Hamiltonian systems of hydrodynamic type in 2+1 dimensions

Antonio Moro

Loughborough University, United Kingdom

Joint work with E. Ferapontov and V. Sokolov

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Why?

$$u_t^i + v_j^{ik}(u)\, u_{x_k}^j = 0 \qquad \begin{array}{l} i = 1,\dots N \\ k = 1,\dots M \end{array}$$

- Limit of Navier-Stokes equations, e.g. Zaboloskaya-Khokhlov (dKP), Benney (hydrodynamic chains).
- Dispersionless limit of soliton equations (Hopf-KdV, dKP-KP, dTL-TL, dVN-VN)
- Leading order of genus expansion in Topological Field Theories
- Slow modulations of Soliton Lattices (Bogolyubov-Whitham averaging method)
- Laplacian growth (Hele-Shaw cells) and Random Matrices models in QFT.
- Nonlinear nonlocal geometric optics – dVN equation
- Quasiclassical Weierstrass representations (highly corrugated surfaces)

1+1D Hamiltonian quasilinear systems

Let us consider the system

$$u_t^i + v_j^i(u)\,u_x^j = 0 \qquad (0.1)$$

where $\quad u = \left(u^1, u^2, \dots, u^N\right)$

Representable in the form

$$u_t^i = \left\{u^i(x), H[u]\right\}$$

where

$$\left\{u^i(x), u^j(y)\right\} = g^{ij}(u(x))\,\delta'(x-y) + b_k^{ij}(u(x))\,u_x^k\,\delta(x-y)$$

It can be written, equivalently, as follows

$$u_t^i + P^{ij}\frac{\partial h}{\partial u^j} = 0$$

where

$$P^{ij} = g^{ij}(u)\frac{d}{dx} + b_k^{ij}(u)u_x^k$$

THEOREM (Dubrovin-Novikov)

• In the non-degenerate case $\det g^{ij} \neq 0$, under local changes of coordinates $u = u(w)$, g^{ij} transforms as a tensor with upper indices and $b_k^{ij} = -g^{is}\Gamma_{sk}^j$ where Γ_{sk}^j is the Christoffel symbol of a differential geometric connection.

•The metric is pseudo-Riemannian, i.e g^{ij} is symmetric and the connection is compatible with the metric (anti-symmetry of the bracket).

•The connection has zero curvature and torsion (Jacobi identity).

There exist local coordinates such that $g^{ij} = const$ and $b_k^{ij} = 0$.

In flat coordinates, the operator P^{ij} takes a constant coefficient form

$$P^{ij} = \varepsilon^i \delta^{ij} \frac{d}{dx}$$

and the Hamiltonian systems take the Hessian form

$$u^i_t + \varepsilon^i \partial_{ij} h \, u^j_x = 0$$

Notation

$$\partial_{ij} h = \frac{\partial^2 h}{\partial u^i \partial u^j}$$

Many important quasilinear systems are diagonalizable, i.e. reducible to the Riemann Invariant form

$$R_t^i = v^i(R) R_x^i \qquad \text{where} \qquad \partial_k\left(\frac{\partial_j v^i}{v^j - v^i}\right) = \partial_j\left(\frac{\partial_k v^i}{v^k - v^i}\right) \qquad (0.2)$$

semi-Hamiltonian property

THEOREM (Tsarev – Generalized hodograph method)

Let us be given a diagonal semi-Hamiltonian system of hydrodynamic type (0.2), and

an arbitrary solution of the linear system

$$\frac{\partial_j w^i}{w^j - w^i} = \frac{\partial_j v^i}{v^j - v^i}$$

A smooth solution R of the system

$$w^i(R) = v^i(R)t + x \qquad (0.3)$$

is a solution of the diagonal semi-Hamiltonian system (0.2). Conversely any solution of (0.2) can be locally represented as a solution of (0.3) in a neighborhood of a point where $R_x^i \neq 0$

Method of hydrodynamic reductions applied to 2+1D systems

Consider

$$u_t + A(u)u_x + B(u)u_y = 0 \qquad (1.1)$$

look for multi-phase solutions in the form

$$u = u\left(R^1(x,y,t),\ldots, R^n(x,y,t)\right) \qquad (1.1)$$

where the phases satisfy

$$R^i_t = v^i(R)R^i_x \qquad\qquad R^i_y = \mu^i(R)R^i_x$$

Solutions are given in terms of the implicit generalized hodograph formula

$$w^i(R) = x + v^i(R)t + \mu^i(R)\ y$$

where

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j w^i}{w^j - w^i}$$

Under these assumptions the equation assumes the form

$$\left(\nu^i I_n + \mu^i A + B\right)\partial_i u = 0$$

where $\partial_i u = \dfrac{\partial u}{\partial R^i}$

The characteristic speeds μ^i, λ^i satisfy the nonlinear dispersion relation

$$\det\left(\nu^i I_n + \mu^i A + B\right) = 0$$

DEFINITION (Ferapontov-Khusnutdinova, 2003)

A (2 + 1)-dimensional quasilinear system is said to be **integrable** if it possesses

n-component reductions of the form

$$R_t^i + \lambda^i(R) R_x^i = 0 \qquad R_y^i + \mu^i(R) R_x^i = 0$$

parametrized by n arbitrary functions of a single argument.

Example. Dispersionless KP equation

Consider $$(u_t - uu_x)_x = u_{yy}$$

introduce the auxiliary field such that $u = \varphi_x$ and has $v = \varphi_y$

$$u_y = w_x, \qquad w_y = -uu_x + u_t$$

interchange $y \leftrightarrow t$

$$u_t = w_x, \qquad w_t = -uu_x + u_y$$

Looking for solutions of the form $u = u(R^1, \ldots, R^N), \; v = v(R^1, \ldots, R^N)$

one gets

$$\partial_i w = v^i \partial_i u, \quad \mu^i = u + (v^i)^2 \qquad \partial_i = \frac{\partial}{\partial R^i}$$

Condition $\partial_i \partial_j w = \partial_j \partial_i w$ together with the commutativity of the Riemann invariants leads us to the Gibbons-Tsarev system

$$\partial_j v_i = \frac{\partial_j u}{v_j - v_i} \qquad\qquad i, j = 1, \ldots, N$$

$$\partial_i \partial_j u = \frac{2 \,\partial_i u \,\partial_j u}{\left(v_j - v_i\right)^2}$$

which is in involution. Given the solution v_i and u of Gibbons-Tsarev system, one can reconstruct v_i and w via

$$\partial_i w = v_i \partial_i u, \quad \mu_i = u + \left(v_i\right)^2$$

Diagonalizability criterion

Consider a matrix $\left(v_j^i(u)\right)_{\substack{i=1,..,n \\ j=1,...n}}$ associated with th system

$$u_t^i + v_j^i(u)\, u_x^j = 0 \qquad (*)$$

Nijenhuis tensor

$$N_{jk}^i = v_j^p \partial_{u^p} v_k^i - v_k^p \partial_{u^p} v_j^i - v_p^i \left(\partial_{u^j} v_k^p - \partial_{u^k} v_j^p\right)$$

Haantjes tensor

$$\mathrm{H}_{jk}^i = N_{pr}^i v_j^p v_k^r - N_{jr}^p v_p^i v_k^r - N_{rk}^p v_p^i v_j^r + N_{jk}^p v_r^i v_p^r$$

THEOREM (Haantjes- reformulated)

The hydrodynamic type system $(*)$ with mutually distinct characteristic speeds is diagonalizable if and only if the corresponding Haantjes tensor vanishes identically

Diagonalizability of 2+1D systems

Take the system

$$u_t + A(u)u_x + B(u)u_y = 0$$

1+1D reductions associated with A and B are required to be diagonalizable.

Perform a linear change on the independent variables

$$\tilde{t} = a_{11}t + a_{12}x + a_{13}y,\ \tilde{x} = a_{21}t + a_{22}x + a_{23}y,\ \tilde{y} = a_{31}t + a_{32}x + a_{33}y$$

one has

$$u_{\tilde{t}} + \tilde{A}(u)u_{\tilde{x}} + \tilde{B}(u)u_{\tilde{y}} = 0$$

$$\tilde{A}(u) = (a_{11}I_n + a_{12}A + a_{13}B)^{-1}(a_{21}I_n + a_{22}A + a_{23}B)$$
$$\tilde{B}(u) = (a_{11}I_n + a_{12}A + a_{13}B)^{-1}(a_{31}I_n + a_{32}A + a_{33}B)$$

We require that the 1+1D reductions associated to $\widetilde{A}$ and $\widetilde{B}$ are diagonalizable, that is

$$(cI_n + aA + bB)^{-1}\left(\widetilde{c}\, I_n + \widetilde{a}\, A + \widetilde{b}\, B\right) \qquad (1)$$

are diagonalizable. (Matrices A and B have necessarily to be diagonalizable)

DEFINITION.

A 2+1D hydrodynamic type system is said to be diagonalizable if an arbitrary matrix of the family (1) is diagonalizable

It can be shown that some parameter in (1) are redundant and it is sufficient to verify diagonalizability and semi-Hamiltonian property for an arbitrary matrix in the smaller family

$$(kI_n + A)^{-1}(lI_n + B)$$

Diagonalizability of matrices

$$(kI_n + A)^{-1}(lI_n + B)$$

for arbitrary values of the parameters, In the case of $N \geq 3$ components represents a very restrictive **necessary** condition for the integrability of 2+1-dimensional hydrodynamic type systems.

We will show that for Hamiltonian systems in 3-components this condition guarantees the integrability, in other words

Diagonalizability + Hamiltonian structure = Integrability

Hamiltonian 2+1D systems

The theory of multidimensional Poisson brackets was constructed by Dubrovin-Novikov, Mokhov. Difference with the one dimensional case is that the metrics can no longer be reduced simultaneously to a constant coefficient form.

Consider the Poisson bracket associated with the operator

$$P^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k(u)u^k_x + \tilde{g}^{ij}(u)\frac{d}{dy} + \tilde{b}^{ij}_k(u)u^k_y$$

We require that both metrics must necessarily be flat, but in general they can no longer be reduced to a constant coefficient form at the same time because of obstruction tensors [Mokhov]. Obstruction tensors vanish only iff one of the metrics is positive definite or the pair of metrics is nonsingular (pair-wise distinct eigenvalues)

Two component case

The Poisson operator P can be reduced to a constant coefficient form

$$P = \begin{pmatrix} d/dx & 0 \\ 0 & d/dy \end{pmatrix}$$

the corresponding Hamiltonian system take the form

$$u_t^1 + (\partial_1 h)_x = 0$$
$$u_t^2 + (\partial_2 h)_y = 0$$

Notation

$$\partial_i = \frac{\partial}{\partial u^i}$$

Three component case

Consider Hamiltonian operators of the following form

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \frac{d}{dy}$$

and the corresponding systems of the form

$$u_t^i + \left(\partial_i h\right)_x + \lambda^i \left(\partial_i h\right)_y = 0$$

Hamiltonian systems in Godunov form

DEFINITION

A system is reducible to the Godunov form if it possesses the conservative representation

$$\left(\partial_i p\right)_t + \left(\partial_i q\right)_x + \left(\partial_i r\right)_y = 0$$

Applying a Legendre transformation to the system $u_t^i + \left(\partial_i h\right)_x + \lambda^i \left(\partial_i h\right)_y = 0$

$$u_i = \partial_i h, \qquad \partial_i H = u^i, \qquad H = L(h) = u^i \partial_i h - h$$

one gets

$$\left(\partial_i H\right)_t + \left(\partial_i u\right)_x + \lambda^i \left(\partial_i u\right)_y = 0$$

Relabeling the independent variables

$$\left(\partial_i u\right)_t + \lambda^i \left(\partial_i u\right)_x + \left(\partial_i H\right)_y = 0$$

This system can be viewed as describing *n* linear waves nonlinearly coupled along the *y* direction

Integrable Hamiltonian systems: two-components

$$v_t + (\partial_v H)_x = 0, \qquad w_y + (\partial_w H)_x = 0 \tag{1.2}$$

$v = u_1, w = u_2$

THEOREM.
Integrability conditions for the system (0.2) are [Ferapontov-Khusnutdinova, Comm.Math.Phys.2004]

$$\begin{aligned} H_{vw}H_{vvvv} &= 2H_{vvv}H_{vvw} \\ H_{vw}H_{vvvw} &= 2H_{vvv}H_{vww} \\ H_{vw}H_{vvww} &= H_{vvw}H_{vww} + H_{vvv}H_{www} \\ H_{vw}H_{vwww} &= 2H_{vvw}H_{www} \\ H_{vw}H_{wwww} &= 2H_{vww}H_{www} \end{aligned} \tag{1.3}$$

Solution space is 10-dimensional

$$v \to av + b, w \to cw + d, H \to \alpha H + \beta v^2 + \gamma w^2 + \mu v + \tau w + \delta$$

THEOREM. The ‘generic’ solution of the system (2) is given by the formula

$$H(v,w)=Z(v+w)+\varepsilon Z(v+\varepsilon w)+\varepsilon^2 Z\left(v+\varepsilon^2 w\right) \qquad (1.4a)$$

where $\varepsilon=e^{2\pi i/3}, \varepsilon^3=1,$ $Z''(s)=\varsigma(s)$ and ς is the Weierstrass zeta function, i.e. $\varsigma'=-\wp, (\wp')^2=4\wp^3-g_3$. Degenerations of this solutions are

$$H(v,w)=\frac{1}{2}v^2\varsigma(w), \qquad (1.4b)$$

$$H(v,w)=(v+w)\log(v+w), \qquad (1.4c)$$

$$H(v,w)=v^2w^2, \qquad (1.4d)$$

$$H(v,w)=vw^2+\frac{\alpha}{5}w^5, \quad \alpha=\text{const} \qquad (1.4e)$$

$$H(v,w)=vw+\frac{1}{6}w^3 \qquad (1.4f)$$

The elliptic example possesses the following specialization for $g_3 = 0$

$$H(v,w) = (v+w)\log(v+w) + \varepsilon(v+w)\log(v+\varepsilon w) + \varepsilon^2(v+w)\log(v+\varepsilon^2 w) \quad (1.4g)$$

and

$$H(v,w) = \frac{v^2}{2w} \quad (1.4h)$$

Dispersionless Lax pairs

Potential $H(v,w)=vw+\frac{1}{6}w^3$

The correponding system arises in the genus zero case of the Whitham hierarchy.

$$v_t+w_x=0,\quad w_y+ww_x+v_x=0$$

The dispersionless Lax pair is

$$\psi_t=\frac{1}{2}\log\left(\psi_x+\frac{w}{2}\right),\ \ \psi_Y=\psi_X^2\,|+\frac{v}{2}$$

After the Legendre transformation

$$u_1=H_v,\ \ u_2=H_w,\ \ h(u_1,u_2)=vH_v+wH_w-h$$

the potential is still polynomial

$$h(v,w)=u_1u_2-\frac{1}{6}(u_2)^3$$

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Potential $H(v,w)=vw^2+\frac{\alpha}{5}w^3$

The corresponding system is a particular case of the generalized dispersioless Harry-Dym equation

$$v_t+\left(w^2\right)_x=0,\quad w_y+2(vw)_x+\alpha\left(w^4\right)_x=0$$

If $\alpha=0$ the Lax pair is

$$\psi_t=-\frac{w^2}{2\psi_x^2},\quad \psi_y=\psi_x^4-2v\psi_x$$

If $\alpha\neq 0$ the Lax pair modifies as follows

$$\psi_t=f\left(\frac{w}{\psi_x}\right),\quad \psi_y=\psi_x^4-2v\psi_x$$

$$f(s)=\frac{1}{3}\left(\log(s-1)+\varepsilon^2\log(s-\varepsilon)+\varepsilon\log\left(s-\varepsilon^2\right)\right)\qquad \varepsilon^3=1$$

Potential $H(v,w)=v^2w^2$

The correponding system is

$$v_t+2\left(vw^2\right)_x=0,\quad w_y+2\left(v^2w\right)_x=0$$

The Lax pair is

$$\psi_t=w^2\wp(\psi_x),\quad \psi_y=-v^2\wp(\psi_x+c)$$

where c is the zero of the Weierstrass $\wp$ function $(\wp')^2=4\left(\wp^3+1\right)$

Potential $H(v,w)=(v+w)\log(v+w)$

The system also appear in the genus zero universal Whitham hyerarchy

$$v_t+\frac{v_x+w_x}{v+w}=0,\quad w_y+\frac{v_x+w_x}{v+w}=0$$

And the Lax pair is

$$\psi_t=-\log(w+\psi_x),\quad \psi_y=\log(v-\psi_x)$$

Potential $H(v,w)=\frac{1}{2}v^2\varsigma(w),$

The system is

$$v_t+\varsigma(w)v_x-v\wp(w)w_x=0,\qquad w_y-\wp(w)vv_x-\frac{1}{2}v^2\wp'(w)w_y=0$$

The Lax pair is

$$\psi_t=\frac{1}{3}\log\sigma(\psi_x+w)+\frac{\varepsilon}{3}\log\sigma(\psi_x+\varepsilon w)+\frac{\varepsilon^2}{3}\log\sigma(\psi_x+\varepsilon^2 w)$$

$$\psi_y=-\frac{1}{2}v^2\wp(\psi_x)$$

where σ is the Weierstrass sigma function $\frac{\sigma'}{\sigma}=\varsigma$

Potential $$H(v,w)=Z(v+w)+\varepsilon Z(v+\varepsilon w)+\varepsilon^2 Z(v+\varepsilon^2 w)$$

The equations associated with $H/3$ are

$$v_t+\left(\varsigma(w)+\frac{2\wp^2(w)}{\wp'(v)+\wp'(w)}\right)v_x+\frac{2\wp(v)\wp(w)}{\wp'(v)+\wp'(w)}w_x=0$$

$$w_y+\frac{2\wp(v)\wp(w)}{\wp'(v)+\wp'(w)}v_x+\left(\varsigma(w)+\frac{2\wp^2(v)}{\wp'(v)+\wp'(w)}\right)w_x=0$$

Lax pair

$$\psi_t=-\frac{1}{3}\log\sigma(\psi_x+w)-\frac{\varepsilon}{3}\log\sigma(\psi_x+\varepsilon w)-\frac{\varepsilon^2}{3}\log\sigma(\psi_x+\varepsilon^2 w)$$

$$\psi_y=\frac{1}{3}\log\sigma(\psi_x-v)+\frac{\varepsilon}{3}\log\sigma(\psi_x-\varepsilon v)+\frac{\varepsilon^2}{3}\log\sigma(\psi_x-\varepsilon^2 v)$$

Potential

$$H(v,w)=(v+w)\log(v+w)+\varepsilon(v+w)\log(v+\varepsilon w)+\varepsilon^2(v+w)\log(v+\varepsilon^2 w)$$

the system associated with $H/3$ is

$$v_t+\frac{w^2}{v^3+w^3}v_x-\frac{vw}{v^3+w^3}w_x=0,\quad w_y-\frac{vw}{v^3+w^3}v_x+\frac{v^2}{v^3+w^3}w_x=0,$$

Lax pair

$$\psi_t=\frac{1}{3}\left[\log(s-1)+\varepsilon^2\log(s-\varepsilon)+\varepsilon\log(s-\varepsilon^2)\right]$$

$$\psi_y=-\frac{1}{3}\left[\log(r-1)+\varepsilon^2\log(r+\varepsilon)+\varepsilon\log(r+\varepsilon^2)\right]$$

where

$$s=\frac{w}{\psi_x},\qquad r=\frac{v}{\psi_x}$$

Integrable Hamiltonian systems: three-components

Classification if three-component integrable systems of the form

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_t + \begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_x + \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_y = 0 \qquad (2.1)$$

Hessian

where eigenvalues λ^i are assumed to be <u>pairwise distinct.</u>

Necessary condition for the system (2.1) to be integrable is that it is diagonalizable, i.e. the Haantjes tensor of the family of matrices

$$(kA + I_3)^{-1}(lB + I_3)$$

where $A = diag(\lambda^i)$ and $B = (H_{ij})$, vanishes identically for any k and l.

Assume the dispersion relation $\det(\nu I_3 + \mu A + B) = 0$ to be **irreducible.**

Introduce the notation

$$\Delta := (\lambda_2 - \lambda_3) H_{11} + (\lambda_3 - \lambda_1) H_{22} + (\lambda_1 - \lambda_2) H_{33}$$

$$R_1 := \frac{H_{12} H_{13}}{H_{23}} \left(\lambda^2 - \lambda^3\right), \quad R_2 := \frac{H_{23} H_{12}}{H_{13}} \left(\lambda^3 - \lambda^1\right), \quad R_3 := \frac{H_{13} H_{23}}{H_{12}} \left(\lambda^1 - \lambda^2\right)$$

$$J := R_1 + R_2 + R_3 - \Delta$$

$$I := \Delta^2 - 4\left(\lambda^2 - \lambda^3\right)\left(\lambda^3 - \lambda^1\right) H_{12}^2 - 4\left(\lambda^3 - \lambda^1\right)\left(\lambda^1 - \lambda^2\right) H_{23}^2 - 4\left(\lambda^1 - \lambda^2\right)\left(\lambda^2 - \lambda^3\right) H_{13}^2$$

THEOREM

The system (2.1) with an irreducible dispersion relation is diagonalizable if and only if the potential

H satisfies the relations

$$
\begin{gathered}
J=0, \qquad H_{123}=0 \\
\partial_1\left(\left(\lambda^3-\lambda^2\right)H_{11}+R_2+R_3\right)=0 \\
\partial_2\left(\left(\lambda^1-\lambda^3\right)H_{22}+R_1+R_3\right)=0 \\
\partial_3\left(\left(\lambda^2-\lambda^1\right)H_{33}+R_1+R_2\right)=0
\end{gathered}
\tag{2.2}
$$

The condition $J=0$ has a very simple geometric interpretation. Given a vector $\mathbf{g}=(g_1,g_2,g_3)$ let us consider the left characteristic cone associated with the system (2.1)

$$\mathbf{g}\cdot\left(\nu I_3+\mu A+B\right)=0$$

Eliminating ν and μ one obtains

$$\begin{aligned}&\left[H_{13}g_1^2g_2+H_{23}g_2^2g_1+H_{33}g_1g_2g_3\right]\left(\lambda^1-\lambda^2\right)+\\&\left[H_{21}g_2^2g_3+H_{13}g_3^2g_2+H_{11}g_1g_2g_3\right]\left(\lambda^2-\lambda^3\right)+\\&\left[H_{23}g_3^2g_1+H_{12}g_1^2g_3+H_{22}g_1g_2g_3\right]\left(\lambda^3-\lambda^1\right)=0\end{aligned}$$

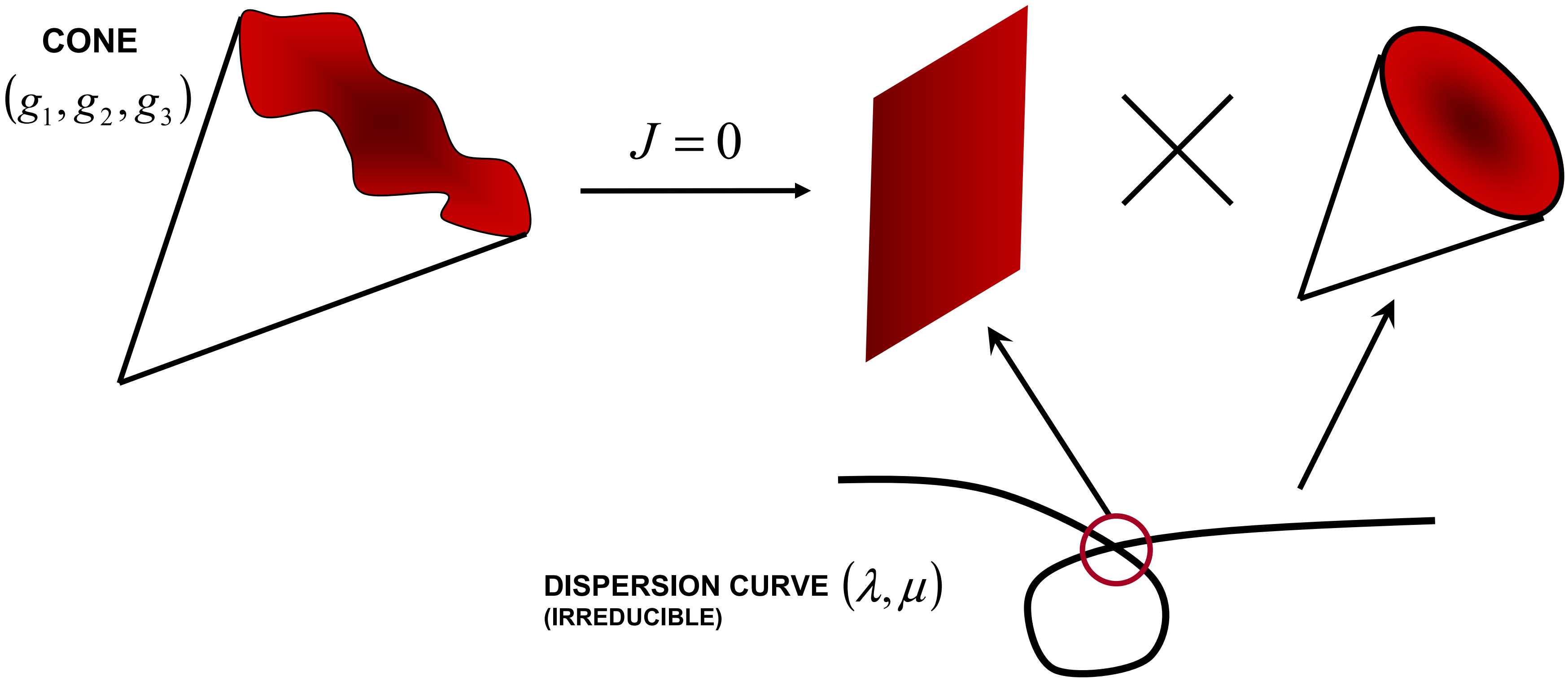
i.e. the components of $\mathbf{g}$ parametrize the left characteristic cone

Note

ν **and** μ **are rational expressions of** g_1, g_2 **and** g_3 **.**

The condition $J=0$ is equivalent to the degeneration into a line and a conic

$$\left[H_{12}H_{13}g_1+H_{12}H_{23}g_2+H_{13}H_{23}g_3\right]$$
$$\left[H_{13}H_{23}\left(\lambda^1-\lambda^2\right)g_1g_2+H_{12}H_{23}\left(\lambda^3-\lambda^1\right)g_1g_3+H_{12}H_{13}\left(\lambda^2-\lambda^3\right)g_2g_3\right]=0$$



THEOREM

The generic solution to the equations (2.1) is given by the formula

$$H = -\sum_{i \neq j} \frac{\lambda^i - \lambda^j}{6a_i^2 a_j^2} V(u_i, u_j)$$

where

$$V(u_i, u_j) = Z(a_i u_i - a_j u_j) + \varepsilon Z(a_i u_i - \varepsilon a_j u_j) + \varepsilon^2 Z(a_i u_i - \varepsilon^2 a_j u_j)$$

$\varepsilon^3 = 1$ and $Z'' = \varsigma$ where ς is the Weierstrass zeta-function: $\varsigma' = -\wp$

Degenerations of this solution correspond to

$$H=-\sum_{i\neq j}\frac{\lambda^i-\lambda^j}{3a_i^2a_j^2}\tilde{V}(u_i,u_j)$$

where

$$\tilde{V}(u_i,u_j)=(a_iu_i-a_ju_j)\log(a_iu_i-a_ju_j)+\varepsilon(a_iu_i-a_ju_j)\log(a_iu_i-\varepsilon a_ju_j)$$
$$+\varepsilon^2(a_iu_i-a_ju_j)\log(a_iu_i-\varepsilon^2a_ju_j)$$

and

$$H=-\sum_{i\neq j}\frac{\lambda^i-\lambda^j}{a_i^2a_j^2}(a_iu_i-a_ju_j)\log(a_iu_i-a_ju_j)$$

Further examples

$$H=\frac{\lambda^1-\lambda^2}{a_2^2}u_1^2\varsigma(a_2u_2)+\frac{\lambda^1-\lambda^3}{a_3^2}u_1^2\varsigma(a_3u_3)-\frac{2}{3}\frac{\lambda^2-\lambda^3}{a_2^2a_3^2}V(u_2,u_3)$$

This potential possesses the following degeneration

$$H=(\lambda^1-\lambda^2)u_1^2u_2^2+(\lambda^2-\lambda^3)\varsigma(u_3+c)u_2^2-(\lambda^3-\lambda^1)\varsigma(u_3)u_1^2$$

where $\varsigma'=-\wp$ and c is such that $\wp(c)=0$ and $\wp'(c)=2$

Quadratic degeneration

$$H=(\lambda^1-\lambda^2)u_1^2u_2^2+(\lambda^2-\lambda^3)u_2^2u_3^2+(\lambda^3-\lambda^1)u_1^2u_3^2$$

Non-symmetric examples include

$$H=(pu_1+qu_3)\log(pu_1+qu_3)-\frac{1}{6}p\left(\lambda^1-\lambda^2\right)\left(\lambda^1-\lambda^3\right)u_1^3-\frac{1}{6}q\left(\lambda^3-\lambda^1\right)\left(\lambda^3-\lambda^2\right)u_3^3$$
$$+p\left(\lambda^3-\lambda^2\right)u_2u_3+q\left(\lambda^2-\lambda^1\right)u_1u_2$$

and

$$H=\alpha\, u_1^2u_2+\beta\, u_2u_3^2+\gamma\, u_1^5+\delta\, u_3^5+u_3G\left(\frac{u_1}{u_3}\right)$$

where

$$G(x)=(px+q)\log(px+q)+\varepsilon(px+q)\log(px+\varepsilon q)+\varepsilon^2(px+q)\log\left(px+\varepsilon^2q\right)$$

$$\alpha=\lambda^2-\lambda^1,\ \beta=\lambda^2-\lambda^3,\ \gamma=\frac{p}{15q^2}\left(\lambda^2-\lambda^1\right)\left(\lambda^1-\lambda^3\right),\ \delta=\frac{q}{15p^2}\left(\lambda^2-\lambda^3\right)\left(\lambda^3-\lambda^1\right)$$

THEOREM

All the Hamiltonian systems listed above possesses a Lax pair of the form

$$\psi_t = f(\mathbf{u}, \psi_x)$$
$$\psi_y = g(\mathbf{u}, \psi_x)$$

THEOREM

It is also possible to prove that these systems admit infinitely many n-component reductions parametrized by n-functions of one variable (**Integrability**).

THEOREM

Any integrable 3+1 dimensional Hamiltonian system in the two and three component case is either linear or reducible.

Conjecture

There exists no non-trivial integrable Hamiltonian systems of hydrodynamic type in 3+1 dimensions corresponding to a local Poisson bracket of hydrodynamic type and a local Hamiltonian density.

Conclusions….

- For Hamiltonian 2+1D systems diagonalizability is necessary and sufficient for the integrability
- Broad class of non-trivial Hamiltonians.

…. and open problems

- Structure of the corresponding Hamiltonian hierarchies
- "Quantization"
- Classification of the general system

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_t + \begin{pmatrix} \lambda^1(u) & 0 & 0 \\ 0 & \lambda^2(u) & 0 \\ 0 & 0 & \lambda^3(u) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_x + \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_y = 0$$