Padé approximations of the Painlevé transcendents

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Outline

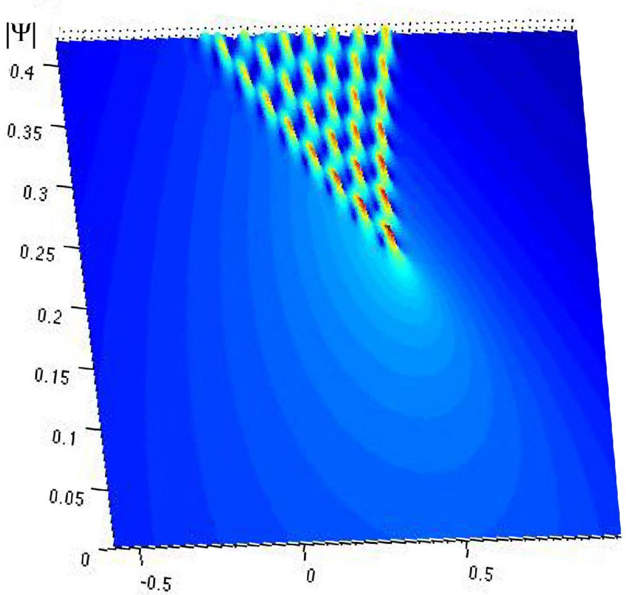
- Motivation: focusing NLS equation and PI
- Poles of PI *tritronqueé* solution
- Fast algorithm for Padé
- *Tronqueé* vs. *tritronqueé*
- Exponentially vanishing terms
- Dubrovin’s conjecture
- Padé approximations for PII

Focusing NLS equation

A model for incipient modulation instability

$$\varepsilon\Psi_t + \frac{\varepsilon^2}{2}\Psi_{xx} + |\Psi|^2\Psi = 0, \qquad \varepsilon << 1$$

with initial profile $\Psi(x,0) = A(x)\exp\{iS(x)/\varepsilon\}$. Take $u = |\Psi|^2,\ v = \frac{\varepsilon}{2i}\left(\frac{\Psi_x}{\Psi} - \frac{\bar{\Psi}_x}{\bar{\Psi}}\right)$,



$$A(x) = A_0 \operatorname{sech} x, \quad S'(x) = \mu \tanh x$$

$$\begin{cases} u_t + (uv)_x = 0, \\ v_t + vv_x - u_x = \frac{\varepsilon^2}{4}\left(\frac{u_{xx}}{u} - \frac{u_x^2}{2u^2}\right)_x \end{cases}$$

Elliptic umbilic catastrophe [1] :

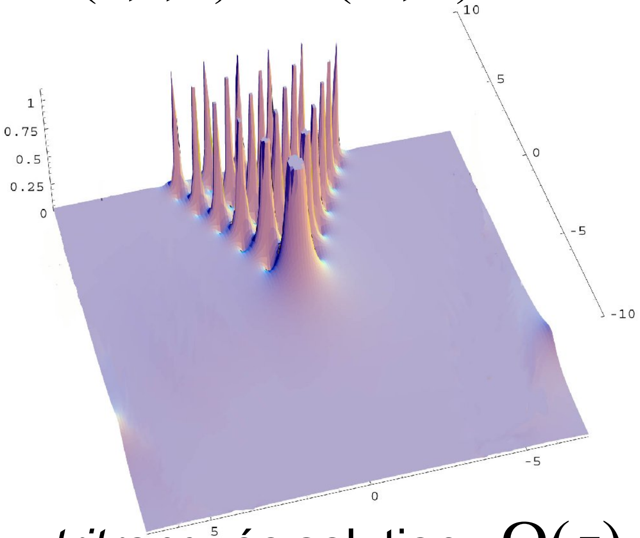
$$H = \frac{1}{2}\left(uv^2 - u^2\right) + \frac{\varepsilon^2}{8u}u_x^2,$$

$$u_t + (H_v)_x = 0, \quad v_t + (H_u)_x = 0.$$

[1] B.Dubrovin, T.Grava and C.Klein, arXiv:0704.0501v2, (2007).

Universality of critical behavior: PI special solution

$\Psi(x,t,\varepsilon) \approx \Phi(X,T) + \varepsilon^{2/5}\Omega(z), \quad z = z(X,T)/\varepsilon^{4/5},$ where X,T – slow space-time,

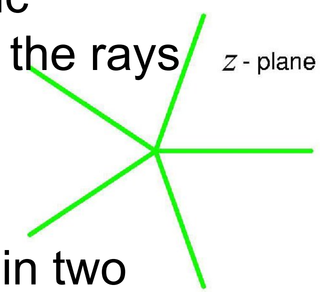


tritronquée solution $\Omega(z)$

$$\boxed{\Omega_{zz} = 6\Omega^2 - z}$$

- Painlevé I equation

- All solutions are meromorphic
- The poles accumulate along the rays



$$\arg z = \frac{2\pi i}{5} n, \quad n = 0, \pm 1, \pm 2.$$

- *Tronquée* solution: no poles in two consecutive sectors (say, along $z \to +\infty$)
- *Trironquée* solution: no poles along <u>three</u> consecutive rays [2]
- Asymptotics at infinity:

$$\Omega(z) = -\sqrt{\frac{z}{6}}\left\{1 + O(z^{-2})\right\}, \quad z \to +\infty.$$

Pierre Boutroux 1880 - 1922

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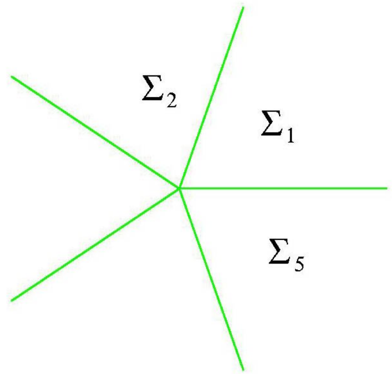
[2] 'P.Boutroux,. Ann.Ecole Norm., 30 (1913) 265 - 375.

PI and isomonodromic deformations

$$\begin{cases} \Psi_\lambda = \begin{pmatrix} \Omega_z & 2\lambda^2 + 2\Omega\lambda - z + 2\Omega^2 \\ 2(\lambda - \Omega) & -\Omega_z \end{pmatrix}\Psi, \\ \Psi_z = -\begin{pmatrix} 0 & \lambda + 2\Omega \\ 1 & 0 \end{pmatrix}\Psi. \end{cases}$$

The Lax pair



$$\Psi_k(\lambda, z) \approx \frac{1}{\sqrt{2}}\begin{pmatrix} \lambda^{1/4} & \lambda^{1/4} \\ \lambda^{-1/4} & -\lambda^{-1/4} \end{pmatrix} e^{\theta(\lambda,z)\sigma_3}, \quad |\lambda| \to \infty, \quad \lambda \in \Sigma_k$$

normalization

$$\theta(\lambda, z) = \frac{4}{5}\lambda^{5/2} - z\,\lambda^{1/2}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\Psi_{k+1}(\lambda, z) = \Psi_k(\lambda, z)\, S_k, \quad \lambda \in \Sigma_k \cap \Sigma_{k+1}. \quad S_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}, \quad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}$$

$$\Omega(z) = \frac{\partial}{\partial z}\left[\lim_{\lambda\to\infty} \lambda^{1/2}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\lambda^{-\frac{1}{4}\sigma_3}\Psi(\lambda, \zeta)\, e^{-\theta(\lambda,z)\sigma_3} - 1\right)\right]_{11}$$

Inversion formula

Tritronquée solution: $s_1 = -s_2 = -s_3 = s_4 = i, \quad s_5 = 0$

Padé approximations for PI solutions

Q: How to find poles of PI function?
A: For large $|z|$ by Isomonodromy Method, else – numerically.

Main diagonal Padé approximation:

$$\Omega(z)=\cfrac{\alpha_0}{1+\cfrac{\alpha_1 z}{1+\cfrac{\alpha_2 z}{1+\ldots}}}$$

The idea of algorithm: an invariance of Painlevé equations under Möbius transformation.

$$\left(A+Bu\right)u''_{zz}+\left(C+Du\right)u'_z-2B\left(u'_z\right)^2+E+Fu+Gu^2+Hu^3=0,$$

where $A\ldots H$ are polynomials of z vanishing at zero, and $u\mapsto\dfrac{\alpha u+\beta}{\gamma u+\delta}$

retains the form of equation.

The Fair-Luke algorithm [3]

Let $u_n = \dfrac{\alpha_n}{1 + z u_{n+1}}$, then

$$\left(A_n + B_n u_n\right) u_n'' + \left(C_n + D_n u_n\right) u_n' - 2B_n \left(u_n'\right)^2 + E_n + F_n u_n + G_n u_n{}^2 + H_n u_n{}^3 = 0,$$

and the recurrence holds

$$A_{n+1} = -A_n - \alpha_n B_n, \quad B_{n+1} = -z A_n, \quad C_{n+1} = -2\frac{A_n + \alpha_n B_n}{z} - C_n - \alpha_n D_n,$$

$$D_{n+1} = 2A_n - zC_n, \quad F_{n+1} = -\frac{C_n + \alpha_n D_n}{z} + 3\frac{E_n}{\alpha_n} + 2F_n + \alpha_n G_n,$$

$$E_{n+1} = z^{-1}\left(\alpha_n^{-1} E_n + F_n + \alpha_n G_n + \alpha_n^2 H_n\right), \quad H_{n+1} = \alpha_n^{-1} z^2 A_n,$$

$$G_{n+1} = 2z^{-1} A_n - C_n + z\left(3\alpha_n^{-1} E_n + F_n\right),$$

and $\alpha_{n+1} = -\dfrac{E_{n+1}(0)}{F_{n+1}(0)}$.

If $u_{n+1} \equiv 0$, then

$$u(z) \approx u_0(z) = \cfrac{\alpha_0}{1 + \cfrac{\alpha_1 z}{1 + \cfrac{\alpha_2 z}{1 + \ldots}}} = \frac{P_n(z)}{Q_n(z)}$$

[3] W.Fair, Y.Luke, Math.Comp. 20 (1968), p.602-605

Padé approximation for PI

$$\Omega_{zz} = 6\Omega^2 - z,$$
$$\Omega(0) = \Omega_0,$$
$$\Omega'(0) = \Omega_1$$

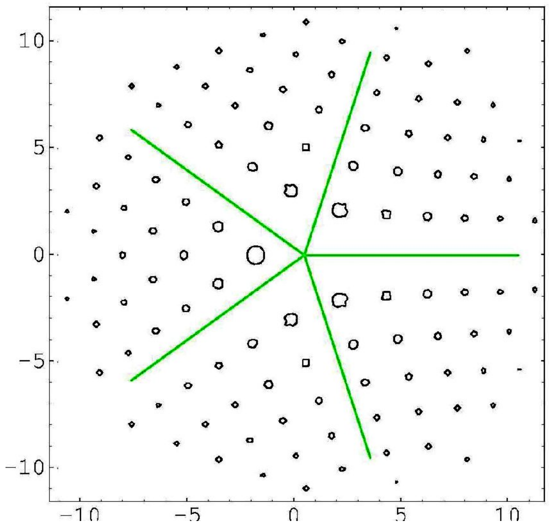
Put $\Omega(z) = \Omega_0 + \Omega_1 z + z^2 u(z)$, then

$$z^2 u'' + 4zu' + 2\left(1 - 6\Omega_0 z^2 - 6\Omega_1 z^3\right)u - 6z^4 u^2 + z - 6\left(\Omega_0 + \Omega_1 z\right)^2 = 0$$

$$u(z) = \cfrac{\alpha_0}{1 + \cfrac{\alpha_1 z}{\cfrac{1+\ldots}{1}}} = \frac{P_n(z)}{Q_n(z)}$$

n-th order Pade fraction

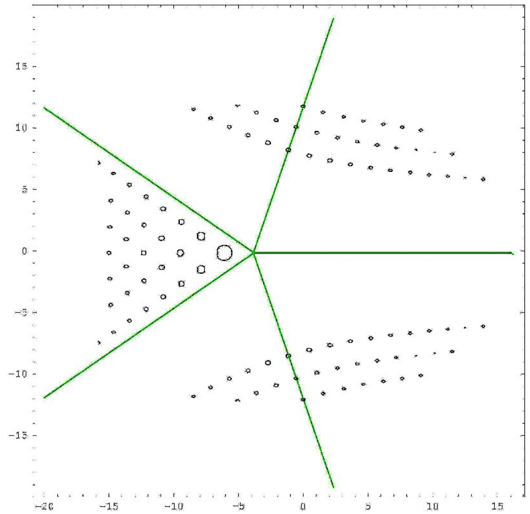
generic



$\Omega_0 = -0.1875$

$\Omega_1 = 0.3049$

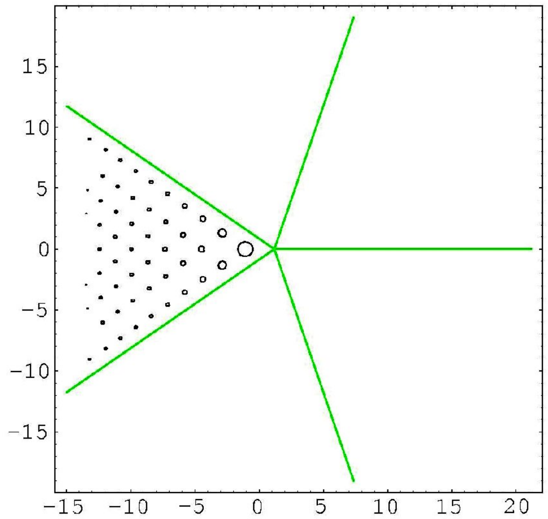
tronquée



$\Omega_0 = -0.1875543083411885$

$\Omega_1 = 0.3049055602602021$

tritronquée

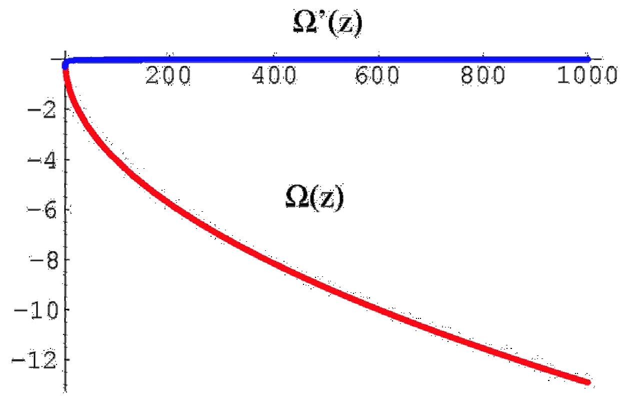


$\Omega_0 = -0.187554308341188526517222532488 35\ldots$

$\Omega_1 = 0.304905560260202120243739945607 56\ldots$

Exponentially vanishing terms

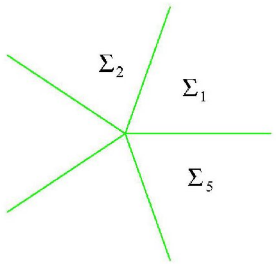
Why the precision of Ω_0, Ω_1 is crucial?



$$\Omega(z) = -\sqrt{\frac{z}{6}} - \frac{1}{48z^2} + \dots$$

$$\boxed{+\frac{a}{z^{5/8}} \exp\left\{-\frac{8}{5}\sqrt[4]{\frac{3}{2}} z^{5/4}\right\}\left(1 + O(z^{-4/5})\right),} \quad |z| \to \infty$$

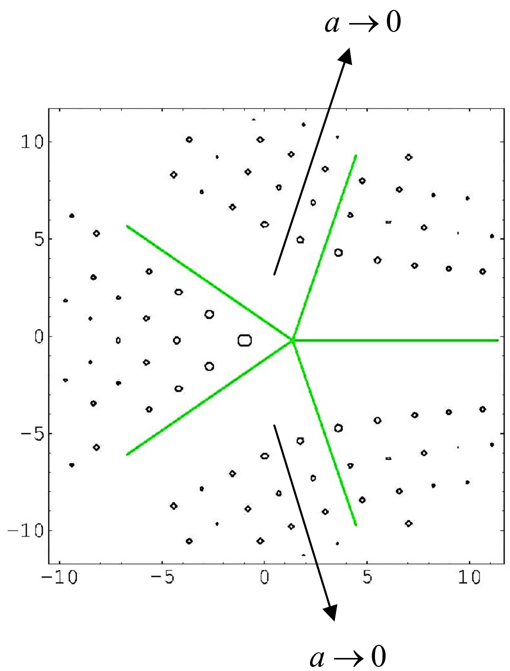
$$a = \frac{3^{-1/8}}{2^{7/8}\sqrt{\pi}}\left(\arg z - \pi\right)(s_1 - s_4), \qquad z \in \Sigma_1 \cup \Sigma_5$$



$a \neq 0$ - tronquée, $a = 0$ - tritronquée

Exponential terms are hard to control numerically.

Initial conditions Ω_0, Ω_1 are almost the same while a is changing.



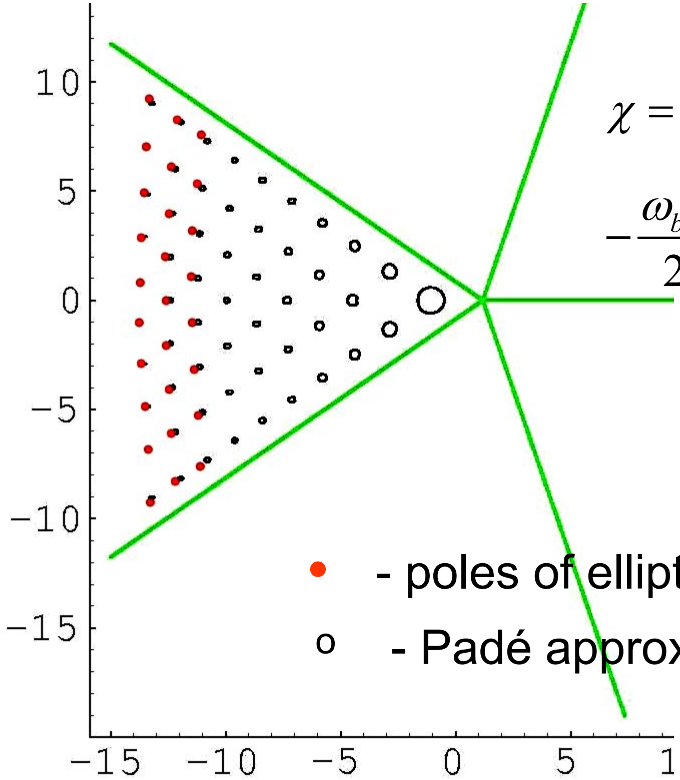
But not the distribution of the poles!

Elliptic function asymptotics at infinity

$$\Omega(z) \approx \wp(e^{i\varphi} z^{5/4} + \chi; g_2(\varphi), g_3(\varphi)), \quad |z| \to \infty$$

where $\wp$ is Weierstrass function with modules

$$g_2(\varphi) = -2e^{i\varphi}, \quad g_3(\varphi) = -A(\varphi) \equiv 4\lambda_1\lambda_2\lambda_3 \,,$$



$$\chi = -\frac{\omega_a(\varphi) - \omega_b(\varphi)}{2\pi i} \ln(is_{2+2k}) - \frac{\omega_b(\varphi)}{2\pi i} \ln(is_{5+2k}) + O(z^{-1}),$$

$$\frac{4\pi}{5} < \arg z < \frac{6\pi}{5}$$

[4]

The Boutroux problem

Find deformation of the elliptic curve

$$w^2(\lambda) = \lambda^3 + \frac{1}{2}\lambda e^{i\varphi} + \frac{1}{4}A(\varphi) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3),$$

such that

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \frac{1}{2}e^{i\varphi}, \\ \mathrm{Re}\int_{a,b} w(\lambda)d\lambda = 0 \end{cases}$$

$$\lambda_{1,3} \to -\frac{1}{\sqrt{6}} \quad \text{as} \quad \varphi \to \frac{4\pi}{5},$$

$$\lambda_{2,3} \to \frac{1}{\sqrt{6}} e^{-i\pi/5} \quad \text{as} \quad \varphi \to \frac{6\pi}{5}.$$

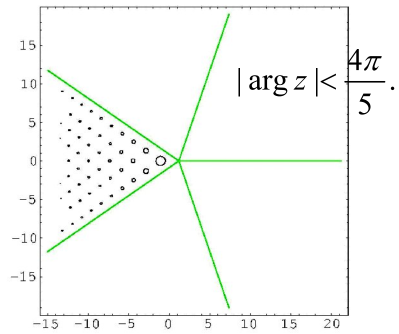
[4] A.A.Kapaev, J. Phys.A:Math.Gen.37 (2004)11149–11167

Dubrovin's conjecture

$$\Omega_{zz} = 6\Omega^2 - z,$$

$$\Omega(z) = -\sqrt{\frac{z}{6}} - \frac{1}{48z^2} + \ldots, \quad |z| \to \infty$$

$\Omega(z)$ is *tritronquée* solution



There are no poles of at infinity as $|\arg z| < \dfrac{4\pi}{5}$.

Proof: Kapaev's theorem, [4]

Conjecture. [1] $\Omega(z)$ *has no poles in the sector* $|\arg z| < \dfrac{4\pi}{5}$.

A way to prove: Let z_0 a pole, put $\Omega(z) = \dfrac{1}{\varepsilon^2} + \dfrac{z_0}{10}\varepsilon^2 + \ldots,\ \varepsilon = z - z_0 \to 0$

Reduce the Lax pair equation to a scalar ODE as $\varepsilon \to 0$

$$\Psi_\lambda = \begin{pmatrix} -2\varepsilon^{-3} & 2\lambda^2 + 2\varepsilon^{-2}\lambda - \zeta_0 + 2\varepsilon^{-4} \\ 2(\lambda - \varepsilon^{-2}) & 2\varepsilon^{-3} \end{pmatrix}\Psi$$

$$Y_{\lambda\lambda} + \left(V(\lambda) + z_0\right)Y = 0,$$

$$Y \to s_1\lambda^{1/4}\exp i\left\{\frac{4}{5}\lambda^{5/2} + z_0\lambda^{1/2}\right\},\ \lambda \to +\infty, \quad Y \to s_4|\lambda|^{-1/4}\exp i\left\{-\frac{4}{5}|\lambda|^{5/2} - z_0|\lambda|^{1/2}\right\},\ \lambda \to -\infty.$$

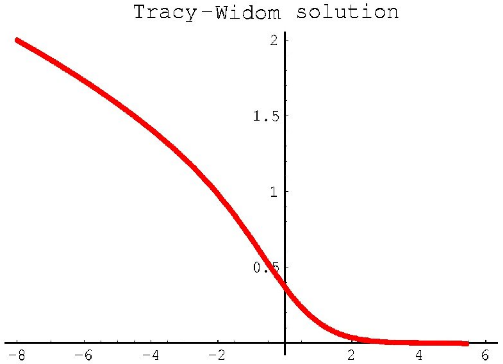
Prove that spectrum z_0 does not contain $|\arg z_0| < \dfrac{4\pi}{5}$.

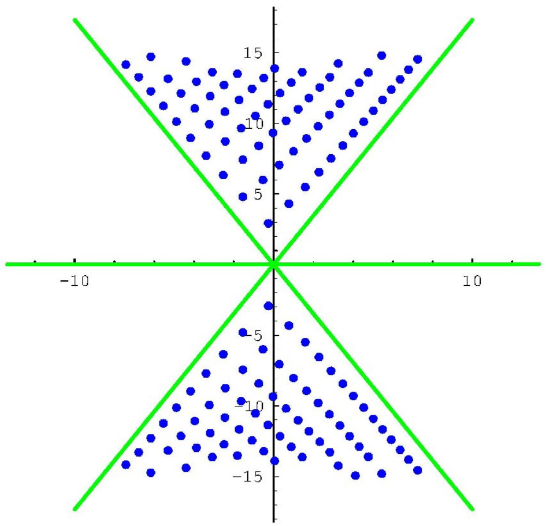
Painlevé II equation. Tracy-Widom solution [5]

$$u_{zz} = zu + 2u^3$$

$$u(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right), \qquad z \to +\infty,$$

$$u(z) \sim \sqrt{-\frac{z}{2}}\left(1 + \frac{1}{8z^3} - \frac{73}{128z^6} + \ldots\right), \quad z \to -\infty.$$





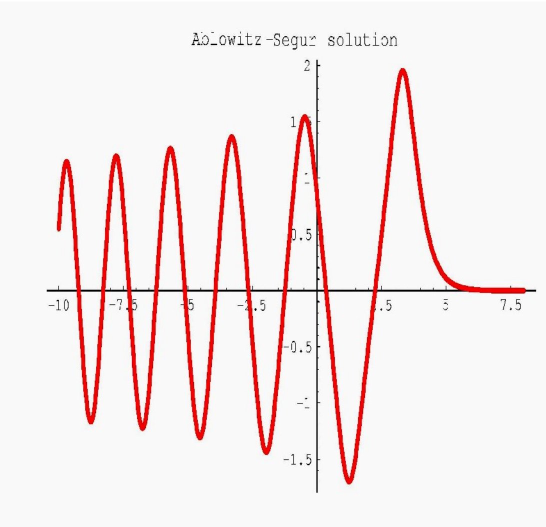
[5] C.Tracy and H.Widom, Comm.Math.Phys. 177 (1996) 727–754..

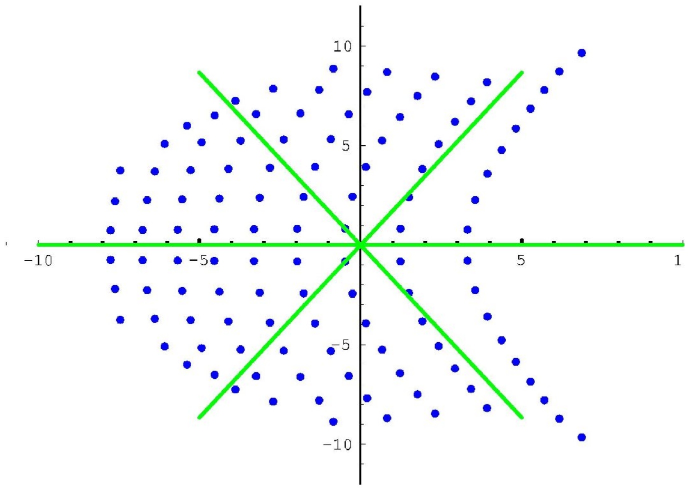
Painlevé II equation. Ablowitz-Segur solution [6]

$$u_{zz} = zu - 2u^3$$

$$u(z) \sim \frac{a}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right), \qquad z \to +\infty,$$

$$u(z) \sim b(-z)^{-1/4} \sin\left(\frac{2}{3}(-z)^{3/2} - \frac{3}{4} b^2 \log(-z) + \theta\right), \qquad z \to -\infty.$$





[6] M.J.Ablowitz and H.Segur, Stud.Appl.Math.57 (1977) 13–44.

Conclusion

- Padé approximations has been found for PI and PII solutions
- The Boutroux *lines of poles* are visualized
- Distribution of poles at infinity has a good correlation with that of $|z| \cong 10$
- Dubrovin's conjecture justified
- Applications to focusing NLS equation