The fixed energy spectral transform for elliptic discrete 2-D Schrödinger operator and the periodic problem for the 2-D Toda hierarchy (discrete Novikov-Veselov hierarchy)

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Recently a new hierarchy associated with the elliptttic discretization of the Schrödinger operator was constructed:

$$A_{m,n}\Psi_{m+1,n} + A_{m-1,n}\Psi_{m-1,n} +$$

$$+B_{m,n}\Psi_{m,n+1} + B_{m,n-1}\Psi_{m,n-1} = F_{m,n}\Psi_{m,n}$$

M. Nieszporski, P. M. Santini and A. Doliwa, Darboux transformations for 5-point and 7-point self-adjoint schemes and an integrable discretization of the 2D Schrödinger operator, Phys. Lett. A **323** (2004), 241–250.
Santini, P. M.; Nieszporski, M.; Doliwa, A. Integrable generalization of the Toda law to the square lattice. *Phys. Rev. E (3)* **70** (2004), no. 5, 056615, 6 pp.
Grushevsky. Samuel, Krichever Igor, Integrable discrete Schrödinger equations and a characterization of Prym varieties by a pair of quadrisecants. arXiv:0705.2829.
P. M. Santini, A. Doliwa and M. Nieszporski, Integrable dynamics of Toda-type on the square and triangular lattices arXiv: 0710.5543v1.

The first flow from this hierarchy has the following form:

$$\frac{dF_{m,n}}{dt} + \xi_{m+1,n}A^2_{m,n} - \xi_{m-1,n}A^2_{m-1,n} +$$

$$+\eta_{m,n+1}B^2_{m,n} - \eta_{m,n-1}B^2_{m,n-1} = 0,$$

$$\frac{d\ln(A_{m,n})}{dt} + \frac{1}{2}[\xi_{m+1,n}F_{m+1,n} - \xi_{m,n}F_{m,n}] = 0,$$

$$\frac{d\ln(B_{m,n})}{dt} + \frac{1}{2}[\eta_{m,n+1}F_{m,n+1} - \eta_{m,n}F_{m,n}] = 0,$$

$$A_{m,n}B_{m,n}[\xi+\eta]_{m,n} = A_{m,n+1}B_{m+1,n}[\xi+\eta]_{m+1,n+1},$$

$$A_{m,n+1}B_{m,n}[\xi+\eta]_{m,n+1} = A_{m,n}B_{m+1,n}[\xi+\eta]_{m+1,n}.$$

One-dimensional reduction of this flow coincides with the Toda lattice equation.

In contrast with the famous 2-dimensuonal generalization of the Toda lattice this one has **2 discrete variables**.

The symmetry algebra for the fixed energy spectral problem for the 2-D Schrödinger operator is defined by the Novikov-Veselov hirarchy.
This hierarchy with 2 discrete spatial variables defines the symmetry algebra for the elliptic discretization of the 2-D Schrödinger operator at one energy. Therefore it also can be interpreted as a discrete analog of the Novikov-Veselov hierarchy.
The fixed-energy discretization of the 2-D double-periodic Schrödinger operator is integrable only at a special submanifold in the phase space (only at this submanifold the hierarchy is well-defined)

The Korteveg-de Vries equation

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x$$

was integrated using the sctattering transform for the 1-D Scrödinger operator

$$-\partial_x^2\Psi + u(x)\Psi = E\Psi$$

The both problems are very important for physics.

- The scattering thransform for 1-d Scrödinger operator linearises KdV.
- The KdV heirarchy generates an infinite-dimensional abel algebra of symmetries for the 1-D Scrödinger operator.

The Novikov-Veselov equations:

$$u_t = 8\partial_z^3 u + 8\partial_{\bar z}^3 u + 2\partial_{\bar z}(uw) + 2\partial_z(u\bar w),$$

$$u(z) = \bar u(z), \qquad \partial_z w(z) = -3\partial_{\bar z} u(z),$$

$$z = x + iy,\ \bar z = x - iy,\ \partial_z = \frac{1}{2}(\partial_x - i\partial_y),\ \partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$$

was integrated using the fixed-energy scattering transform for the 2-D Schrödinger operator

$$L = -4\partial_z\partial_{\bar z} + u(z), \quad L\Psi = E_0\Psi.$$

No important physical applications of NV heirarchy are known.

- The NV heirarchy generates an infinite-dimensional abel algebra of symmetries for the fixed-energy scattering problem for the 2-D Scrödinger operator.

B.A.Dubrovin, I.M.Krichever, S.P.Novikov, The Schrödinger equation in a periodic field and Riemann surfaces, *Sov. Math. Dokl.*, **17** (1976), 947-951.

- It was shown, that the fixed-energy scattering problem for the 2-D Scrödinger operator (with an extra magnetic term)

$$L = -4\partial_z\partial_{\bar z} + V(z)\partial_z + u(z), \quad L\Psi = E_0\Psi$$

is integrable.

The problem of selecting spectral data corresponding to the pure potential operators $V(z) = 0$ turned out to be rather non-trivial. It was solved only 8 year later:

S. P. Novikov and A. P. Veselov, Finite-zone, two-dimensional Schrödinger operators. Potential operators, *Sov. Math. Dokl.*, **30** (1984) 705–708;
Finite-zone, two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations, *Sov. Math. Dokl.*, **30** (1984) 588–591.

The solution was based on the ideas by Cherednik:

I. V. Cherednik Reality conditions in "finite-zone" integration, *Dokl. Akad. Nauk SSSR* **252** (1980) 1104–1108.

By analogy with integrable and non-integrable equations one can define **integrable and non-inegrable spectral problems.**

There is no strict definiton of an integrable system. We also have no strict definition of an integrable spectral problems, but typically they possess some distinguished properties,

Example 1. The spectral transform for the 1-D Scrödinger opeartor

$$-\partial_x^2 + u(x)$$

is integrable.

Example 2. The spectral transform for the multidimensional Scrödinger opeartor

$$-\partial_{x_1}^2 - \ldots - \partial_{x_n}^2 + u(x_1, \ldots, x_n), \;\; n > 1$$

is non-integrable.

Integrable transformations	Non-integrable transformations
1. All constraints on the scattering data can be solved explicitly.	1. Constraints on the scattering data are complicated.
2. Sufficiently many commuting operators.	2. Not enough commuting operators.
3. Infinite-dimensional algebra of symmetries, generated by soliton equations	3. Smaller number of symmetries.
4. Can be used to integrate soliton equations.	4. No associated soliton equation are known.
5. For “good” potentials all wave functions can be calculated explicitly.	5. No explicit formulas for the wave functions are known.

Integrable and non-integrable discretizations:

Let us recall the situation for the 1-D Schödinger operator:

$$L = -\partial_x^2 + u(x).$$

The discretization

$$(Lf)_n = \frac{f_{n+1} + f_{n-1} - 2f_n}{h^2} + u_n f_n$$

is non-integrable. No good characterization of the spectral data in known. No no-trivial symmetries are known.

An integrable discretization (Lax operator for Toda):

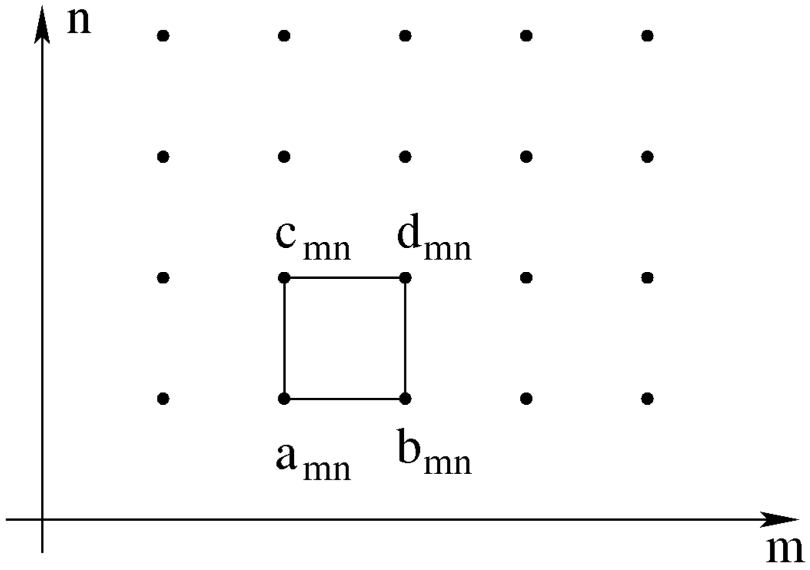
$$(Lf)_n = a_{n+1} f_{n+1} + a_n f_{n-1} + c_n f_n.$$

Manakov, S. V. Complete integrability and stochastization of discrete dynamical systems. *Soviet Physics JETP* **40** (1974), no. 2, 269–274 (1975).
Flaschka, H. On the Toda lattice. II. Inverse-scattering solution. *Progr. Theoret. Phys.* **51** (1974), 703–716.
An integrable reduction:

$$c_n \equiv 0.$$

Hyperbolic discretization of fixed energy 2-D Scrödinger problem, admitting finite-gap solutions

I.M.Krichever, Two-dimensional periodic difference operators and algebraic geometry, *Dokl. Akad. Nauk SSSR*, **285** (1985)



$$(L\Psi)_{m,n} = a_{m,n}\Psi_{m,n} + b_{m,n}\Psi_{m+1,n} + c_{m,n}\Psi_{m,n+1} + d_{m,n}\Psi_{m+1,n+1},$$
$$L\Psi = 0.$$

The direct spectral transform

Assume operator L to be double-periodic

$$a_{m,n} = a_{m+N,n} = a_{m,n+N}, \quad b_{m,n} = b_{m+N,n} = b_{m,n+N},$$
$$c_{m,n} = c_{m+N,n} = c_{m,n+N}, \quad d_{m,n} = d_{m+N,n} = d_{m,n+N}.$$

The space of Bloch functions $\mathcal{B}(\kappa_1, \kappa_2)$:

$$\Psi(m+N, n) = \kappa_1 \Psi(m,n), \quad \Psi(m, n+N) = \kappa_2 \Psi(m,n).$$

The spectral curve:

$$\Gamma : \qquad \det L|_{\mathcal{B}(\kappa_1,\kappa_2)} = 0.$$

Singular point for the Bloch multipliers:

$$P_j^- : \kappa_1 \longrightarrow \infty, \kappa_2 \sim 1, \qquad P_j^+ : \kappa_1 \longrightarrow 0, \kappa_2 \sim 1,$$
$$Q_j^- : \kappa_2 \longrightarrow \infty, \kappa_1 \sim 1, \qquad Q_j^+ : \kappa_2 \longrightarrow 0, \kappa_1 \sim 1,$$

Near P^-_{m+1}

$$a_{m,n}\Psi_{m,n} + b_{m,n}\Psi_{m+1,n} + c_{m,n}\Psi_{m,n+1} + d_{m,n}\Psi_{m+1,n+1} = 0$$

is approximated by

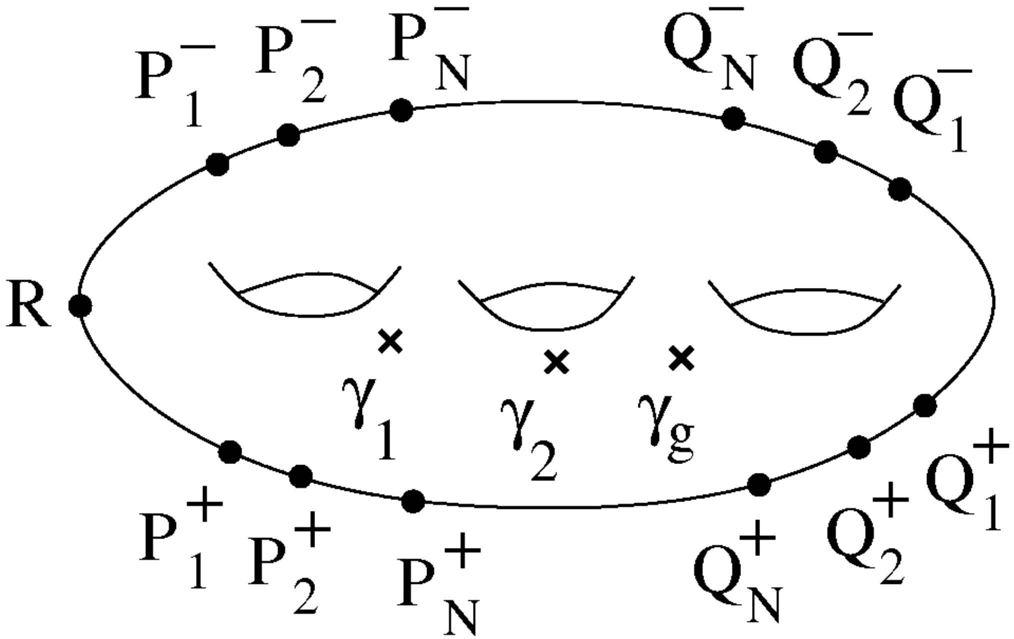
$$c_{m,n}\Psi_{m,n+1} + d_{m,n}\Psi_{m+1,n+1} \sim 0$$

$$\kappa_2^{(m+1)} \sim \prod_{n=1}^{N}\left(-\frac{c_{m,n}}{d_{m,n}}\right)$$

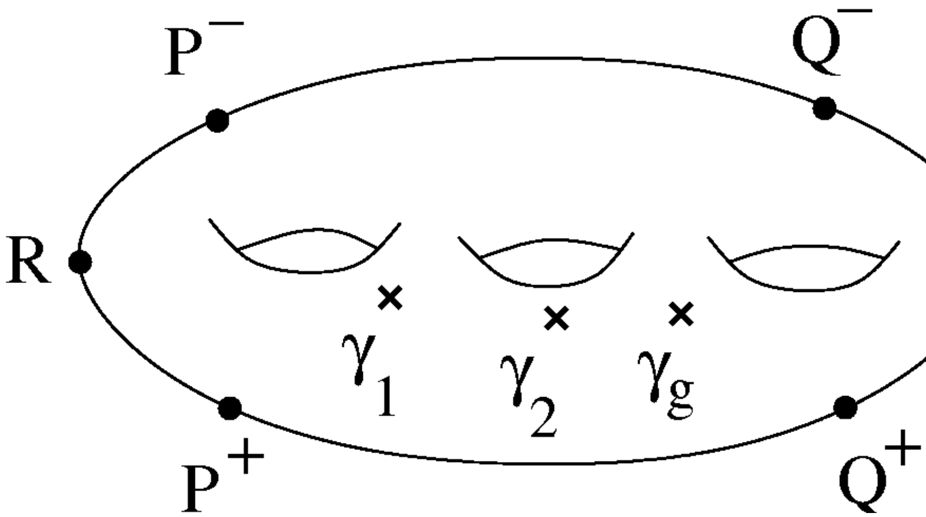
All P^-_{m+1} coincide iff $\kappa_2^{(m+1)}$ does not depend on m

In the original paper of Krichever the generic situation was discussed. Ususlly, nevertheless, it is assumed, that

$$P_j^- = P^-, \quad P_j^+ = P^+, \quad Q_j^- = Q^-, \quad Q_j^+ = Q^+, \quad \text{for all } j.$$



This spectral curve Γ corresponds to a generic operator L.

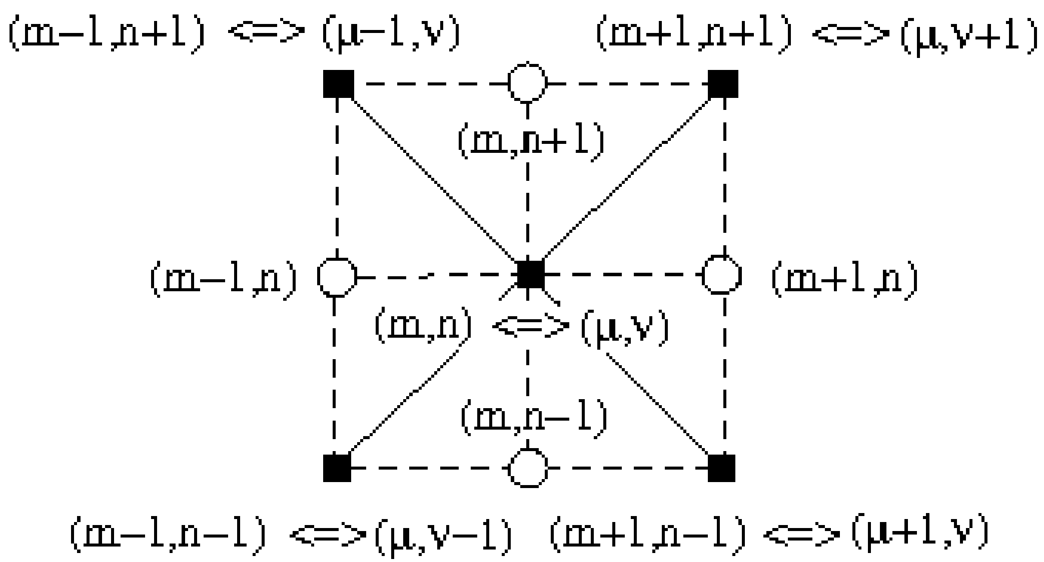


This spectral curve Γ corresponds to an operator L with additional constraints on the coefficients..

Elliptic discretization of the 2-D Scrödinger problem

Elliptic discretization was obtained from a special reduction of the hyperbolic one.

Doliwa A., Grinevich P.G., Nieszporski M., Santini P.M. Integrable lattices and their sub-lattices: from the discrete Moutard (discrete Cauchy-Riemann) 4-point equation to the self-adjoint 5-point scheme. *Journal of Mathematical Physics*, **48**, No.1 (2007), 28 pages.



$$a_{m,n}\Psi_{m,n}+b_{m,n}\Psi_{m+1,n}+c_{m,n}\Psi_{m,n+1}+d_{m,n}\Psi_{m+1,n+1}=0 \quad (1)$$

Using the gauge transformations:

$$L \longrightarrow f_{m,n} L g_{m,n}$$

one can gauge out 2 coefficients of 4.

Problem: For which coefficients $a_{m,n}$, $b_{m,n}$,$c_{m,n}$, $d_{m,n}$, all solutions of (1) satisfy the same linear equation on the even sublattice: $m+n=2k$.

Answer: It is necessary and sufficient that (1) can be gauged the following form:

$$\psi_{m+1,n+1}-\psi_{m,n}=if_{m,n}(\psi_{m+1,n}-\psi_{m,n+1}).$$

The corresponding 5-point equation has the form:

$$A_{\mu,\nu}(\Psi_{\mu+1,\nu} - \Psi_{\mu,\nu}) + A_{\mu-1,\nu}(\Psi_{\mu-1,\nu} - \Psi_{\mu,\nu}) +$$
$$+ B_{\mu,\nu}(\Psi_{\mu,\nu+1} - \Psi_{\mu,\nu}) + B_{\mu,\nu-1}(\Psi_{\mu,\nu-1} - \Psi_{\mu,\nu}) = 0$$

where

$$A_{m,n} = f_{m,n-1}, \qquad B_{m,n} = \frac{1}{f_{m,n}}.$$

The finite-gap potentials correspond to a special reduction of the hyperbolic discretization. This reduction is very similar to the Novikov and Veslov reduction for potential operators.

First finite-gap solutions were constructed in
Doliwa A., Grinevich P.G., Nieszporski M., Santini P.M. Integrable lattices and their sub-lattices: from the discrete Moutard (discrete Cauchy-Riemann) 4-point equation to the self-adjoint 5-point scheme. *Journal of Mathematical Physics*, **48**, No.1 (2007), 28 pages.
They correspond to involutions of spectral curves with two fixed points.

Grushesky and Krichever pointed out, that generic solutions correspond to involutions without fixed points.
Grushevsky. Samuel, Krichever Igor. Integrable discrete Schrödinger equations and a characterization of Prym varieties by a pair of quadrisecants. arXiv:0705.2829.

This discretization was used by Grushesky and Krichever to solve an analog of Shottky-Prym problem for the prymians of the curves with involutions without fixed points:
How to obtain a characterization of the period matrices for the prymians of these curves?
It was a classical problem.

- 2-D continuous Schrödinger $\leftrightarrow$ Novikov-Veselov hierarchy.
- Elliptic discretization of 2-D Schrödinger $\leftrightarrow$ 2-D Toda with 2 discrete spatial variables (discrete Novikov-Veselov).

M. Nieszporski, P. M. Santini and A. Doliwa, Darboux transformations for 5-point and 7-point self-adjoint schemes and an integrable discretization of the 2D Schrödinger operator, Phys. Lett. A **323** (2004), 241–250.
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Grushevsky. Samuel, Krichever Igor, Integrable discrete Schrödinger equations and a characterization of Prym varieties by a pair of quadrisecants. arXiv:0705.2829.
P. M. Santini, A. Doliwa and M. Nieszporski, Integrable dynamics of Toda-type on the square and triangular lattices arXiv: 0710.5543v1.

The Lax representation:

The spectral problem:

$$A_{m,n}\Psi_{m+1,n} + A_{m-1,n}\Psi_{m-1,n} +$$

$$+B_{m,n}\Psi_{m,n+1} + B_{m,n-1}\Psi_{m,n-1} = F_{m,n}\Psi_{m,n}$$

The time evolution:

$$\frac{d\Psi_{m,n}}{dt} = \frac{\xi_{m,n}}{2}\left(A_{m,n}\Psi_{m+1,n} - A_{m-1,n}\Psi_{m-1,n}\right) +$$

$$+\frac{\eta_{m,n}}{2}\left(B_{m,n}\Psi_{m,n+1} - B_{m,n-1}\Psi_{m,n-1}\right)$$

The functions $\xi_{m,n}$, $\eta_{m,n}$ are defined by:

$$A_{m,n}B_{m,n}[\xi + \eta]_{m,n} = A_{m,n+1}B_{m+1,n}[\xi + \eta]_{m+1,n+1},$$

$$A_{m,n+1}B_{m,n}[\xi + \eta]_{m,n+1} = A_{m,n}B_{m+1,n}[\xi + \eta]_{m+1,n}.$$

Equations have the following form:

$$\frac{dF_{m,n}}{dt} + \xi_{m+1,n} A^2_{m,n} - \xi_{m-1,n} A^2_{m-1,n} +$$

$$+\eta_{m,n+1} B^2_{m,n} - \eta_{m,n-1} B^2_{m,n-1} = 0,$$

$$\frac{d\ln(A_{m,n})}{dt} + \frac{1}{2}[\xi_{m+1,n} F_{m+1,n} - \xi_{m,n} F_{m,n}] = 0,$$

$$\frac{d\ln(B_{m,n})}{dt} + \frac{1}{2}[\eta_{m,n+1} F_{m,n+1} - \eta_{m,n} F_{m,n}] = 0,$$

Starting from now we shall assume, that operator L is double-periodic

$$\begin{gathered}
A_{m+M,n} = A_{m,n+N} = A_{m,n}, \\
B_{m+M,n} = B_{m,n+N} = B_{m,n}, \\
F_{m+M,n} = F_{m,n+N} = F_{m,n},
\end{gathered} \tag{2}$$

elliptic:

$$A_{m,n} > 0, \;\; B_{m,n} > 0, F_{m,n} > 0,$$

and some additional non-degeneracy condions are fulfilled (they are fulfilled at least in the neighbourhood of the constant operator $A_{m,n} = 1, \;\; B_{m,n} = 1, F_{m,n} = 4$).

Lemma. The space of solutions of the equations

$$A_{m,n}B_{m,n}[\xi+\eta]_{m,n} = A_{m,n+1}B_{m+1,n}[\xi+\eta]_{m+1,n+1},$$

$$A_{m,n+1}B_{m,n}[\xi+\eta]_{m,n+1} = A_{m,n}B_{m+1,n}[\xi+\eta]_{m+1,n}.$$

compatible with the periodicity is at most 2-dimensional. It is convenient to choose the following basis:

$$[\xi^{(1)} - \eta^{(1)}]_{m,n} \equiv 0$$

$$[\xi^{(2)} + \eta^{(2)}]_{m,n} \equiv 0$$

Theorem. The both flows defined above are compatible with the periodicity constraints (2) and isospectral if and only if all $P_k^+ = P^+$, $P_k^- = P^-$, $Q_k^+ = Q^+$, $Q_k^- = Q^-$ do not depend on k.

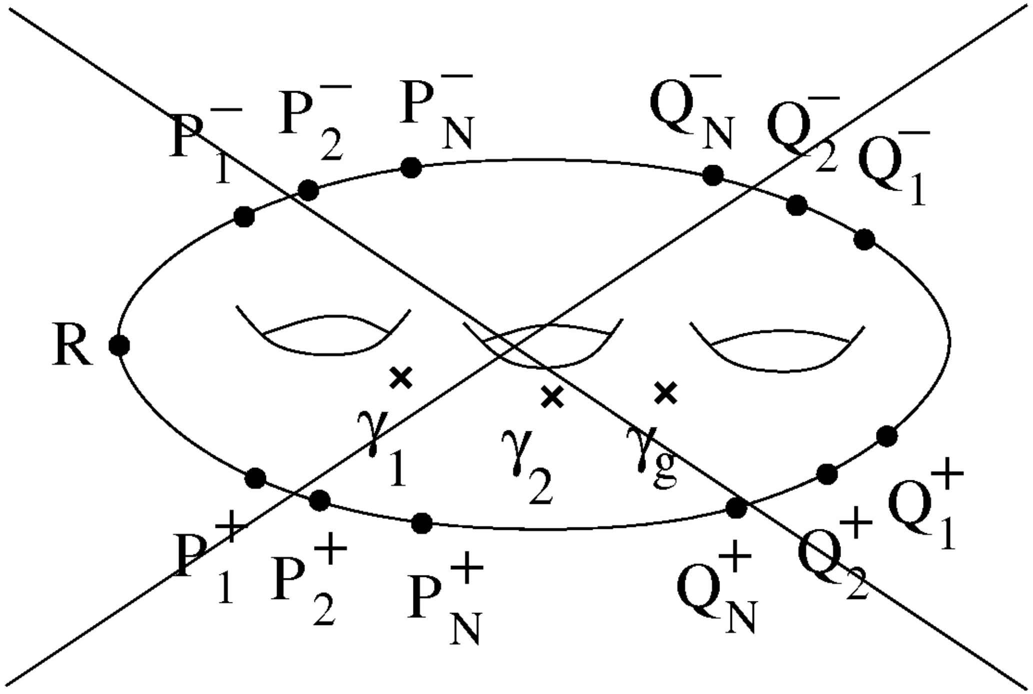
Equivalently, let us define

$$J_1(l) = \prod_{k=0}^{N-1} \frac{A_{k,l-k}}{B_{k,l-k}}, \quad J_2(l) = \prod_{k=0}^{N-1} \frac{A_{k,l+k}}{B_{k+1,l+k}}$$

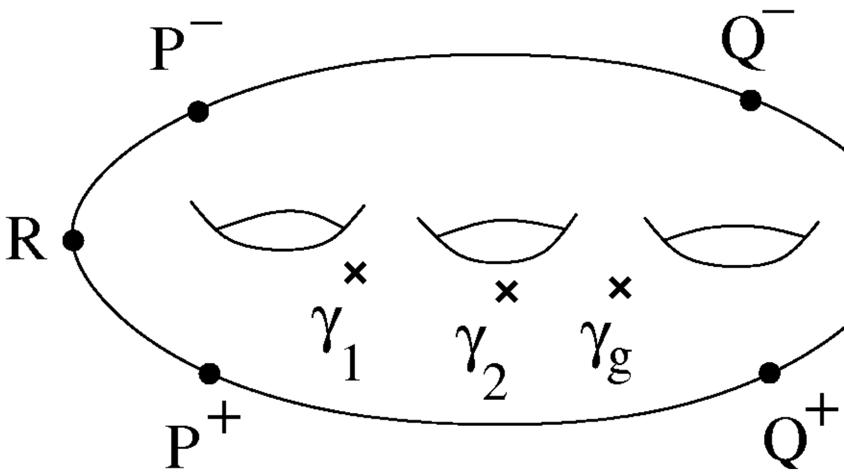
The both basic flows are compatible with the periodicity and isospectral if and only if

$$J_1(1) = J_1(2) = \ldots = J_1(N), \quad J_2(1) = J_2(2) = \ldots = J_2(N).$$

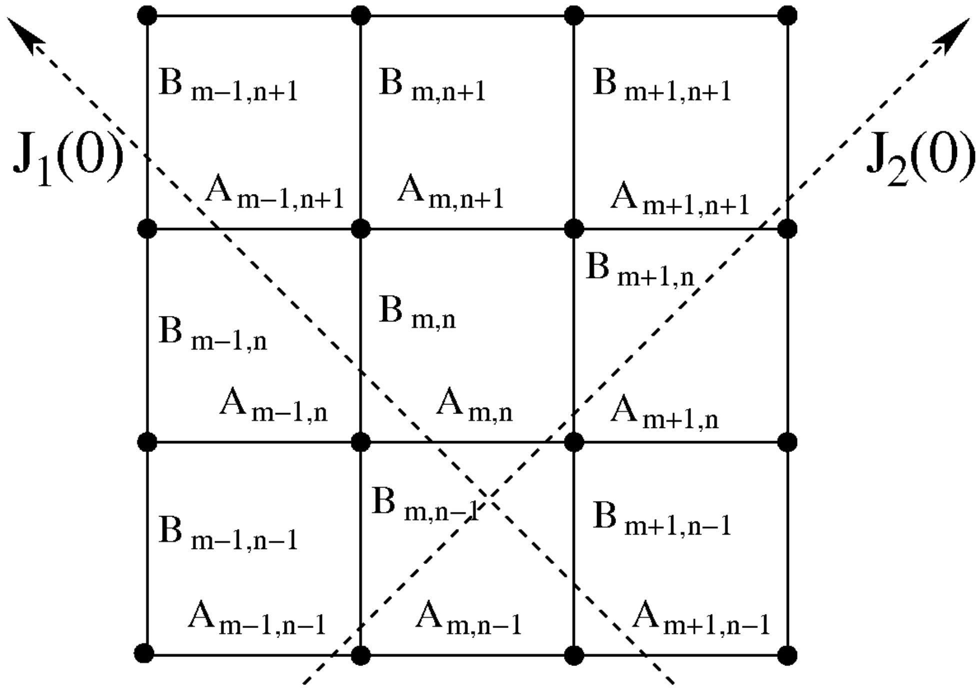
The spectral transform for the 5-point scheme is integrable on a submanifold of the phase space only.



These operators does not possess the full algebra of symmetries – they are non-integrable .



These operators possess the full algebra of symmetries – they are integrable .



Sketch of the proof:

Existens of 2 flows follows immediately from explicit formulas (a simple Lemma in the text by Grushesky and Krichever).

Lemma: If dynamics preservs the periodicty, the both $\xi_{m,n}$, $\eta_{m,n}$ should be double-periodic. It is evident for generic operetors, the case of degeneration was also analized, but it is a little tricky.

$$\frac{d}{dt}\log J_1(l) = \frac{1}{2}\sum_{k=0}^{N-1}\left[(\xi+\eta)_{k,l+1-k}F_{k,l+1-k} - (\xi+\eta)_{k,l-k}F_{k,l-k}\right]$$

$$\frac{d}{dt}\log J_2(l) = \frac{1}{2}\sum_{k=0}^{N-1}\left[(\xi+\eta)_{k+1,k+l}F_{k+1,k+l} - (\xi+\eta)_{k,k+l}F_{k,k+l}\right]$$

If the both $\xi_{m,n}$, $\eta_{m,n}$ are double-periodic, the both derivatives vanish.

Another naural discretization of 2-D Schrödinger operator corresponds to the triangular regular lattice.

I. A. Dynnikov and S. P. Novikov, Geometry of the triangle equation on two-manifolds, *Mosc. Math. J.* **3** (2003) 419-438.

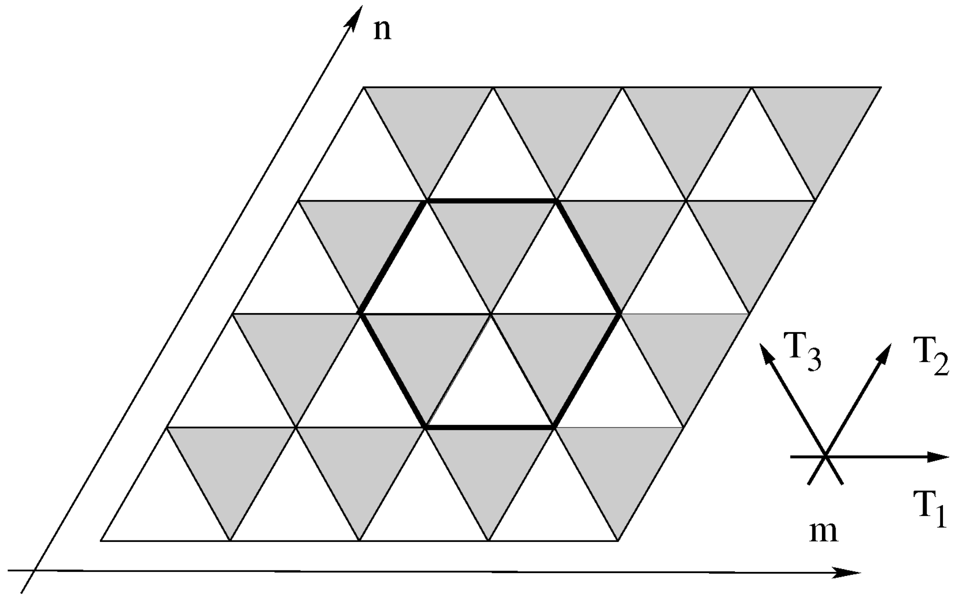
It has some important advantages. The 2-D Toda (Novikov-Veselov hierarchy) waq constrcuted in

M. Nieszporski, P. M. Santini and A. Doliwa, Darboux transformations for 5-point and 7-point self-adjoint schemes and an integrable discretization of the 2D Schrödinger operator, Phys. Lett. A **323** (2004), 241–250.

Santini, P. M.; Nieszporski, M.; Doliwa, A. Integrable generalization of the Toda law to the square lattice. *Phys. Rev. E (3)* **70** (2004), no. 5, 056615, 6 pp.

P. M. Santini, A. Doliwa and M. Nieszporski, Integrable dy-

namics of Toda-type on the square and triangular lattices arXiv: 0710.5543v1.



$$(L\Psi)_{m,n} = A_{m,n}\Psi_{m+1,n} + A_{m-1,n}\Psi_{m-1,n} + B_{m,n}\Psi_{m,n+1} +$$
$$+B_{m,n-1}\Psi_{m,n-1} + C_{m,n}\Psi_{m-1,n+1} + C_{m+1,n-1}\Psi_{m+1,n-1} - F_{m,n}\Psi_{m,n}$$

Analogous results are valid for the corresponding equations.