Algebraic origins of integrability of nonlinear evolution equations and dressing procedure

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"Nonlinear Physics. Theory and Experiment. V"
Gallipoli, June 12–21, 2008

$$\forall A,\ B: \quad \exists A^{-1},\ (A-I)^{-1},\ (A-aI)^{-1}$$

$$(a-1)(A-a)(A-I)B(A-I)^{-1}(A-a)^{-1}-$$
$$-a(A-a)ABA^{-1}(A-a)^{-1}+$$
$$+(A-I)ABA^{-1}(A-I)^{-1}+$$
$$+(A-a)B(A-a)^{-1}-a(A-I)B(A-I)^{-1}+$$
$$+(a-1)ABA^{-1}=0$$

$$B(m_1,m_2,m_3)=A^{m_1}(A-I)^{m_2}(A-a)^{m_3}B\times$$
$$\times A^{-m_1}(A-I)^{-m_2}(A-a)^{-m_3}$$

$$\Delta_j B(m)=B(m_j+1)-B(m)$$

$$[(a-1)\Delta_2\Delta_3-a\Delta_1\Delta_3+\Delta_1\Delta_2]B(m)=0$$

$$(a-1)[B(m_2+1,m_3+1)+B(m_1+1)]-$$
$$-a[B(m_1+1,m_3+1)+B(m_2+1)]+$$
$$+[B(m_1+1,m_2+1)+B(m_3+1)]=0$$

$$(a-1)(A-a)(A-I)B(A-I)^{-1}(A-a)^{-1}-$$
$$-a(A-a)ABA^{-1}(A-a)^{-1}+$$
$$+(A-I)ABA^{-1}(A-I)^{-1}+$$
$$+(A-a)B(A-a)^{-1}-a(A-I)B(A-I)^{-1}+$$
$$+(a-1)ABA^{-1}=0$$

$$a\to\frac{1}{a},\qquad \frac{\partial}{\partial a}\dots\Big|_{a=0}\quad\Rightarrow$$

$$\Big[A, ABA^{-1}-(A-I)B(A-I)^{-1}\Big]+$$
$$+(A-I)ABA^{-1}(A-I)^{-1}-(A-I)B(A-I)^{-1}-$$
$$-ABA^{-1}+I=0$$

$$B(m_1,m_2,t_3)=A^{m_1}(A-I)^{m_2}e^{t_3A^3}B\times$$
$$\times A^{-m_1}(A-I)^{-m_2}e^{-t_3A^3}$$

$$[\Delta_1\Delta_2+\partial_{t_3}(\Delta_1-\Delta_2)]B(m,t_3)=0$$

$$\Big[A, ABA^{-1} - (A-I)B(A-I)^{-1}\Big] +$$
$$+ (A-I)ABA^{-1}(A-I)^{-1} - (A-I)B(A-I)^{-1} -$$
$$- ABA^{-1} + I = 0$$

$$A \to \alpha A, \qquad \alpha \to 0$$

$$[A, [A^{-1}, B]] = 2B - ABA^{-1} - A^{-1}BA$$

$$B_{m_1}(t_2, t_3) = e^{t_2 A + t_3 A^{-1}} A^{m_1} B A^{-m_1} e^{-t_2 A - t_3 A^{-1}}$$

$$\partial_{t_2}\partial_{t_3} B_{m_1}(t_2, t_3) =$$
$$= 2B_{m_1}(t_2, t_3) - B_{m_1+1}(t_2, t_3) - B_{m_1-1}(t_2, t_3)$$

$$[A,[A^{-1},B]] = 2B - ABA^{-1} - A^{-1}BA$$

$$A \rightarrow e^{\alpha A}, \quad \left(\frac{\partial}{\partial\alpha}\right)^4 \ldots\Big|_{\alpha=0} \quad \Rightarrow$$

$$[A^3,[A,B]] - \frac{3}{4}[A^2,[A^2,B]] - \frac{1}{4}[A,[A,[A,[A,B]]]] = 0$$

$$B(t_1,t_2,t_3) = e^{t_1 A + t_2 A^2 + t_3 A^3} B e^{-t_1 A - t_2 A^2 - t_3 A^3}$$

$$[4\partial_{t_1}\partial_{t_3}^2 B(t) - 3\partial_{t_2}^2 B(t) - \partial_{t_1}^4] B(t) = 0$$

$$B:\ B(x,y), \qquad A: A(x,y) = x\delta(x-y)$$

$$ABA^{-1} \to \frac{x}{y}B(x,y), \qquad [A,B] \to (x-y)B(x,y)$$

$$(A-I)B(A-I)^{-1} \to \frac{x-1}{y-1}B(x,y)$$

$$\frac{x}{y} = u, \qquad \frac{x-1}{y-1} = v$$

$$\frac{x-a}{y-a} = \frac{P(u,v)}{Q(u,v)} \qquad \frac{x-a}{y-a}Q(u,v) = P(u,v)$$

$$(a-1)(A-a)(A-I)B(A-I)^{-1}(A-a)^{-1}-$$
$$-a(A-a)ABA^{-1}(A-a)^{-1}+$$
$$+(A-I)ABA^{-1}(A-I)^{-1}+$$
$$+(A-a)B(A-a)^{-1} - a(A-I)B(A-I)^{-1}+$$
$$+(a-1)ABA^{-1} = 0$$

$$B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$$

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_3[(AI_1)^2, B] = [AI_1, [AI_1, B]],$$

$$\sigma_3[(AI_2)^2, B] = -[AI_2, [AI_2, B]],$$

$$B_j(t) = e^{iI_j(t_1 A + t_2 A^2)} B e^{-iI_j(t_1 A + t_2 A^2)},$$

$$i\sigma_3 \partial_{t_2} B_j(t) = (-1)^{j+1} \partial_{t_1}^2 B_j(t)$$

$$x^2 = xx, \qquad y^2 = yy$$

$m = (m_1, m_2), m' = (m'_1, m'_2),\ m_j, m'_j \in \mathbb{Z}$

$F_{m,m'} \quad F_{m,m'} \to F(\zeta_1, \zeta_2; z_1, z_2) = F(\zeta; z)$

$$F(\zeta; z) = \sum_{m,m'} z_1^{m'_1 - m_1} z_2^{m'_2 - m_2} \zeta_1^{-m_1} \zeta_2^{-m_2} F_{m,m'}$$

$\zeta = (\zeta_1, \zeta_2),\ z = (z_1, z_2),\ z_j, \zeta_j \in \mathbb{C},\ |\zeta_j| = 1$

$$(FG)_{m,m'} = \sum_{n=(n_1,n_2)} F_{m,n} G_{n,m'} \quad \to$$

$$(FG)(\zeta; z) = \oint_{|\zeta'|=1} \left(\frac{d\zeta'}{2\pi i \zeta'} \right)^2 F(\zeta \overline{\zeta}'; z\zeta') G(\zeta'; z)$$

$F(\zeta_1, \zeta_2; z_1, z_2) \in \mathcal{S}'$

$$I(\zeta; z) = \delta_c(\zeta), \qquad \delta_c(\zeta) = \delta_c(\zeta_1)\delta_c(\zeta_2)$$

$$\delta_c(\zeta_j) = \sum_{n=-\infty}^{\infty} \zeta_j^n$$

$$(\mathcal{T}_1)_{m,m'} = \delta_{m_1-1,m'_1}\delta_{m_2,m'_2} \Rightarrow T_1(\zeta; z) = z_1\delta_c(\zeta)$$

$$\overline{\partial}_1 F: \quad (\overline{\partial}_1 F)(\zeta, z) = \frac{\partial F(\zeta, z)}{\partial \overline{z}_1}$$

$$B(m_1+1) = AB(m)A^{-1}$$
$$B(m_2+1) = (A-I)B(m)(A-I)^{-1}$$
$$B(m_3+1) = (A-a)B(m)(A-a)^{-1}$$
$$B(m_j+1) = T_j B(m) T_j^{-1}, \quad j=1,2$$

$$B_{n_1,n_2,n_1',n_2'}(m_1,m_2,m_3) =$$
$$= B_{n_1+m_1,n_2+m_2,n_1'+m_1,n_2'+m_2}(m_3)$$

$$j=1: \qquad A = T_1, \quad A(\zeta;z) = z_1\delta_c(\zeta)$$

$j=2:$

$$B_{m_2+1} = T_2 B(m) T_2^{-1} = (T_1 - I)B(m)(T_1-I)^{-1}$$

$$L_0 = T_2 - T_1 + I$$

$$L_0 B = T_2 B T_2^{-1} L_0$$

$$B(\zeta,z) = B(\zeta;z_1)$$

$$\nu(\zeta;z)=\nu(\zeta;z_1):$$

$$\overline{\partial}_1\nu=\nu B \qquad \lim_{z_1\to\infty}\nu(\zeta;z)=\delta_c(\zeta)$$

$$\overline{\partial}_1\nu(m)=\nu(m)B(m)$$

$$\nu(m_1+1)=T_1\nu(m)T_1^{-1},\ \ \nu(m_2+1)=T_2\nu(m)T_2^{-1}$$

$$\begin{aligned}&\overline{\partial}_1\Big(L_0\nu-T_2\nu T_2^{-1}L_0\Big)\nu=\\ &=(L_0\nu-T_2\nu T_2^{-1}L_0\Big)B-\\ &-T_2\nu T_2^{-1}\Big(L_0B-T_2BT_2^{-1}L_0\Big)\end{aligned}$$

$$L_0=T_2-T_1+I$$

$$L\nu=T_2\nu T_2^{-1}L_0 \qquad L=L_0+T_1uT_1^{-1}-T_2uT_2^{-1}$$

$$\nu(m,\zeta;z)=\delta_c(\zeta)+\frac{u(m,\zeta)}{z_1}+\ldots$$

$$\nu(m_2+1) = (T_1 - I)\nu(m)(T_1 - I)^{-1} + \\ + \Big[u(m_2+1) - u(m_1+1)\Big]\nu(m)(T_1 - I)^{-1}$$

$$\nu(m_3+1) = (T_1 - a)\nu(m)(T_1 - a)^{-1} + \\ + \Big[u(m_3+1) - u(m_1+1)\Big]\nu(m)(T_1 - a)^{-1}$$

$$\Big[1 + \Delta_1 u - \Delta_2 u\Big]\Delta_1\Delta_2 u - \\ - \Big[a + \Delta_1 u - \Delta_3 u\Big]\Delta_1\Delta_3 u + \\ + \Big[a - 1 + \Delta_2 u - \Delta_3 u\Big]\Delta_2\Delta_3 u = 0$$

$$u = u(m_1, m_2, m_3)$$

$$\Big[\Delta_1\Delta_2 - a\Delta_1\Delta_3 + (a-1)\Delta_2\Delta_3\Big]B(m) = 0$$

$$\varphi_m(z) = z^{m_1}(z-1)^{m_2} \oint\limits_{|\zeta_j|=1} \left(\frac{d\zeta\, \zeta^{m-1}}{2\pi i}\right)^2 \nu(\zeta; z)$$

$$\varphi_{m_2+1} = \varphi_{m_1+1} + \Big(u_{m_2+1} - u_{m_1+1} - 1\Big)\varphi_m$$
$$\varphi_{m_3+1} = \varphi_{m_1+1} + \Big(u_{m_3+1} - u_{m_1+1} - a\Big)\varphi_m$$

$$\Big[1 + u_{m_1+1} - u_{m_2+1}\Big]\Big[u_{m_1+1,m_2+1} + u_{m_3+1}\Big]-$$
$$-\Big[a + u_{m_1+1} - u_{m_3+1}\Big]\Big[u_{m_1+1,m_3+1} + u_{m_2+1}\Big]+$$
$$+\Big[a - 1 + u_{m_2+1} - u_{m_3+1}\Big]\Big[u_{m_2+1,m_3+1} + u_{m_1+1}\Big] =$$
$$=0$$

$$\Big[b_{m_1+1,m_2+1} + b_{m_3+1}]-$$
$$-a\Big[b_{m_1+1,m_3+1} + b_{m_2+1}]+$$
$$+(a-1)\Big[b_{m_2+1,m_3+1} + b_{m_1+1}] = 0$$