Asymptotic analysis of the cyclotron autoresonance phenomenon *

L. Kalyakin

Institute of mathematics RAS,
Ufa, RUSSIA
http://www.anrb.ru/matem/

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1. Equations

2. Resonance and autoresonance in one dimension system

3. First integral and cyclotron resonance

4. Small parameter and adiabatic approach

5. First integrals in the case of circle polarization

6. Resonance solutions

7. General solutions

$$
\begin{cases}
\dfrac{dP_z}{dt} = N\dfrac{d\gamma}{dt} = \dfrac{NP_\perp}{2\gamma}[\mathcal{E}_+ \cos\theta_- + \mathcal{E}_- \cos\theta_+], \\[2ex]
\dfrac{dP_\perp}{dt} = \dfrac{1}{2}\Big(1 - \dfrac{NP_z}{\gamma}\Big)[\mathcal{E}_+ \cos\theta_- + \mathcal{E}_- \cos\theta_+], \\[2ex]
\dfrac{d\theta_c}{dt} = -\dfrac{\Omega}{\gamma} + \dfrac{1}{2P_\perp}\Big(1 - \dfrac{NP_z}{\gamma}\Big)[\mathcal{E}_+ \sin\theta_- - \mathcal{E}_- \sin\theta_+], \\[2ex]
\dfrac{d\theta}{dt} = -1 + \dfrac{NP_z}{\gamma}, \qquad (\theta_\pm = \theta \pm \theta_c).
\end{cases}
$$

Resonance

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon \cos(\Omega t), \quad \omega, \Omega, \varepsilon = \text{const} > 0.$$

$$x(t;\varepsilon) = A\cos(\omega t + S) - \varepsilon \cos(\Omega t)/(\Omega^2 - \omega^2),$$

$$\forall\, A, S = \text{const},\ t \in \mathbb{R},\ 0 < \varepsilon < 1.$$

$$x(t;\varepsilon) = A\cos(\omega t + S) + \varepsilon t \sin(\Omega t)/(2\Omega),\ \omega = \Omega$$

Autoresonance

$$\frac{d^2x}{dt^2} + \sin x = \varepsilon \cos\varphi(t)$$

$$\varphi(t) = \Omega t + \alpha t^2/2$$

Main resonance equations

$$\frac{dR}{dT} = \sin\Psi, \qquad \frac{d\Psi}{dT} = R - \lambda T. \tag{1}$$

$$\frac{dR}{dT} = \sin\Psi, \qquad \frac{d\Psi}{dT} = R^2 - \lambda T. \tag{2}$$

$$\frac{dR}{dT} = \sin\Psi, \qquad R\Big[\frac{d\Psi}{dT} - R^2 + \lambda T\Big] = b\cos\Psi. \tag{3}$$

$$\frac{dR}{dT} = R\sin\Psi, \qquad \frac{d\Psi}{dT} - R + \lambda T = b\cos\Psi. \tag{4}$$

Here $\lambda, b = \mathrm{const} \neq 0$**, functions** R, Ψ **depend on** $T \in \mathbb{R}$**.**

Problem: are there solutions which have unlimited amplitude

$$R(T) \to \infty \ as\ T \to \infty.$$

Initial data: a particle with a charge e and weight m_0; a longitudinal constant magnetic field

$$\bar{B}_0 = (0, 0, B_0) = \text{const};$$

a flat cross electromagnetic wave

$$\mathbf{E} = (E_1 \cos\theta, E_2 \cos\theta, 0), \quad (E_1, E_2 = \text{const}),$$

$$\mathbf{H} = (-NE_1 \cos\theta, NE_2 \cos\theta, 0).$$

The wave is characterized by constant frequency ω and wave number k. Interaction with a particle is described by a Doppler shift of frequency in the phase equation

$$\frac{d\theta}{d\tilde{t}} = -\omega + kv_z.$$

The parameter of refraction

$$N = kc/\omega \equiv c/v_{ph} \neq 0$$

is the ratio of speed of light to phase speed of a wave.

Dimensionless impulse, time, amplitude are entered:

$$\mathbf{P} = \mathbf{p}/m_0c, \;\; t = \omega\tilde{t}, \;\; \mathcal{E}_i = eE_i/(m_0c\omega)$$

The important constant

$$\Omega = eB_0/(m_0 c\omega)$$

is the ratio of gyrofrequency to frequency of a wave. The transverse impulse is considered in polar coordinates

$$(P_x, P_y) = P_\perp(\cos\theta_c, \sin\theta_c).$$

Thus, the equations of movement and interaction with a wave is considered for five functions, including the relativistic factor γ:

$$\begin{cases} \dfrac{dP_z}{dt} = N\dfrac{d\gamma}{dt} = \dfrac{NP_\perp}{2\gamma}[\mathcal{E}_+\cos\theta_- + \mathcal{E}_-\cos\theta_+], \\[2ex] \dfrac{dP_\perp}{dt} = \dfrac{1}{2}\Big(1 - \dfrac{NP_z}{\gamma}\Big)[\mathcal{E}_+\cos\theta_- + \mathcal{E}_-\cos\theta_+], \\[2ex] \dfrac{d\theta_c}{dt} = -\dfrac{\Omega}{\gamma} + \dfrac{1}{2P_\perp}\Big(1 - \dfrac{NP_z}{\gamma}\Big)[\mathcal{E}_+\sin\theta_- - \mathcal{E}_-\sin\theta_+], \\[2ex] \dfrac{d\theta}{dt} = -1 + \dfrac{NP_z}{\gamma}, \quad (\theta_\pm = \theta \pm \theta_c). \end{cases} \tag{5}$$

$\mathcal{E}_{\pm} = \mathcal{E}_1 \pm \mathcal{E}_2 = \mathrm{const}$

The first integral

$$N\gamma - P_z = Y \equiv \mathrm{const}.$$

The condition of a resonance between a wave and a particle:

$$\gamma - NP_z = \Omega.$$

The case $N = 1$ **corresponds to a "vacuum"wave. In this case a resonance preserves. It provides an increase of the impulse.**

$$\begin{cases} \dfrac{dP_z}{dt} = \dfrac{d\gamma}{dt} = \dfrac{P_\perp}{2\gamma}[\mathcal{E}_+ \cos\theta_- + \mathcal{E}_- \cos\theta_+], \\[2ex] \dfrac{dP_\perp}{dt} = \dfrac{1}{2}\Big(1 - \dfrac{P_z}{\gamma}\Big)[\mathcal{E}_+ \cos\theta_- + \mathcal{E}_- \cos\theta_+], \\[2ex] \dfrac{d\theta_c}{dt} = -\dfrac{\Omega}{\gamma} + \dfrac{1}{2P_\perp}\Big(1 - \dfrac{P_z}{\gamma}\Big)[\mathcal{E}_+ \sin\theta_- - \mathcal{E}_- \sin\theta_+], \\[2ex] \dfrac{d\theta}{dt} = -1 + \dfrac{P_z}{\gamma}, \quad (\theta_\pm = \theta \pm \theta_c). \end{cases} \tag{6}$$

The first integral is $\gamma - P_z = Y \equiv \mathrm{const}$, and a condition of a resonance is $Y = \Omega$.

It is not known - whether this system of the equations is integrated.

The asymptotic analysis with small amplitudes electromagnetic field is possible.

$$\mathcal{E}_1 + \mathcal{E}_2 = 2\varepsilon a,\ \mathcal{E}_1 - \mathcal{E}_2 = 2\varepsilon b, \quad (a, b = \mathrm{const}).$$

The equations with small parameter

$$\begin{cases} \dfrac{dP_z}{dt} = \dfrac{d\gamma}{dt} = \varepsilon \dfrac{P_\perp}{\gamma}[a\cos\theta_- + b\cos\theta_+], \\ \\ \dfrac{dP_\perp}{dt} = \varepsilon \dfrac{\gamma - P_z}{\gamma}[a\cos\theta_- + b\cos\theta_+], \\ \\ \dfrac{d\theta_c}{dt} = -\dfrac{\Omega}{\gamma} + \varepsilon \dfrac{\gamma - P_z}{\gamma}\dfrac{1}{P_\perp}[a\sin\theta_- - b\sin\theta_+], \\ \\ \dfrac{d\theta}{dt} = -\dfrac{\gamma - P_z}{\gamma}, \quad (\theta_\pm = \theta \pm \theta_c). \end{cases} \tag{7}$$

There are trying problems with application of averaging method because of

two fast phases. Well known classical averaging method can be applied in the resonance case, when both fast phases coincide. But it occur under specific value of the first integral $\gamma - P_z = \Omega$.

There is a specific case when the equations are integrated precisely. It is the case of circular polarization $E_1 = E_2$, when $b = 0$.

The theory of a cyclotron autoresonance [1] (an autoresonance between a wave and a particle) is based on the analysis of the first integrals of system of the differential equations in this case

$$
\begin{cases}
\dfrac{dP_z}{dt} = \dfrac{d\gamma}{dt} = \dfrac{P_\perp}{\gamma} A\cos\theta_-, \\[2ex]
\dfrac{dP_\perp}{dt} = \Big(1 - \dfrac{P_z}{\gamma}\Big) A\cos\theta_-, \\[2ex]
\dfrac{d\theta_c}{dt} = -\dfrac{\Omega}{\gamma} + \dfrac{1}{P_\perp}\Big(1 - \dfrac{P_z}{\gamma}\Big) A\sin\theta_-, \\[2ex]
\dfrac{d\theta}{dt} = -1 + \dfrac{P_z}{\gamma},\ \ \theta_- = \theta - \theta_c, \quad (\Omega, A = \mathrm{const}).
\end{cases}
\tag{8}
$$

One of the equations for a phase can be separated, having put $\psi = \theta_- = \theta - \theta_c$ and writing out four equations for four functions

$$
\begin{cases}
\dfrac{dP_z}{dt} = \dfrac{d\gamma}{dt} = \dfrac{P_\perp}{\gamma} A \cos\psi, \\[2ex]
\dfrac{dP_\perp}{dt} = \dfrac{\gamma - P_z}{\gamma} A \cos\psi, \\[2ex]
\dfrac{d\psi}{dt} = -\dfrac{\gamma - P_z - \Omega}{\gamma} - \dfrac{1}{\gamma P_\perp}(\gamma - P_z) A \sin\psi.
\end{cases}
\tag{9}
$$

The known set for this system of the first integrals turns out from the first three of the equations:

$$
\gamma - P_z = Y = \text{const}, \quad P_\perp^2 - 2Y\gamma = X = \text{const}. \tag{10}
$$

To write out the explicit solution, an additional integral is required. It turns out from last pair of the equations:

$$
P_\perp \sin\psi + \gamma \frac{Y - \Omega}{A} = Z = \text{const}. \tag{11}
$$

Resonant solutions correspond to a specific value of the first integral: $Y = \Omega$.

Just this case was analyzed in 1962 by Kolomenski in 1962. In that case

$$\sin\varphi = Z/P_\perp,$$

$$P_\perp \cos\varphi = \sqrt{P_\perp^2 - Z^2} = \sqrt{X - Z^2 - 2Y\gamma}.$$

The equation for γ is reduced to the form

$$\frac{d\gamma}{dt} = \frac{A\sqrt{X - Z^2 - 2Y\gamma}}{\gamma} = \frac{\alpha\sqrt{\gamma - \beta}}{\gamma}$$

$$\alpha = A\sqrt{2Y},\ \beta = (Z^2 - X)/2Y$$

The equation is integrated in elementary functions:

$$\frac{2}{3}(\gamma - \beta)^{3/2} 2\beta\sqrt{\gamma - \beta} = \alpha t \mathrm{const}$$

An asymptotics is obtained from the exact solution and shows unlimited growth

$$\gamma(t) \approx (\frac{3}{2}\alpha t)^{2/3},\ t \to \infty.$$

The factor α corresponds to amplitude electromagnetic wave. It was done in 1962 by Kolomenski.

The family of autoresonance solutions with growing longitudinal impulse

$$P_z(t) \approx (\frac{3}{2}\alpha t)^{2/3},\ t \to \infty, \quad (\text{at } \gamma - P_z = \Omega).$$

is very poor because of a restriction on the first integral. For this reason the phenomenon of a cyclotron autoresonance "in the pure state"cannot be found out neither in numerical, nor in physical experiments.

We find the first integral in general case which correspond to rich enough family of the solutions closed to autoresonance. These solutions correspond to physically observable phenomenon.

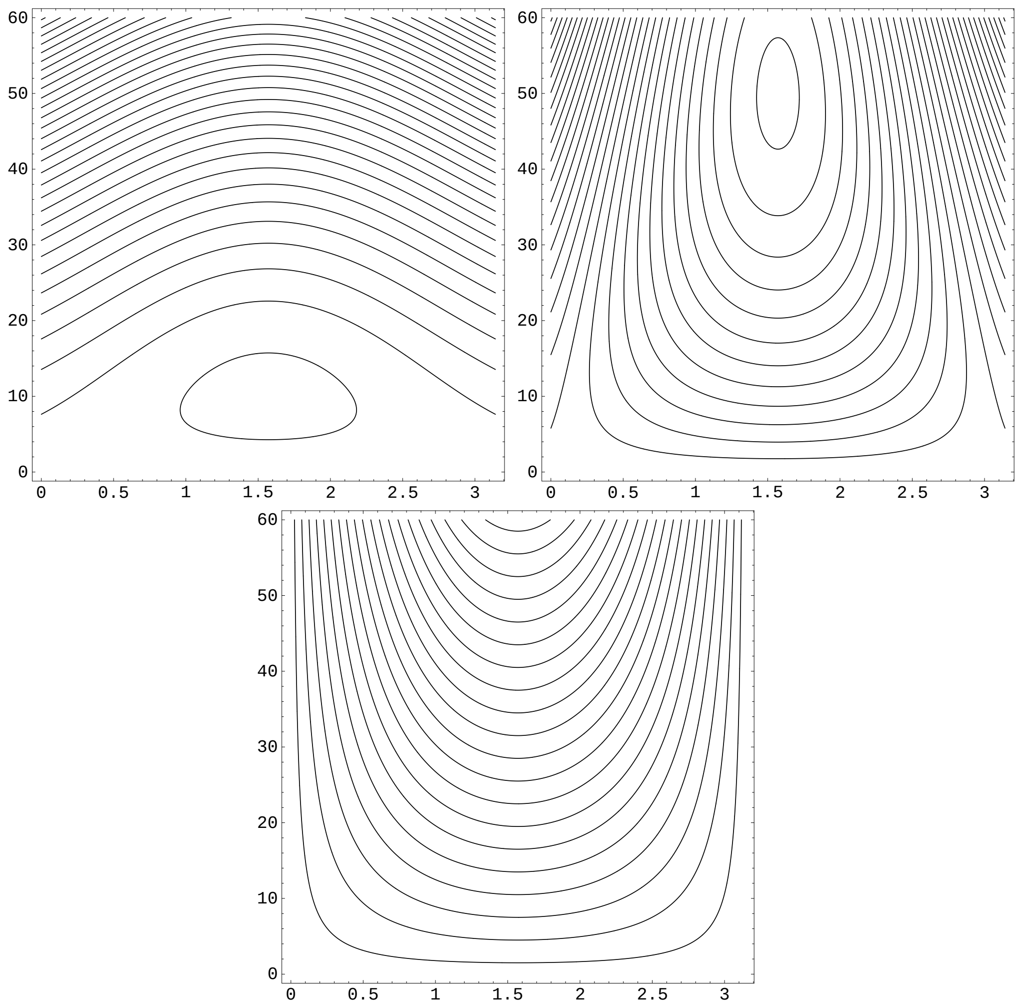
Solutions in general case.

$$P_\perp \sin\psi (P_\perp^2 - X)\frac{Y - \Omega}{2YA} = Z = \text{const}.$$

The phase portrait consists of the family of circles on a plane $x = P_\perp \cos\psi,\ y = P_\perp \sin\psi$. The center of circles in a point and $x = 0,\ y = YA/(Y - \Omega)$.

Deformation of the phase portrait in

plane ψ, γ depanding on the first integral $Y - \Omega \to 0$



[1] V.P.Milant'ev, Cyclotron autoresonance phenomenon and its applications. Uspekhi Fis. Nauk. V.167: 1 (1997). P. 3–16.