n^2}{\lambda - v_n^2}\right),$$

where the functions u, v depend only on times of the hierarchy. This anzats is consistent with Dunajski equation hierarchy and defines $(1+1)$-dimensional reduction. The first system arises from $\det J = 1$.