Scattering of weakly nonlinear dispersive wave on a parametric resonance

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Outlines

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1. Parametric driven nonlinear Klein-Gordon equation
2. Small amplitude solution

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Mathematical model

- An object of this talk is solutions of parametric perturbed nonlinear Klein-Gordon equation:

$$\partial_t^2 U - \partial_x^2 U + \left(1 + \varepsilon f \cos\left(\frac{S(\varepsilon^2 x, \varepsilon^2 t)}{\varepsilon^2}\right)\right) U + \gamma U^3 = 0.$$

Mathematical model

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- Here ε is small parameter, $\gamma, f \in \mathbf{R}$ and $S(y, z)$ is smooth function.

Small amplitude solution

- We study a small amplitude solution in the form of a modulated oscillating wave:

$$U(x,t,\varepsilon) \sim \varepsilon u_1(x_1,t_1,x_2,t_2)\exp\{i(kx+\omega t)\} + c.c..$$

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- This solution depends on groups of scaled variables:

  - fast variables are x, t;
  - slow variables are $x_1 = \varepsilon x,\ t_1 = \varepsilon t$;
  - very slow variables are $x_2 = \varepsilon^2 x$ and $t_2 = \varepsilon^2 t$.

Problem

- Background

- In a general approach the shape of the weak nonlinear wave is defined by Nonlinear Schrödinger equation.
- There exist resonant curves on the plane (x_2, t_2) such that this approach is not valid near these curves and the main role plays the perturbation.

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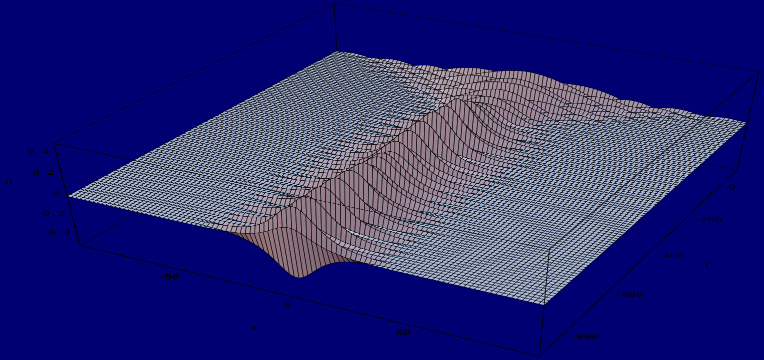
- Goals

  - To control of weak nonlinear dispersive waves.
  - To find a connection formula for the solution before and after the resonance.

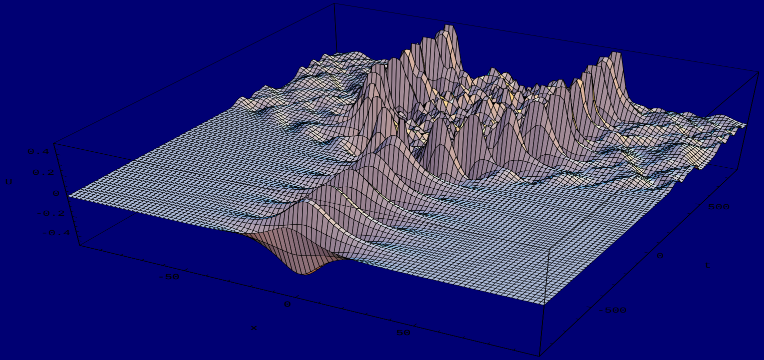
Simulations

Annihilation of NLSE soliton



Simulations

Generation of NLSE soliton



Bounds of NLSE approach

It is well-known that the weak nonlinear waves are defined by NLSE.

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Further we show the borders of this approach for solutions of the parametric driven equation.

Theorem about external expansion

- The asymptotic solution of PNGKE has the form

- $U = \varepsilon\big(u_1(t_1, x_1, x_2, t_2)\exp\{i(kx_2 + \omega t_2)/\varepsilon^2\} + c.c.\big) +$

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- where

$$u_1 = \Psi \exp\left\{ -i\frac{f^2}{4\omega}\int^{t_2}\left[\frac{1}{L[\phi_-]} + \frac{1}{L[\phi_+]}\right] dt_2 \right\}.$$

- Function Ψ is determined by the NLSE:

$$i\omega\partial_{t_2}\Psi - \partial_\zeta^2\Psi + 3\gamma|\Psi|^2\Psi = 0, \quad \zeta = \omega x_1 + kt_1.$$

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- $\varepsilon^2\big(u_2^+ \exp(i\phi_+(x_2, t_2)/\varepsilon^2) + u_2^- \exp(i\phi_-(x_2, t_2))/\varepsilon^2) + c.c.\big) + \quad \dots.$

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- The expansion is valid in the domains

$$-\varepsilon^{-1}\left(-(\partial_{t_2}\phi_{\pm})^2 + (\partial_{x_2}\phi_{\pm})^2 + 1\right) \gg 1,$$

Local parametric resonance

- The condition of primary local parametric resonance is an equation:

$$-\left(\partial_{t_2}(kx_2+\omega t_2 \pm S(x_2,t_2))\right)^2+\left(\partial_{x_2}(kx_2+\omega t_2 \pm S(x_2,t_2))\right)^2+1=0$$

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- A typical local resonance generates new harmonics (see Glebov, Kiselev, Lazarev, 2006) with

$$k_1=k\pm\partial_{x_2}S|_{L[\chi_{1,\pm 1}]=0},\quad \text{as}\quad |k_1|\neq k$$

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- Firstly the local resonance was studied since of 1970's:

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- Here we consider a special case which is called by local parametric resonance

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In this case new harmonics are not generated in the leading-order term but the envelope function changes.

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- The local parametric resonance was studied for ordinary differential equations by

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Local parametric resonance

- In a neighborhood of the parametric resonant curve the formal asymptotic expansion is

$$U(x,t,\varepsilon) = \varepsilon\big(w_1(x_1,t_1,x_2,t_2)\exp(iS(x_2,t_2)/\varepsilon) + c.c.\big) +$$

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- where $w_1(x_1,t_1,x_2,t_2)$ is a solution of

Local Parametric Resonance Equation

$$i\partial_{x_2}S\partial_{x_1}w_1 - i\partial_{t_2}S\partial_{t_1}w_1 + \lambda w_1 + \frac{f}{2}\overline{w_1} = 0.$$

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- and λ is a function which depends on slow and very slow variables:

$$\lambda(x_1,t_1,x_2,t_2) = \varepsilon^{-1}\left(-\left(\partial_{t_2}\phi_\pm\right)^2 + \left(\partial_{x_2}\phi_\pm\right)^2 + 1\right)$$

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- $$\varepsilon^2\big(w_{2,1}\exp(iS(x_2,t_2)/\varepsilon)+w_{2,2}\exp(2iS(x_2,t_2)/\varepsilon)+c.c.\big)+\dots.$$

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Scattering problem

Our goal is to solve a scattering problem for the local parametric resonance equation. This problem is solved by four steps.

- Obtain an asymptotic reduction to

the ordinary primary parametric resonance equation

$i\frac{dW}{d\sigma} + \sigma W + F\overline{W} = 0$

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- Solve this equation using the parabolic cylinder functions.
- Use formula for the solution to solve a scattering problem.
- Estimate a domain of validity for the internal asymptotic solution.

Matching asymptotic expansions and connection formula

In the domain $l > 0$ the formal asymptotic solution has the same form

$$U(x,t,\varepsilon) \sim \varepsilon v_1(x_1,t_1,t_2)\exp\{i(kx+\omega t)\} + c.c..$$

The amplitude $v_1 = \Psi_+ \exp\left\{ i\frac{f^2}{4\omega} G(x_2,t_2)\right\}$ and Ψ is determined by the nonlinear Schrödinger equation also and initial datum on the curve $l = 0$

$$\Psi_+(t_2,\zeta)|_{l=0} = e^{\frac{f^2\pi}{8}}\Psi_- + \frac{(1+i)e^{\frac{f^2\pi}{16}}e^{i\frac{f^2}{16}\ln(2)} f\sqrt{\pi}}{2\Gamma(1-i\frac{f^2}{8})}\overline{\Psi_-},$$

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- The shape of this solution defines by NSE.
- This approximation is valid before and after the parametric resonant curve.
- On this curve the solution has a jump. The solution of NSE before the line and after the line are connected by:

$$\Psi_+(t_2,\zeta)|_{l=0} =$$
$$e^{\frac{f^2\pi}{8}}\Psi_-(t_2,\zeta)|_{l=0} + \frac{(1+i)e^{\frac{f^2\pi}{16}}e^{i\frac{f^2}{16}\ln(2)}f\sqrt{\pi}}{2\Gamma(1-i\frac{f^2}{8})}\overline{\Psi_-(t_2,\zeta)}|_{l=0},$$

New results

- **Primary parametric resonance partial differential equation**

$$i\partial_{x_2}S\partial_{x_1}w_1 - i\partial_{t_2}S\partial_{t_1}w_1 + \lambda w_1 + \frac{f}{2}\overline{w_1} = 0.$$

New results

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- Connection formula:

$$\Psi_+(t_2,\zeta)|_{l=0} = e^{\frac{f^2\pi}{8}}\Psi_-(t_2,\zeta)|_{l=0} + \frac{(1+i)e^{\frac{f^2\pi}{16}} e^{i\frac{f^2}{16}\ln(2)} f\sqrt{\pi}}{2\Gamma(1-i\frac{f^2}{8})}\overline{\Psi_-(t_2,\zeta)}|_{l=0},$$

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