Extended resolvent of the heat operator for a Darboux transformed potential

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Building Inverse Scattering in two dimensions

Let us consider the KPI and KPII equations

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = \pm 3u_{x_2x_2}, \qquad x = (x_1, x_2),$$

prototype of integrable equations in 2+1 dimensions, with $u = u(x,t)$ real.

They are associated, respectively, to the Nonstationary Schrödinger operator

$$\mathcal{L}(x, i\partial_x) = i\partial_{x_2} + \partial_{x_1}^2 - u(x),$$

and to the heat operator

$$\mathcal{L}(x, i\partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x)$$

Being a generalization of the KdV equation, they admit solutions behaving at space infinity like the solutions of the KdV equation.

Their Inverse Scattering theory for solutions decaying at large spaces was developed by Manakov, Ablowitz, Fokas and Bar Yakoov at the beginning of the eighties.

In the case of solutions with constant behavior along some rays, the integral equations defining the Jost solutions of $\mathcal{L}$ are divergent, nor can be regularized.

One needs to proceed in two successive steps: first by considering the pure N soliton solution u_N and afterwards by adding, via a dressing procedure, an arbitrary smooth decaying background u' getting $u = u_N + u'$.

In order to perform this program, one needs to explore the spectral nature of $\mathcal{L}$ by using a more general mathematical tool than the resolvent, precisely, what we call the **extended resolvent**. This corresponds to consider a general class of Green's functions of $\mathcal{L}$, depending on a two dimensional spectral parameter.

First, one gets explicitly the extended resolvent for the N soliton potential and studies its singularities, due to the constant behavior of the potential at large space along some rays.

Afterwards, one deals with the perturbed resolvent and its singularities, that, now, can be studied, since they are, in some sense inherited from the unperturbed resolvent. All traditional mathematical entities of the Inverse Scattering theory, i.e., Jost solutions and spectral data are, then, obtained via a reduction procedure from the resolvent.

In the KPI case, this procedure was already performed successfully by Boiti, Pempinelli, Pogrebkov. The Green's function $G_N(x, x'; \mathbf{k})$ of the N soliton potential was obtained as a reduction of the resolvent and it turned out to be analytical in the complex $\mathbf{k}$-plane with a discontinuity across the real axis and along N segments with a log singularity at the end points. The perturbed Jost solutions have, consequently, the same analytical properties. Spectral data are relating the limiting values of the Jost solution at the two sides of these cuts. Direct and inverse spectral problem was contructed and time evolution of spectral data was determined.

In the KPII case, this procedure was performed by Boiti, Pempinelli, Pogrebkov, Prinari only for a bidimensionally perturbed one soliton potential. Here, again, the Green's function $G_1(x, x'; \mathbf{k})$ of the one soliton potential, obtained as a reduction of the resolvent, resulted to have unexpected singular properties. Precisely, besides being non analytical, it has not just poles, but pole-like discontinuities at two points. Notwithstanding, spectral data were defined and direct and inverse problem solved.

Successively, Ablowitz and Villaroel considered the same problem by using the standard Inverse Scattering theory. However, they required a special and inexplicit condition on the potential, in order to deal with less singular objects.

Here, we report on the first step for the heat operator, that is on the construction of the resolvent for the pure N soliton potential u_N. In fact, we construct the resolvent for the more general case of N solitons superimposed *à la* Darboux to a generic smooth background.

The problem turned out to be unexpectedly more complicated than in the case of the Nonstationary Schrödinger operator and, up to now, some technical details are not yet solved. Singularities of the Green's function $G_N(x, x'; \mathbf{k})$ remain to be explored.

We think that the difficulties are mainly due to the fact that the heat operator is not autodual.

Extended operators and resolvent

For any differential operator $\mathcal{L}(x, i\partial_x)$ we introduce its **extension**

$$\mathbf{L}(x, x'; \mathbf{q}) \equiv \mathcal{L}(x, i\partial_x + \mathbf{q})\delta(x - x'), \qquad \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{C}^2.$$

By using the Fourier transform we can write

$$\mathbf{L}(x, x'; \mathbf{q}) = \frac{1}{(2\pi)^2} \int d\alpha\, e^{-i\alpha(x-x')} \mathcal{L}(x, \alpha + \mathbf{q}), \quad \alpha = (\alpha_1, \alpha_2).$$

In the KPII case

$$\mathcal{L}(x, \mathbf{q}) = i\mathbf{q}_2 - \mathbf{q}_1^2 - u(x).$$

Notice that for $\mathbf{q} = \ell(\mathbf{k}) \equiv (\mathbf{k}, -i\mathbf{k}^2)$ we have $\mathcal{L}(x, \ell(\mathbf{k})) = 0$.

By considering not just a polynomial $\mathcal{L}(x, \mathbf{q})$ in $\mathbf{q}$ but a tempered distribution $\mathcal{P}(x, \mathbf{q})$ in its six real variables we introduce more general operators, the **extended operators**

$$\mathbf{A}(x, x'; \mathbf{q}) = \frac{1}{(2\pi)^2} \int d\alpha\, e^{-i\alpha(x-x')} \mathcal{P}(x, \alpha + \mathbf{q})$$

Notice that

$$\mathbf{A}(x, x'; \mathbf{q}) = e^{i\mathbf{q}_{\Re}(x-x')} A(x, x'; q), \qquad q \equiv \mathbf{q}_{\Im}.$$

The extended operators generalize the pseudo-differential operators in two respects: they depend on a spectral parameter $\mathbf{q}$ and belong to the space of tempered distributions, which is larger than the functional space generally used.

M. Kruskal, first, noted that $\mathcal{P}(x, \mathbf{q})$ generalizes what the mathematicians call the symbol of a pseudo-differential operator. This generalization is essential since just its dependence on the spectral parameter $\mathbf{q}$ allows us to get the Jost solutions from some special symbols after the reduction $\mathbf{q} = \ell(\mathbf{k})$.

It is useful to consider also the Fourier transform of the symbol $\mathcal{P}(x, \mathbf{q})$

$$A(p; \mathbf{q}) = \frac{1}{(2\pi)^2} \int dx e^{ipx} \mathcal{P}(x, \mathbf{q}) =$$

$$= \frac{1}{(2\pi)^2} \int dx \int dx' \, e^{i(p+\mathbf{q}_{\Re})x - i\mathbf{q}_{\Re}x'} A(x, x'; \mathbf{q}_{\Im}), \quad p = (p_1, p_2).$$

Then, we can consider $A(x, x'; q)$ and $A(p; \mathbf{q})$ as two representations of the operator $A(q)$ in the x and in the p-space. The p-space is more suitable for studying analyticity properties, while boundedness is more easily studied in the x-space.

For generic operators $A(q)$ and $B(q)$ with kernels $A(x, x'; q)$ and $B(x, x'; q)$ we introduce the standard composition law

$$(AB)(x, x'; q) = \int dx'' \, A(x, x''; q) \, B(x'', x'; q),$$

if the integral exists in terms of distributions.

The composition in the p-space becomes a sort of shifted convolution

$$(AB)(p; \mathbf{q}) = \int dp' A(p - p'; \mathbf{q} + p') B(p'; \mathbf{q}).$$

The main object of our approach is the extended resolvent (or resolvent for short) M of the operator L, which is defined as the inverse of the operator L, i.e.,

$$LM = ML = I,$$

where,correspondingly, in the x and in the p-space,

$$I(x, x'; q) = \delta(x - x'), \quad \text{and} \quad I(p; \mathbf{q}) = \delta(p).$$

The Hilbert identity

$$M' - M = -M'(L' - L)M,$$

satisfied by two extended differential operators L and L' and their resolvents M and M' is the main instrument of our construction.

Notice that

$$\mathcal{L}(x, \partial_x)\widehat{M}(x, x'; q) = \delta(x - x') = \mathcal{L}^{\mathrm{d}}(x', \partial_{x'})\widehat{M}(x, x'; q),$$

where $\widehat{M}(x, x'; q) = e^{q(x-x')}M(x, x'; q)$ and $\mathcal{L}^{\mathrm{d}}(x', \partial_{x'})$ is the dual to $\mathcal{L}(x, \partial_x)$.

That is, the extended resolvent $\widehat{M}((x, x'; q)$ can be considered as defining a two-parameter set of Green's functions of $\mathcal{L}$.

Dressing operators ν and ω

In the p-space the bare operators L_0 and M_0 are given by

$$L_0(p;\mathbf{q}) = \delta(p)(i\mathbf{q}_2 - \mathbf{q}_1^2), \qquad M_0(p;\mathbf{q}) = \frac{\delta(p)}{i\mathbf{q}_2 - \mathbf{q}_1^2}.$$

We start by considering the case of a rapidly decaying potential $u(x)$. The resolvent M can also be defined as the solution of the integral equations

$$M = M_0 + M_0 u M, \qquad M = M_0 + M u M_0,$$

which can be directly derived from the Hilbert identity.

Looking at these integral equations written in the p space, recalling the explicit form of $M_0(p;\mathbf{q})$, we recognize that the kernel $M(p;\mathbf{q})$ is singular at $\mathbf{q} = \ell(\mathbf{k})$ and at $\mathbf{q} + p = \ell(\mathbf{k} + p_1)$ for any choice of the complex parameter $\mathbf{k}$, $\ell(\mathbf{k})$ being the two-component vector $\ell(\mathbf{k}) = (\mathbf{k}, -i\mathbf{k}^2)$

Therefore, it is natural to introduce the following truncations and reductions of the resolvent

$$\nu(p;\mathbf{q}) = (ML_0)(p;\mathbf{q})\Big|_{\mathbf{q}=\ell(\mathbf{q}_1)},$$

$$\omega(p;\mathbf{q}) = (L_0 M)(p;\mathbf{q})\Big|_{\mathbf{q}=\ell(\mathbf{q}_1+p_1)-p}.$$

From the Hilbert identity one gets that the operator $L(q)$ and its resolvent $M(q)$ admit the following bilinear representations in terms of ν and ω

$$L = \nu L_0 \omega, \qquad M = \nu M_0 \omega.$$

Then, the operators ν and ω are called dressing operators since they "dress" the bare operators L_0 and M_0.

In order to define Jost solutions by means of the dressing operators we introduce operators $\chi(\mathbf{k})$ and $\xi(\mathbf{k})$ depending on the complex parameter $\mathbf{k} \in \mathbb{C}$ with kernels in p-space

$$\chi(p, \mathbf{k}; \mathbf{q}) = \nu(p; \mathbf{k}), \quad \xi(p, \mathbf{k}; \mathbf{q}) = \omega(p; \mathbf{k} - p_1),$$

independent of the $\mathbf{q}$-variable. Taking their Fourier transforms

$$\chi(x, \mathbf{k}) = \int dp\, e^{-ipx} \nu(p; \mathbf{k}), \quad \xi(x, \mathbf{k}) = \int dp\, e^{-ipx} \omega(p; \mathbf{k} - p_1),$$

one recovers the standard Jost solutions

$$\Phi(x, \mathbf{k}) = e^{-i\ell(\mathbf{k})x} \chi(x, \mathbf{k}), \quad \Psi(x, \mathbf{k}) = e^{i\ell(\mathbf{k})x} \xi(x, \mathbf{k}).$$

Scalar product and completeness in this operatorial language are written, respectively,

$$\omega\nu = I, \quad \nu\omega = I.$$

Darboux transformation via twisting operators ζ and η

In order to build the extended resolvent corresponding to a two-dimensional potential describing N solitons superimposed to a generic smooth background, we need to use the operator formulation previously introduced and to bypass the recursive procedure building directly the final Darboux transformation. The main tools in doing this is what we call the twisting operators.

The most general Darboux transformation from the operator L to a new operator of the same form

$$L' = L_0 - u', \qquad u'(x, x'; q) = u'(x)\delta(x - x'),$$

can be obtained by means of an operator pair ζ, η according to the formulae

$$L'\zeta = \zeta L, \qquad \eta L' = L\eta,$$

"twisting" L to L'. We require η to be the left inverse of ζ

$$\eta\zeta = I,$$

so that

$$L = \eta L' \zeta.$$

However, in order to generate solitons by this twisting transformation, the product $\zeta\eta$ cannot be equal to I and we have

$$P = I - \zeta\eta,$$

with P a projector since

$$P^2 = P.$$

The twisting operators ζ and η generate a Darboux transformation since they not only generate the new potential u', but also the new dressing operators

$$\nu' = \zeta\nu, \quad \omega' = \omega\eta.$$

These dressing operators are orthogonal

$$\omega'\nu' = I,$$

but not complete, since,

$$\nu'\omega' + P = I.$$

Construction of ζ and η

In order to obtain a Darboux transformation we must specify the analyticity properties of the kernels $\zeta(p;\mathbf{q})$ and $\eta(p;\mathbf{q})$ of the twisting operators with respect to the variables $\mathbf{q}$.

First, we note that, thanks to the completeness of the dressing operators ν and ω, from

$$\nu' = \zeta\nu, \quad \omega' = \omega\eta,$$

we have

$$\zeta = \nu'\omega, \qquad \eta = \nu\omega'.$$

We assume

1. that $\nu'(p;\mathbf{q})$ and $\omega'(p;\mathbf{q})$, as $\nu(p;\mathbf{q})$ and $\omega(p;\mathbf{q})$, are independent of $\mathbf{q}_2$ and have the same asymptotic behavior

$$\lim_{\mathbf{q}_1\to\infty} \nu'(p;\mathbf{q}) = \delta(p), \qquad \lim_{\mathbf{q}_1\to\infty} \omega'(p;\mathbf{q}) = \delta(p).$$

2. that $\nu'(p;\mathbf{q})$ and $\omega'(p;\mathbf{q})$ have (correspondingly, right and left) simple poles with respect to the variable $\mathbf{q}_1$

$$\nu'_{b_l}(p) = \lim_{\mathbf{q}_1\to ib_l} \nu'(p;\mathbf{q})(\mathbf{q}_1 - ib_l),$$
$$\omega'_{a_j}(p) = \lim_{\mathbf{q}_1\to -p_1+ia_j} \omega'(p;\mathbf{q})(\mathbf{q}_1 + p_1 - ia_j),$$

where $a_1,\ldots,a_{N_a}$, $b_1,\ldots,b_{N_b}$ are N_a+N_b real parameters, which we choose to be all different.

Then, the kernels $\zeta(p;\mathbf{q})$ and $\eta(p;\mathbf{q})$ are given by means of the following representations

$$\zeta(p;\mathbf{q}) = \delta(p) + \sum_{l=1}^{N_b} \int dp' \, \frac{\nu'_{b_l}(p-p')\omega(p';ib_l - p'_1)}{\mathbf{q}_1 + p'_1 - ib_l},$$

$$\eta(p;\mathbf{q}) = \delta(p) + \sum_{j=1}^{N_a} \int dp' \, \frac{\nu(p-p';ia_j)\omega'_{a_j}(p')}{\mathbf{q}_1 + p'_1 - ia_j}.$$

By imposing $\eta\zeta = I$ and, then, working in the x-space, we get for the transformed Jost solutions

$$\Phi'(x,\mathbf{k}) = \Phi(x,\mathbf{k}) - \sum_{j=1}^{N_a}\sum_{l=1}^{N_b} \Phi(x,ia_j)m_{jl}(x)\mathcal{F}(x,ib_l,\mathbf{k}),$$

$$\Psi'(x,\mathbf{k}) = \Psi(x,\mathbf{k}) - \sum_{j=1}^{N_a}\sum_{l=1}^{N_b} \mathcal{F}(x,\mathbf{k},ia_j)m_{jl}(x)\Psi(x,ib_l),$$

where $m(x)$ is the $N_a \times N_b$ matrix

$$m(x) = (E_{N_a} + c\mathcal{F}(x))^{-1}c = c(E_{N_b} + \mathcal{F}(x)c)^{-1}.$$

c is a real constant $N_a \times N_b$ matrix, E_{N_a} the unity $N_a \times N_a$ matrix, E_{N_b} and the unity $N_b \times N_b$ matrix,

$$\mathcal{F}(x,\mathbf{k},\mathbf{k}') = \int_{(\mathbf{k}'_{\Im} - \mathbf{k}_{\Im})\infty}^{x_1} dx'_1 \Psi(x',\mathbf{k})\Phi(x',\mathbf{k}')\Big|_{x'_2 = x_2}$$

and $\mathcal{F}(x)$ is the $N_b \times N_a$ matrix with elements $\mathcal{F}(x)_{lj} = \mathcal{F}(x,ib_l,ia_j)$.

For the transformed potential we get

$$u'(x) = u(x) - 2\partial_{x_1}^2 \ln\det(E_{N_b} + \mathcal{F}c) =$$
$$= u(x) - 2\partial_{x_1}^2 \ln\det(E_{N_a} + c\mathcal{F}).$$

In the case $u(x) \equiv 0$ one gets the general N soliton solution.

This solution was already obtained in 2001, in an equivalent form, by Boiti, Pempinelli, Pogrebkov and Prinari, but only the structure of the 2 soliton solution was studied in its generality.

Successively, Biondini, Kodama and Chakrabarty, using τ functions, obtained an equivalent but more symmetric form and, in a series of papers, studied the general structure of the N soliton solution.

Getting the transformed resolvent

Once obtained the transformed operator L'

$$L' = L_0 - u',$$

if M_Δ is a solution of the operator equations

$$L' M_\Delta = P, \qquad M_\Delta L' = P,$$

then, one can show that resolvent M' of L' is given by

$$M' = \zeta M \eta + M_\Delta.$$

The main difficulty is to find a bounded solution M_Δ.

Taking advantage of the experience made in the KPI case one would write

$$(\zeta M \eta)(x, x'; q) = -\frac{\operatorname{sgn}(x_2 - x_2')}{2\pi} e^{-q(x-x')} \times$$
$$\times \int dp_1 \theta\left((q_2 + p_1^2 - q_1^2)(x_2 - x_2')\right) \Phi'(x; p_1 + iq_1) \Psi'(x'; p_1 + iq_1).$$

and

$$M_\Delta(x, x'; q) = \operatorname{sgn}(x_2 - x_2') e^{-q(x-x')} \times$$
$$\times \sum_{j=1}^{N_a} \sum_{l=1}^{N_b} [\theta(q_1 - b_l) - \theta(q_1 - a_j)] \times$$
$$\times \theta((x_2 - x_2')(q_2 - (a_j + b_l) q_1 + a_j b_l)) \Phi'(x; ia_j) c_{jl} \Psi'(x', ib_l).$$

One can show easily that $(\zeta M\eta)(x,x';q)$ is bounded.

In the special case in which $N_a = N_b$ and the matrix c_{ij} is diagonal one can show that also $M_\triangle(x,x';q)$ is bounded.

However, in the general case this expression for $M_\triangle(x,x';q)$ is not bounded and needs to be modified, writing an alternative more symmetric form in the spectral parameters a_j and b_l.

We succeeded on doing this in the special cases where N_a has any value and $N_b = 1$ and $N_b = 2$.

However, the general case is not yet solved.

I could write this formulas, but we prefer to wait for having the explicit formula for the general case, also because we think that in getting it we will discover what is the simplest and more transparent way of writing $M_\triangle$.

The main difficulty in doing this job is due to the fact that, in spite of the fact that a symmetric formulation of the potential is possible as shown by Biondini, Kodama and Chakrabarty, the Jost solutions $\Phi'(x;\mathbf{k})$ and $\Psi'(x',\mathbf{k})$ are solutions of two different spectral problems and the asymmetry in the role played by the spectral parameters a_j and b_l in the resolvent cannot be totally removed.

However, in the search of the correct expression for $M_\triangle(x,x';q)$ it seems that a crucial role is played by the identity

$$\sum_{j=1}^{N_a}\sum_{l=1}^{N_b} \operatorname*{res}_{\mathbf{k}=ia_j,ib_l} \Phi'(x;\mathbf{k})\Psi'(x',\mathbf{k}) = 0,$$

which is totally symmetric in the a_j and b_l parameters.