Searching (and finding) Lagrangians

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Lagrange vindicated

In the Avertissement to his "Méchanique Analitique" (1788) Lagrange wrote:

> *Les méthodes que j'y expose ne demandent ni constructions, ni raisonnemens géométriques ou méchaniques, mail seulement des opérations algébriques, assujetties à une marche réguliere & uniforme. Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche, & me sauront gré d'en avoir étendu ainsi le domaine.*

(tr. by J.R. Maddox: The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field.)

It is a joke, isn't it??!!

Jacobi last multiplier

(1842) Sur un noveau principe de la mécanique analytique, *C. R. Acad. Sci. Paris* **15** 202–205

(1844) Sul principio dell'ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica, *Giornale Arcadico di Scienze, Lettere ed Arti* **99** 129-146

(1844) Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi: Pars I, *J Reine Angew Math* **27** 199-268

(1845) Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi: Pars II, *J Reine Angew Math* **29** 213-279 and 333-376

(1866) *Vorlesungen über Dynamik*, Druck und Verlag von Georg Reimer, Berlin

$$Af = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} = 0 \qquad (\star)$$

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \quad (\star\star)$$

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = MAf$$

$\omega_i, (i = 1, \ldots, n-1)$ solutions of $(\star)$ ie first integrals of $(\star\star)$

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0$$

IMPORTANT PROPERTY:

$$\frac{M_1}{M_2} = \text{First Integral}$$

NB: ratio may be quite trivial.

Lie S (1874) Veralgemeinerung und neue Verwerthung der Jacobischen Multiplicator-Theorie, *Fordhandlinger i Videnokabs –Selshabet i Christiania* 255-274

Lie S (1912) *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen*, Teubner, Leipzig

Bianchi L (1918) *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni*, Enrico Spoerri, Pisa

If there exist $n-1$ symmetries of $(\star\star)$, say

$$\Gamma_i = \xi_{ij}\partial_{x_j}, \quad i = 1, n-1$$

then JLM is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}$$

Corollary: if $\exists$ M=const, then Δ is a first integral.

The equation for JLM is

$$\frac{\mathrm{d}\log(M)}{\mathrm{d}t} = -\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}$$

If $\partial a_i / \partial x_i = 0 \Rightarrow M$=const.

How many Lagrangians does one know?

In the case of a second-order ODE, there is a well-known link between a Jacobi Last Multiplier M and a Lagrangian L (Jacobi, 1986), (Whittaker, 1904), i.e.

$$\frac{\partial^2 L}{\partial \dot{q}^2} = M. \tag{1}$$

Consequently a knowledge of the multipliers of a system enables one to construct a number of Lagrangians of that system. How many??

Let us consider the simple harmonic oscillator:

$$\begin{aligned} \dot{u_1} &= u_2 \\ \dot{u_2} &= -k^2 u_1 \end{aligned} \tag{2}$$

Lie symmetry algebra of $\ddot{u} = -k^2 u$:

$$\begin{aligned} \Gamma_1 &= \cos kt \partial_u \\ \Gamma_2 &= \sin kt \partial_u \\ \Gamma_3 &= u \partial_u \\ \Gamma_4 &= \partial_t \\ \Gamma_5 &= \cos 2kt \partial_t - ku \sin 2kt \partial_u \\ \Gamma_6 &= \sin 2kt \partial_t + ku \cos 2kt \partial_u \\ \Gamma_7 &= u \cos kt \partial_t - ku^2 \sin kt \partial_u \\ \Gamma_8 &= u \sin kt \partial_t + ku^2 \cos kt \partial_u \end{aligned} \tag{3}$$

The possible Jacobi Last Multipliers are the reciprocals of the nonzero determinants of the possible matrices.

We denote them by JLM_{ij}, where the i and j refer to the symmetries used in the determinants. For example JLM_{12} is obtained from

$$\Delta_{12} = \det \begin{bmatrix} 1 & u_2 & -k^2 u_1 \\ 0 & \cos kt & -k \sin kt \\ 0 & \sin kt & k \cos kt \end{bmatrix} = k. \qquad (4)$$

In the matrix the second row comes from Γ_1 and the third from Γ_2. Then we obtain

$$\begin{aligned}
JLM_{12} &= k \\
JLM_{13} &= \frac{1}{ku_1 \sin kt + u_2 \cos kt} \\
JLM_{14} &= -\frac{1}{k} JLM_{23} \\
JLM_{15} &= \frac{1}{k} JLM_{23} \\
JLM_{16} &= -\frac{1}{k} JLM_{13} \\
JLM_{17} &= -JLM_{13}^2 = -\frac{1}{(ku_1 \sin kt + u_2 \cos kt)^2} \\
JLM_{18} &= -JLM_{13} \times JLM_{23} \\
&= \frac{1}{(ku_1 \sin kt + u_2 \cos kt)(ku_1 \cos kt - u_2 \sin kt)} \\
JLM_{23} &= \frac{1}{-ku_1 \cos kt + u_2 \sin kt} \\
JLM_{24} &= \frac{1}{k} JLM_{13}
\end{aligned}$$

$$JLM_{25} = \frac{1}{k} JLM_{13}$$
$$JLM_{26} = \frac{1}{k} JLM_{23}$$
$$JLM_{27} = JLM_{18}$$
$$JLM_{28} = -JLM_{23}^2 = -\frac{1}{(-ku_1 \cos kt + u_2 \sin kt)^2}$$
$$JLM_{34} = \left[JLM_{13}^{-2} + JLM_{23}^{-2}\right]^{-1} = \frac{1}{u_2^2 + k^2 u_1^2}$$
$$JLM_{35} = \left[JLM_{13}^{-2} - JLM_{23}^{-2}\right]^{-1}$$
$$JLM_{36} = -\frac{1}{2} JLM_{18}$$
$$JLM_{37} \Rightarrow \text{NO}$$
$$JLM_{38} \Rightarrow \text{NO}$$
$$JLM_{45} = \frac{1}{2k} JLM_{18}$$
$$JLM_{46} = \frac{1}{k} JLM_{35}$$
$$JLM_{47} = JLM_{13} JLM_{34}$$
$$= \frac{1}{(u_2^2 + k^2 u_1^2)(ku_1 \sin kt + u_2 \cos kt)}$$
$$JLM_{48} = JLM_{23} JLM_{34}$$
$$= \frac{1}{(u_2^2 + k^2 u_1^2)(-ku_1 \cos kt + u_2 \sin kt)}$$

$$JLM_{56} = \frac{1}{k} JLM_{34}$$
$$JLM_{57} = JLM_{13} JLM_{35}$$
$$JLM_{58} = JLM_{23} JLM_{35}$$
$$JLM_{67} = \frac{1}{2} JLM_{13}^2 JLM_{23}$$
$$JLM_{68} = \frac{1}{2} JLM_{13} JLM_{23}^2$$
$$JLM_{78} \Rightarrow \text{NO.}$$

Therefore there exist 14 different multipliers and from each of them one can derive a (class of) Lagrangian by means of (1). We use the same subscripts to identify the Lagrangians.
N.B.: multiplicative constants = 1.

$$L_{12} = \tfrac{1}{2} u_2^2 + f_1 u_2 + f_2,$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = k^2 u_1;$$

gauge function $g(t, u_1, u_2)$ such that

$$f_1 = \frac{\partial g}{\partial x}, \quad f_2 = \frac{\partial g}{\partial t} - \frac{1}{2} k^2 u_1^2$$

NOTHING NEW, BUT WAIT...

$$L_{13} = \sec^2 kt \Big[\log(ku_1 \sin kt + u_2 \cos kt)$$

$$\times (ku_1 \sin kt + u_2 \cos kt) - u_2 \cos kt - ku_1 \sin kt \Big]$$

$$(\heartsuit) + f_1 u_2 + f_2,$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0;$$

$$L_{23} = \operatorname{cosec}^2 kt \Big[\log(-ku_1 \cos kt + u_2 \sin kt)$$

$$\times (-ku_1 \cos kt + u_2 \sin kt) - u_2 \sin kt + ku_1 \cos kt \Big]_{\heartsuit}$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0;$$

$$L_{17} = -\sec^2 kt \log (ku_1 \sin kt + u_2 \cos kt)_{\heartsuit}$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0;$$

$$L_{18} = \frac{1}{ku_1 \sin kt \cos kt} \Big[\sin kt (ku_1 \sin kt + u_2 \cos kt)$$

$$\times \log (ku_1 \sin kt + u_2 \cos kt)$$

$$+ \cos kt (-u_2 \sin kt + ku_1) \log (-u_2 \sin kt + ku_1) \Big]_{\heartsuit}$$

$$(u_1 \sin kt \cos kt) \left(\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} \right) = 1;$$

$$L_{28} = \mathrm{cosec}^2 kt \log(ku_1 \cos kt - u_2 \sin kt)_\heartsuit$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0;$$

$$L_{34} = \frac{u_2}{ku_1} \arctan\left(\frac{u_2}{ku_1}\right) - \frac{1}{2}\log\left(\frac{u_2^2}{k^2u_1^2} + 1\right)_\heartsuit$$

$$u_1\left(\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1}\right) = 1;$$

$$L_{35} = \frac{1}{2ku_1 \cos 2kt}\Big[2ku_1 + (u_2 \cos 2kt + ku_1 \sin 2kt$$
$$-ku_1)\log\Big((\sin kt + \cos kt)u_2 + (\sin kt - \cos kt)ku_1\Big)$$
$$-(u_2 \cos 2kt + ku_1 \sin 2kt + ku_1)$$
$$\times \log\Big((\sin kt - \cos kt)u_2 - (\sin kt + \cos kt)ku_1\Big)\Big]_\heartsuit$$

$$u_1 \cos 2kt\left(\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1}\right) = 1;$$

$$L_{47} = \frac{1}{2k^2u_1^2}\Big[-(u_2 \cos kt + ku_1 \sin kt)\log(u_2^2 + k^2u_1^2)$$
$$+2(u_2 \sin kt - ku_1 \cos kt)\arctan\left(\frac{u_2}{ku_1}\right)$$
$$+2(u_2 \cos kt + ku_1 \sin kt)\log(u_2 \cos kt + ku_1 \sin kt)\Big]_\heartsuit$$

$$ku_1^2\left(\frac{\partial f_1}{\partial t}-\frac{\partial f_2}{\partial u_1}\right)=\sin kt;$$

$$L_{48}=\frac{1}{2k^2u_1^2}\Big[-(u_2\sin kt-ku_1\cos kt)\log(u_2^2+k^2u_1^2)$$
$$-2(u_2\cos kt+ku_1\sin kt)\arctan\left(\frac{u_2}{ku_1}\right)$$
$$+2(u_2\sin kt-ku_1\cos kt)\log(u_2\sin kt-ku_1\cos kt)\Big]_{\heartsuit}$$

$$ku_1^2\left(\frac{\partial f_1}{\partial t}-\frac{\partial f_2}{\partial u_1}\right)=-\cos kt;$$

$$L_{57}=\frac{1}{2k^2u_1^2}\Big[\Big((\cos kt-\sin kt)u_2+(\cos kt+\sin kt)ku_1\Big)$$
$$\times\log\Big((\cos kt-\sin kt)u_2+(\cos kt+\sin kt)ku_1\Big)$$
$$+\Big((\cos kt+\sin kt)u_2-(\cos kt-\sin kt)ku_1\Big)$$
$$\times\log\Big((\cos kt+\sin kt)u_2-(\cos kt-\sin kt)ku_1\Big)$$
$$-2(u_2\cos kt+ku_1\sin kt)\log(u_2\cos kt+ku_1\sin kt)\Big]_{\heartsuit}$$

$$\frac{\partial f_1}{\partial t}-\frac{\partial f_2}{\partial u_1}=0;$$

$$L_{58} = \frac{1}{2k^2u_1^2}\Big[\Big((\cos kt - \sin kt)u_2 + (\cos kt + \sin kt)ku_1\Big)$$

$$\times \log\Big((\cos kt - \sin kt)u_2 + (\cos kt + \sin kt)ku_1\Big)$$

$$-\Big((\cos kt + \sin kt)u_2 - (\cos kt - \sin kt)ku_1\Big)$$

$$\times \log\Big((\cos kt + \sin kt)u_2 - (\cos kt - \sin kt)ku_1\Big)$$

$$+2(u_2 \sin kt - ku_1 \cos kt)\ \log(u_2 \sin kt - ku_1 \cos kt)\Big]_\heartsuit$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0;$$

$$L_{67} = \frac{ku_1 \cos kt - u_2 \sin kt}{2k^2u_1^2}\Big[\log(ku_1 \sin kt + u_2 \cos kt)$$

$$- \log(u_2 \sin kt - ku_1 \cos kt)\Big]_\heartsuit$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0;$$

$$L_{68} = \frac{ku_1 \sin kt + u_2 \cos kt}{2k^2u_1^2}\Big[\log(ku_1 \sin kt + u_2 \cos kt)$$

$$- \log(u_2 \sin kt - ku_1 \cos kt)\Big]_\heartsuit$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0.$$

Damping linear harmonic oscillator

$$\ddot{u} + 2c^2\dot{u} + (c^2 + k^2)u = 0$$

namely

$$\begin{aligned} \dot{u_1} &= u_2 \\ \dot{u_2} &= -\left(c^2 + k^2\right) u_1 - 2cu_2 \end{aligned}$$

14 Lagrangians! Let's see 3 at least:

$$L_{12} = \tfrac{1}{2}\exp[2ct]u_2^2 + f_1 u_2 + f_2,$$
$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = \exp[2ct](c^2 + k^2)u_1;$$

$$L_{34} = \left(\frac{u_2}{ku_1} + \frac{c}{k}\right) \arctan\left(\frac{u_2}{ku_1} + \frac{c}{k}\right)$$
$$-\frac{1}{2}\log\left(\left(\frac{u_2}{ku_1} + \frac{c}{k}\right)^2 + 1\right) + f_1 u_2 + f_2, \left(\frac{\partial L_{34}}{\partial t} = 0\right)$$
$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = \frac{1}{u_1};$$

$$L_{67} = -\frac{u_2 \sin 2kt + (c\sin 2kt + 2k\sin^2 kt)u_1}{2\exp[ct]k^2 u_1^2 \sin kt}$$
$$\times \log\left[\frac{u_2\cos kt + (c\cos kt + k\sin kt)u_1}{u_2\sin kt + (c\sin kt - k\cos kt)u_1}\right] + f_1 u_2 + f_2,$$
$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0$$

Noether Symmetries and Integrals

Five Noetherian point symmetries

$$
\begin{array}{lll}
L_{12} & \Gamma_1 \Rightarrow & -u_2 \cos kt - k u_1 \sin kt \\
& \Gamma_2 \Rightarrow & u_2 \sin kt - k u_1 \cos kt \\
& \Gamma_4 \Rightarrow & \frac{1}{2}\left(u_2^2 + k^2 u_1^2\right) \\
& \Gamma_5 \Rightarrow & \frac{1}{2}\left(u_2^2 - k^2 u_1^2\right) \cos 2kt + k u_1 u_2 \sin 2kt \\
& \Gamma_6 \Rightarrow & -\frac{1}{2}\left(u_2^2 - k^2 u_1^2\right) \sin 2kt + k u_1 u_2 \cos 2kt
\end{array}
$$

Three Noetherian point symmetries

$$
\begin{array}{lll}
L_{13} & \Gamma_1 & \Rightarrow \log\left(u_2 \cos kt + k u_1 \sin kt\right) \\
& \Gamma_4 + \Gamma_5 & \Rightarrow 2\left(u_2 \cos kt + k u_1 \sin kt\right) \\
& -k\Gamma_3 + \Gamma_6 & \Rightarrow 2\left(-k u_1 \cos kt + u_2 \sin kt\right)
\end{array}
$$

$$
\begin{array}{lll}
L_{23} & \Gamma_2 & \Rightarrow \log\left(-k u_1 \cos kt + u_2 \sin kt\right) \\
& -\Gamma_4 + \Gamma_5 & \Rightarrow 2\left(k u_1 \cos kt - u_2 \sin kt\right) \\
& k\Gamma_3 + \Gamma_6 & \Rightarrow 2\left(u_2 \cos kt + k u_1 \sin kt\right)
\end{array}
$$

$$
\begin{array}{lll}
L_{28} & \Gamma_2 & \Rightarrow \dfrac{1}{ku_1\cos kt - u_2\sin kt} \\
& \Gamma_3 & \Rightarrow \dfrac{1}{k}\,\dfrac{ku_1\sin kt + u_2\cos kt}{ku_1\cos kt - u_2\sin kt} \\
& \Gamma_5 - \Gamma_4 & \Rightarrow 2\left(\log\left(ku_1\cos kt - u_2\sin kt\right) - 1\right)
\end{array}
$$

$$
\begin{array}{lll}
L_{67} & k\Gamma_3 + \Gamma_6 & \Rightarrow \dfrac{-ku_1\cos kt + u_2\sin kt}{-\frac{1}{2}\left(u_2^2 - k^2u_1^2\right)\sin 2kt + ku_1u_2\cos 2kt} \\
& \Gamma_7 & \Rightarrow \dfrac{1}{2k}\,\log\left(\dfrac{-ku_1\cos kt + u_2\sin kt}{u_2\cos kt + ku_1\sin kt}\right) + \dfrac{1}{2k} \\
& \Gamma_8 & \Rightarrow \dfrac{1}{k}\,\dfrac{ku_1u_2\sin 2kt - \left(u_2^2 - k^2u_1^2\right)\cos^2 kt - u_2^2\sin^2 kt}{-\left(u_2^2 - k^2u_1^2\right)\sin 2kt + 2ku_1u_2\cos 2kt}
\end{array}
$$

$$
\begin{array}{lll}
L_{68} & \Gamma_6 - k\Gamma_3 & \Rightarrow -\dfrac{u_2\cos kt + ku_1\sin kt}{\frac{1}{2}\left(u_2^2 - k^2u_1^2\right)\sin 2kt - ku_1u_2\cos 2kt} \\
& \Gamma_7 & \Rightarrow \dfrac{1}{k}\,\dfrac{ku_1u_2\sin 2kt + k^2u_1^2\sin^2 kt + u_2^2\cos^2 kt}{-\left(u_2^2 - k^2u_1^2\right)\sin 2kt + 2ku_1u_2\cos 2kt} \\
& \Gamma_8 & \Rightarrow \dfrac{1}{2k}\,\log\left(\dfrac{-ku_1\cos kt + u_2\sin kt}{u_2\cos kt + ku_1\sin kt}\right) - \dfrac{1}{2k}
\end{array}
$$

Two Noetherian point symmetries

$$L_{17} \quad \Gamma_1 \quad \Rightarrow \; -\frac{1}{u_2 \cos kt + k u_1 \sin kt}$$

$$k\Gamma_4 + \Gamma_5 \quad \Rightarrow \; 2\left(-\log\left(u_2 \cos kt + k u_1 \sin kt\right) + 1\right)$$

$$L_{18} \quad \Gamma_3 \quad \Rightarrow \; \frac{1}{k} \log\left(\frac{k u_1 \cos kt - u_2 \sin kt}{k u_1 \sin kt + u_2 \cos kt}\right)$$

$$\Gamma_6 \quad \Rightarrow \; -\log\left(-\tfrac{1}{2}(u_2^2 - k^2 u_1^2) \sin 2kt + k u_1 u_2 \cos 2kt\right)$$

$$L_{34} \quad \Gamma_3 \quad \Rightarrow \; -\frac{1}{k} \arctan\left(\frac{u_2}{k u_1}\right) - t$$

$$\Gamma_4 \quad \Rightarrow \; \tfrac{1}{2} \log\left(\frac{u_2^2 + k^2 u_1^2}{k^2}\right)$$

$$L_{35} \quad \Gamma_3 \quad \Rightarrow \; \frac{1}{2k} \log\left(\frac{-(u_2 + k u_1) \cos kt + (u_2 - k u_1) \sin kt}{(u_2 - k u_1) \cos kt + (u_2 + k u_1) \sin kt}\right)$$

$$\Gamma_5 \quad \Rightarrow \; \tfrac{1}{2} \log\left(-(u_2^2 - k^2 u_1^2) \cos 2kt - 2k u_1 u_2 \sin 2kt\right) - 1$$

$$L_{47} \qquad \Gamma_7 \;\Rightarrow\; \frac{1}{k}\,\arctan\left(\frac{u_2}{ku_1}\right)+t$$

$$\Gamma_8 \;\Rightarrow\; \frac{1}{2k}\,\log\left(\frac{u_2^2+k^2u_1^2}{(ku_1\sin kt+u_2\cos kt)^2}\right)$$

$$L_{48} \qquad \Gamma_7 \;\Rightarrow\; -\frac{1}{2k}\,\log\left(\frac{u_2^2+k^2u_1^2}{(-ku_1\cos kt+u_2\sin kt)^2}\right)+\frac{1}{k}$$

$$\Gamma_8 \;\Rightarrow\; \frac{1}{k}\,\arctan\left(\frac{u_2}{ku_1}\right)+t$$

$$L_{57} \qquad \Gamma_7 \;\Rightarrow\; \frac{1}{2k}\,\log\left(\frac{(u_2-ku_1)\cos kt+(u_2+ku_1)\sin kt}{(u_2+ku_1)\cos kt-(u_2-ku_1)\sin kt}\right)$$

$$\Gamma_8 \;\Rightarrow\; \log\left(\frac{(ku_1\sin kt+u_2\cos kt)^2}{[(u_2-ku_1)\cos kt+(u_2+ku_1)\sin kt]\,[(u_2+ku_1)\cos kt-(u_2-ku_1)\sin kt]}\right)$$

$$L_{58} \qquad \Gamma_7 \;\Rightarrow\; \log\left(\frac{(u_2\sin kt-ku_1\cos kt)^2}{[(u_2-ku_1)\cos kt+(u_2+ku_1)\sin kt]\,[(u_2+ku_1)\cos kt-(u_2-ku_1)\sin kt]}\right)$$

$$\Gamma_8 \;\Rightarrow\; \frac{1}{2k}\,\log\left(\frac{(u_2-ku_1)\cos kt+(u_2+ku_1)\sin kt}{(u_2+ku_1)\cos kt-(u_2-ku_1)\sin kt}\right)$$

JLM for multidimensional systems

We assume that we have a Lagrangian, $L(t, q_1, q_2, \dot{q}_1, \dot{q}_2)$. The corresponding Lagrange equations are

$$\ddot{q}_1 = f_1(t, q_1, q_2), \tag{1}$$
$$\ddot{q}_2 = f_2(t, q_1, q_2), \tag{2}$$

namely

$$\frac{\partial^2 L}{\partial \dot{q}_1 \partial t} + \frac{\partial^2 L}{\partial \dot{q}_1 \partial q_1}\dot{q}_1 + \frac{\partial^2 L}{\partial \dot{q}_1 \partial q_2}\dot{q}_2 + \frac{\partial^2 L}{\partial \dot{q}_1^2} f_1 + \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_2} f_2 - \frac{\partial L}{\partial q_1} = 0 \ (\spadesuit)$$

$$\frac{\partial^2 L}{\partial \dot{q}_2 \partial t} + \frac{\partial^2 L}{\partial \dot{q}_2 \partial q_1}\dot{q}_1 + \frac{\partial^2 L}{\partial \dot{q}_2 \partial q_2}\dot{q}_2 + \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_2} f_1 + \frac{\partial^2 L}{\partial \dot{q}_2^2} f_2 - \frac{\partial L}{\partial q_2} = 0 \ (\spadesuit)$$

We follow (Rao, 1940) in defining the connection between the last multiplier and the Lagrangian as

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \ \ i, j = 1,\ 2, \tag{3}$$

Once the multiplier has been calculated, the Lagrangian follows by a double quadrature.

We differentiate $(\spadesuit)$ and $(\spadesuit)$ once with respect to both $\dot{q}_1$ and $\dot{q}_2$. We illustrate the calculation in the case of the differentiation of $(\spadesuit)$ with respect to $\dot{q}_2$. We obtain

$$\begin{aligned} &\frac{\partial^3 L}{\partial \dot{q}_1 \partial \dot{q}_2 \partial t} + \frac{\partial^3 L}{\partial \dot{q}_1 \partial \dot{q}_2 \partial q_1}\dot{q}_1 + \frac{\partial^3 L}{\partial \dot{q}_1 \partial \dot{q}_2 \partial q_2}\dot{q}_2 \\ &\quad + \frac{\partial^2 L}{\partial \dot{q}_1 \partial q_2} + \frac{\partial^3 L}{\partial \dot{q}_1^2 \partial \dot{q}_2} f_1 + \frac{\partial^3 L}{\partial \dot{q}_1 \partial \dot{q}_2^2} f_2 - \frac{\partial^2 L}{\partial \dot{q}_2 \partial q_1} = 0, \end{aligned} \tag{4}$$

We use the definition, (3), for M_{12}. Then (4) becomes

$$\begin{aligned} &\frac{\partial}{\partial t}(M_{12}) + \frac{\partial}{\partial q_1}(M_{12}\dot{q}_1) + \frac{\partial}{\partial q_2}(M_{12}\dot{q}_2) + \frac{\partial}{\partial \dot{q}_1}(M_{12} f_1) \\ &\quad + \frac{\partial}{\partial \dot{q}_2}(M_{12} f_2) + \frac{\partial^2 L}{\partial \dot{q}_1 \partial q_2} - \frac{\partial^2 L}{\partial \dot{q}_2 \partial q_1} = 0. \end{aligned} \tag{5}$$

In the cases of M_{11} and M_{22} the ultimate and penultimate terms in the equations corresponding to (5) cancel. The other equation for M_{12} has the subscripts reversed so that, when the two equations are added, the terms vanish. Consequently each multiplier is a solution of the equation

$$\frac{\partial}{\partial t}(M)+\frac{\partial}{\partial q_1}(M\dot{q}_1)+\frac{\partial}{\partial q_2}(M\dot{q}_2)+\frac{\partial}{\partial \dot{q}_1}(Mf_1)+\frac{\partial}{\partial \dot{q}_2}(Mf_2)=0. \tag{6}$$

This is the equation for the JLM, *ie*, the number of degrees of freedom of the system only affects the number of terms in the equation.

A linear example

$$\begin{aligned}\ddot{q}_1 &= -\Omega_1^2 q_1 - \Omega_2^2(q_1-q_2) \equiv f_1\\ \ddot{q}_2 &= \Omega_2^2(q_1-q_2) - \Omega_1^2 q_2 \equiv f_2\end{aligned}$$

Lie symmetry algebra of dimension 7:

$$\begin{aligned}
\Gamma_1 &= q_2\partial_{q_1} + q_1\partial_{q_2} + \dot{q}_2\partial_{\dot{q}_1} + \dot{q}_1\partial_{\dot{q}_2}\\
\Gamma_2 &= \exp\left[Qit\right]\left[\left(\partial_{q_1} - \partial_{q_2}\right) + iQ\left(\partial_{\dot{q}_1} - \partial_{\dot{q}_2}\right)\right]\\
\Gamma_3 &= \partial_t\\
\Gamma_4 &= q_1\partial_{q_1} + q_2\partial_{q_2} + \dot{q}_1\partial_{\dot{q}_1} + \dot{q}_2\partial_{\dot{q}_2}\\
\Gamma_5 &= \exp\left[-Qit\right]\left(\partial_{q_1} - \partial_{q_2} - iQ\left(\partial_{\dot{q}_1} - \partial_{\dot{q}_2}\right)\right]\\
\Gamma_6 &= \exp\left[i\Omega_1 t\right]\left[\left(\partial_{q_1} + \partial_{q_2}\right) + i\Omega_1\left(\partial_{\dot{q}_1} + \partial_{\dot{q}_2}\right)\right]\\
\Gamma_7 &= \exp\left[-i\Omega_1 t\right]\left[\left(\partial_{q_1} + \partial_{q_2}\right) - i\Omega_1\Omega_1 t\left(\partial_{\dot{q}_1} + \partial_{\dot{q}_2}\right)\right],
\end{aligned}$$

$(Q=\sqrt{\Omega_1^2+2\Omega_2^2})$

We have to evaluate 35 determinants. For example:

$$C_{1235} = \begin{bmatrix} 1 & \dot{q}_1 & \dot{q}_2 & f_1 & f_2 \\ 0 & q_2 & q_1 & \dot{q}_2 & \dot{q}_1 \\ 0 & e^{Qit} & -e^{Qit} & iQe^{Qit} & -iQe^{Qit} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & e^{-Qit} & -e^{-Qit} & -iQe^{-Qit} & iQe^{-Qit} \end{bmatrix}$$

Two possible Lagrangians are:

$$L_{2567} = \tfrac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \tfrac{1}{2}\Big[\left(\Omega_1^2 + \Omega_2^2\right)\left(q_1^2 + q_2^2\right)$$
$$-2\Omega_2^2 q_1 q_2\Big] + \dot{g}$$

which is nothing new, but what's about this other one?

$$L_{1235} = -\frac{1}{2iQ\Omega_1\left(q_1 + q_2\right)}\left(\dot{q}_1 + \dot{q}_2\right)\arctan\left(\frac{\dot{q}_1 + \dot{q}_2}{\Omega_1\left(q_1 + q_2\right)}\right)$$
$$+\frac{1}{4iQ}\log\left[\frac{\Omega_1^2\left(q_1 + q_2\right)^2 + \left(\dot{q}_1 + \dot{q}_2\right)^2}{\Omega_1^2\left(q_1 + q_2\right)^2}\right]$$
$$+\frac{i}{2Q}\log\left(q_1 + q_2\right) + \dot{g}$$

What does it happen if we consider the classical two-dimensional harmonic oscillator?

$$\ddot{u}_1 = -u_1$$
$$\ddot{u}_2 = -u_2$$

In this case the Lie symmetry algebra has dimension 15 ($sl(4, I\!R)$).

Determinants to be calculated: 1365 !!!

A nonlinear example

In (Anderson-Thompson, 1992) the following autonomous second-order two-dimensional nonlinear system was presented on page 5:

$$\ddot{u}_1 = -\frac{u_2}{u_1^2 + u_2^2}, \qquad \ddot{u}_2 = \frac{u_1}{u_1^2 + u_2^2}. \tag{7}$$

The authors introduced a t dependent Lagrangian, i.e.

$$L_1 = \frac{1}{2}(\dot{u}_1^2 + \dot{u}_2^2) + t\frac{u_2\dot{u}_1 - u_1\dot{u}_2}{u_1^2 + u_2^2} \tag{8}$$

and state that "there does not exist a translational invariant Lagrangian". This is not true. In fact it is evident that a solution of JLM equation is simply $M = constant$. We take the constant to be unity for M_{11} and M_{22} and zero for M_{12}. Then the corresponding Lagrangian is

$$L = \frac{1}{2}(\dot{u}_1^2 + \dot{u}_2^2) - \arctan\left(\frac{u_2}{u_1}\right) + \dot{g}(t, u_1, u_2) \tag{9}$$

where $g(t, u_1, u_2)$ is an arbitrary "gauge variant" function. It is obvious that $L_1 \subset L$, with

$$g = t \arctan\left(u_2/u_1\right).$$

Moreover we note that system (7) admits a three-dimensional Lie point symmetry algebra generated by the following operators:

$$X_1 = \partial_t, \qquad X_2 = t\partial_t + u_1\partial_{u_1} + u_2\partial_{u_2},$$
$$X_3 = u_2\partial_{u_1} - u_1\partial_{u_2}.$$

L admits two Noether symmetries and therefore two (autonomous) first integrals of system (7) can be derived from Noether's theorem, i.e.:

$$X_1 \Longrightarrow I_1 = \frac{1}{2}(\dot{u}_1^2 + \dot{u}_2^2) + \arctan\left(\frac{u_2}{u_1}\right), \quad (10)$$
$$X_3 \Longrightarrow I_2 = u_1\dot{u}_2 - u_2\dot{u}_1. \quad (11)$$

Therefore the Lagrangian L in (9) is translational invariant.
Other Lagrangians may be derived from I_1 and I_2 (or any function of them) which are themselves Jacobi Last multipliers of system (7).

A dissipative equation with variable coeff.s

Musielak, Roy, Swift (2008):

$$\ddot{x} + b(x)\dot{x}^2 + c(x)x = 0 \tag{12}$$

with $b(x), c(x)$ arbitrary functions of x.
Jacobi (1845) found his "new multiplier" for the following class of second-order ordinary differential equations:

$$\ddot{x} + \frac{1}{2}\frac{\partial\varphi}{\partial x}\dot{x}^2 + \frac{\partial\varphi}{\partial t}\dot{x} + B = 0 \tag{13}$$

with φ, B arbitrary functions of t and x.

$$M = e^{\varphi(t,x)}, \tag{14}$$

It is obvious that equation (12), for any $b(x)$ and $c(x)$, is a particular case of the class of equations (13) studied by Jacobi. Therefore a Jacobi Last Multiplier (14) is already known, i.e.

$$M_1 = e^{2P_b(x)}, \qquad \text{with} \quad P_b(x) = \int b(x)\, dx, \tag{15}$$

and the corresponding Lagrangian is

$$L_1 = \frac{1}{2}e^{2P_b(x)}\dot{x}^2 + f_1(t,x)\dot{x} + f_2(t,x) \tag{16}$$

with f_1, f_2 functions of t and x satisfying the following equation:

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = e^{2P_b(x)}c(x)x. \tag{17}$$

The Lagrangian derived by Museliak et al. after lengthy calculations, i.e.

$$L = \frac{1}{2}e^{2P_b(x)}\dot{x}^2 - \int e^{2P_b(x)}c(x)x\,dx, \qquad (18)$$

is a subcase of the Lagrangian (16), with $f_1 = 0$ and $f_2 = f_2(x) = \int e^{2P_b(x)}c(x)x\,dx$, which is an obvious particular solution of (17).

Actually we can derive other Lagrangians of (12). Equation (12) admits one trivial Lie point symmetry for any $b(x)$ and $c(x)$, i.e. $\Gamma = \partial_t$, which is also a Noether's symmetry for the Lagrangian (16). Therefore a first integral can be easily obtained from Noether's theorem, i.e.:

$$I_1 = \frac{1}{2}e^{2P_b}\dot{x}^2 + \int e^{2P_b}c(x)x\,dx. \qquad (19)$$

Let us use a property of the Jacobi last multiplier, namely if one knows a Jacobi last multiplier M_1 and a first integral I_1 of equation (12), then their product is another Jacobi last multiplier, i.e.

$$M_2 = M_1 I_1 = \frac{1}{2}\,e^{2P_b}\left(e^{2P_b}\dot{x}^2 + \int e^{2P_b}c(x)x\,dx\right)$$

Consequently we are able to obtain a second Lagrangian of equation (12) for any $b(x)$ and $c(x)$, i.e.

$$L_2 = \frac{1}{24}e^{2P_b}\dot{x}^2\left(e^{2P_b}\dot{x}^2 + 12\int e^{2P_b}c(x)x\,dx\right) + f_1(t,x)\dot{x} + f_2(t,x)$$

with f_1, f_2 functions of t and x satisfying the following equation:

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = e^{2P_b(x)} c(x) \int e^{2P_b} c(x) x \, dx.$$

This Lagrangian admits $\Gamma = \partial_t$ as a Noether's symmetry and the corresponding first integral is just the square of I_1 in (19). Then:

$$M_3 = M_1 I_1^2 = \frac{1}{4} e^{2P_b} \left(e^{2P_b} \dot{x}^2 + \int e^{2P_b} c(x) x \, dx \right)^2$$

yields the following third Lagrangian of equation (12)

$$L_3 = \frac{1}{120} e^{2P_b} \dot{x}^2 \left(e^{4P_b} \dot{x}^4 + 10 e^{2P_b} \dot{x}^2 \int e^{2P_b} c(x) x \, dx \right.$$
$$\left. + 60 \left(\int e^{2P_b} c(x) x \, dx \right)^2 \right) + f_1(t,x) \dot{x} + f_2(t,x)$$

with f_1, f_2 functions of t and x satisfying the following equation:

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = e^{2P_b(x)} c(x) x \left(\int e^{2P_b} c(x) x \, dx \right)^2.$$

Lagrangians for N. Euler's ODE

$$\ddot{v} = \frac{c_1}{2v^3} + \frac{a_0 + a_1 x + a_2 x^2}{2v^5} \tag{24}$$

It possesses one Lie point symmetry:

$$\Gamma = 2(a_0 + a_1 x + a_2 x^2)\partial_x + v(a_1 + 2a_2 x)\partial_v$$

$$M = 1$$

$$L = \frac{1}{2}\dot{v}^2 + f_1(x,v)\dot{v} + f_2(x,v)$$

where

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = -\frac{a_0 + a_1 x + a_2 x^2 + c_1 v^2}{2v^5}$$

or equivalently:

$$L = \frac{1}{2}\dot{v}^2 - \frac{a_0 + a_1 x + a_2 x^2 + 2c_1 v^2}{8v^4} + \dot{g}(x,v)$$

It admits Γ as a Noether's symmetry, which yields the following first integral (also a JLM!):

$$\begin{aligned} I_1 &= (a_0 + a_1 x + a_2 x^2)\dot{v}^2 - (a_1 + 2a_2 x)v\dot{v} + 4a_2 v^2 \\ &\quad + 2c_1 \frac{a_0 + a_1 x + a_2 x^2}{v^2} + \frac{(a_0 + a_1 x + a_2 x^2)^2}{v^4} \end{aligned}$$

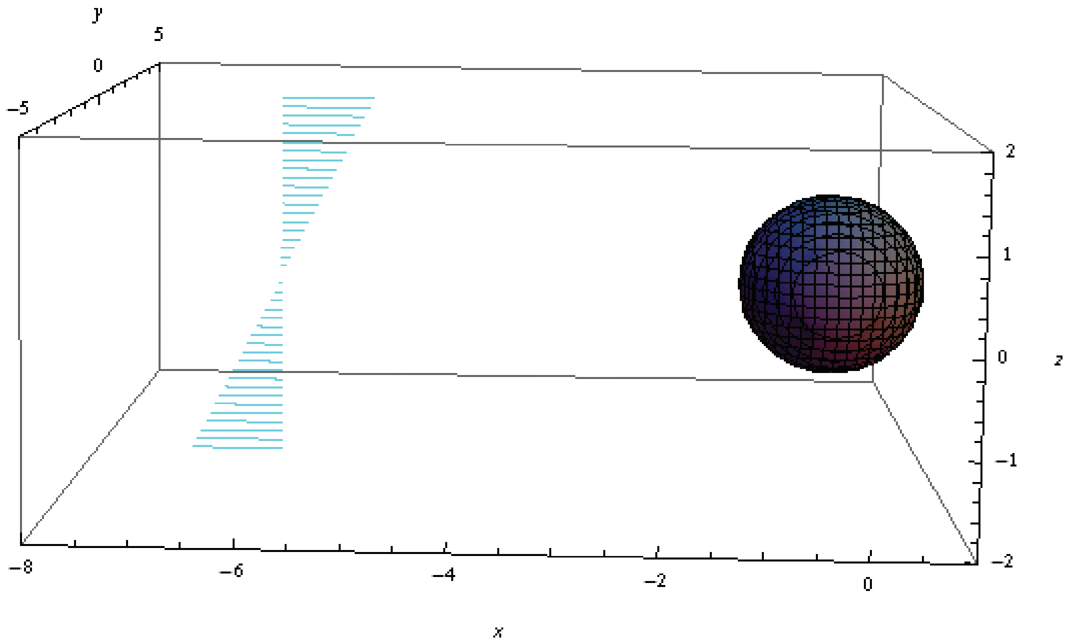
Then $L_2 = \int \left(\int I_1 \, \mathrm{d}\dot{v}\right) \, \mathrm{d}\dot{v} + f_{21}(x,v)\dot{v} + f_{22}(x,v), \ldots$

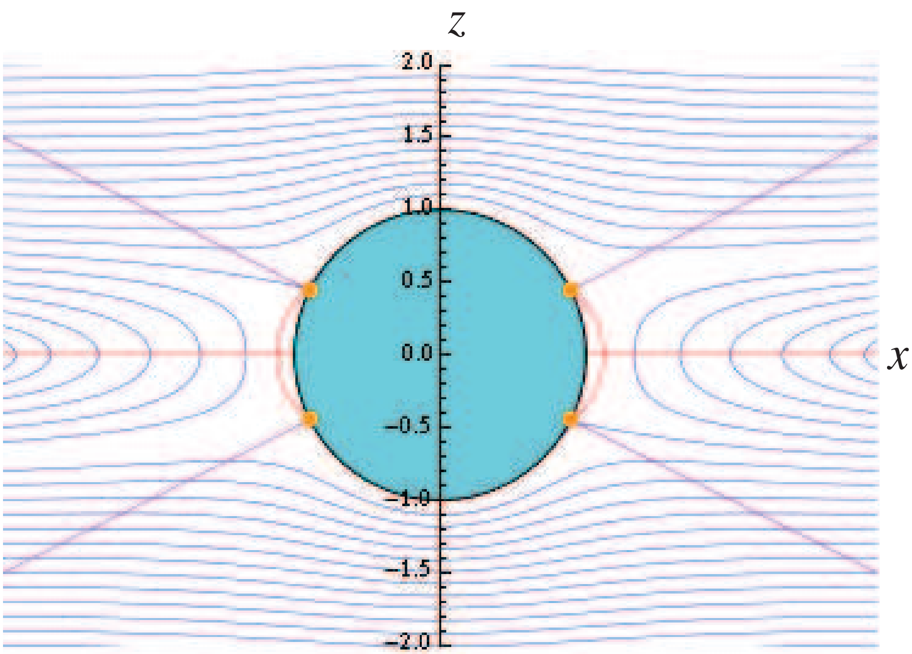
[N.B.: One integral and one JLM implies that equation (24) is completely integrable à la Jacobi]

Lagrangians for first-order ODEs

A sphere immersed in a Stokes flow with linear shear

A. T. Chwang and T. Yao-Tsu Wu, Hydromechanics of low-Reynolds-number flow. Part 2. Singularity method for Stokes flows. JFM 67, pp 787-815 (1975)





$$\dot{z} = 2\frac{4(a^2+ar+r^2)zr^2+(a+r)(4z+1)a^3}{(3a^3+6a^2r+4ar^2+2r^3)(a-r)r}$$

$$\dot{y} = 2\left(a^4+a^3r+2a^2r^2+2ar^3+2r^4\right)\tan\left(\frac{\arccos(2z)}{2}\right) \Big/ \left(r(r-a)(3a^3+6a^2r+4ar^2+2r^3)\sqrt{1-4z^2}\right)$$

$$M = \left(3a^3+6a^2r+4ar^2+2r^3\right)^{4/3}\left(1-\frac{a}{r}\right)^{8/3}$$

$$L = M(y\dot{z} - z\dot{y}) + f_3(r,z,y)$$

$$\begin{aligned}
f_3 = \Big(&\big(a^9+12a^8r+4a^7r^2+4a^6r^3-16a^5r^4-32a^4r^5 \\
&-32a^3r^6+16a^2r^7+16ar^8+16r^9\big)\log\left(\cos\left(\frac{\arccos(2z)}{2}\right)\right) \\
&+24a^9zy+12a^9y+24a^8zyr+12a^8yr-16a^7zyr^2 \\
&-20a^7yr^2-16a^6zyr^3-20a^6yr^3-16a^5zyr^4-24a^4zyr^5 \\
&+8a^4yr^5-24a^3zyr^6+8a^3yr^6+16a^2zyr^7+16azyr^8 \\
&+16zyr^9\Big) \Big/ \left(r^{\frac{11}{3}}(3a^3+6a^2r+4ar^2+2r^3)^{\frac{2}{3}}(r-a)^{\frac{1}{3}}\right) \\
&+s(r).
\end{aligned}$$

Final Remarks

- Any problem of Classical Mechanics becomes an abstract problem where *the real Physics vanishes.* Lagrange is vindicated!
- From each Lagrangian one can $\sim$ easily obtain a corresponding Hamiltonian:

$$L_{34} = \left(\frac{u_2}{ku_1} + \frac{c}{k}\right) \arctan\left(\frac{u_2}{ku_1} + \frac{c}{k}\right) - \frac{1}{2} \log\left(\left(\frac{u_2}{ku_1} + \frac{c}{k}\right)^2 + 1\right) + f_1 u_2 + f_2$$

- Even dissipative systems have many Lagrangians. The application of the Jacobi Last Multiplier extends beyond Lagrangian Mechanics. Therefore calculus of variations may be used in fields where first-order systems dominate as biology, epidemiology, chemistry, etc.
- Do not forget the importance of Lie symmetries.
- Open problem: application to PDEs.
  One may begin with the case of a quasilinear first-order PDE:

$$u_t + a(u)u_x + b(u)u_y = 0$$

$\Longrightarrow\Longrightarrow\Longrightarrow\Longrightarrow\Longrightarrow\Longrightarrow$

As brilliantly stated by William S.:

There are more Lagrangians in heaven and earth, Horatio,
Than are dreamt of in your natural philosophy.