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Commuting vector fields, integrable multidimensional PDEs and the analytic description of the gradient catastrophe of 2D water waves near the shore

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We use the recently developed IST for one-parameter families of vector fields, to study the dynamics of localized waves evolving according to the heavenly equation of Plebanski (describing self-dual Einstein fields) and to the dispersionless Kadomtsev-Petviashvili (dKP) equation (describing the evolution of two-dimensional shallow water waves near the shore). In particular, in the dKP case, we obtain the exact analytic description of the gradient catastrophe of 2D water waves near the shore.

1) Phys. Lett. A **359** (2006) (heavenly)
2) JETP Letters **83** (2006) (dKP system + dKP)
3) Theor. Math. Phys. **152** (2007) (Pavlov's equ.)
4) J.Phys.A: Math.Theor. **41** (2008) 055204.
(asymptotics, solutions and wave breaking in dKP)

Examples of equations solvable by the theory:

Nonlinear PDEs in $4+N$ dimensions (N arbitrary):

$$\vec{U}_{t_1 z_2} - \vec{U}_{t_2 z_1} + \left(\vec{U}_{z_1} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_2} - \left(\vec{U}_{z_2} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_1} = \vec{0}, \qquad (1)$$

Its basic reduction, the (4-dimensional) second heavenly equation of Plebanski:

$$\theta_{zy} - \theta_{tx} + \theta_{xy}^2 - \theta_{xx}\theta_{yy} = 0, \quad \theta = \theta(x,y,z,t) \qquad (2)$$

The dKP system:

$$\begin{array}{l} u_{xt} + u_{yy} = -(uu_x)_x - v_x u_{xy} + v_y u_{xx}, \ \ u, v \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \\ v_{xt} + v_{yy} = -uv_{xx} - v_x v_{xy} + v_y v_{xx} \end{array}$$

$$\begin{array}{llll} v = 0: & u_{xt} + u_{yy} + (uu_x)_x = 0, & u = u(x,y,t) & \text{dKP} \\ u = 0: & v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, & v = v(x,y,t) & \text{Pavlov} \end{array} \qquad (3)$$

The nonlinear wave equation (dToda):

$$v_{tt} = (\ln v)_{z\bar{z}} \quad \Rightarrow \quad (e^{\phi})_{tt} = \phi_{z\bar{z}}, \quad v = e^{\phi}. \qquad (4)$$

Applications: heavenly: (self-dual Einstein fields). dKP: small amplitude, nearly one-dimensional waves in shallow water, near the shore. dKP system: general Einstein-Weyl metrics (M. Dunajski). dToda: Field theory, ..

dKP describes small amplitude, nearly one-dimensional waves in shallow water, near the shore.

$$(u_t + uu_x)_x + u_{yy} = 0, \quad u = u(x, y, t) \in \mathbb{R} \tag{5}$$

If the y-dispersion is negligeable, dKP reduces to the Hopf equation:

$$u_t + uu_x = 0, \tag{6}$$

the universal model describing the gradient catastrophe (breaking) of 1D waves.

NATURAL QUESTIONS:

Is dKP the universal model for describing the gradient catastrophe of 2D waves? More concretely:
1) Do localized waves evolving according to dKP break?
2) If yes, does a small initial datum also break?
3) If yes, does breaking take place in a point of the (x, y) plane or on a line?
4) Do the geometric and analytic aspects of breaking exhibit universal feature, as in the (1+1)-dimensional case?
5) How are these features connected with the dKP initial data?
To answer these basic questions on the 2D-wave breaking of dKP solutions, we have to construct: 1) the IST for one-parameter families of vector fields; 2) the long-time behavior of localized initial waves.

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The commutation of linear, first order, partial differential operators with scalar coefficients (vector fields) leads to integrable quasi-linear PDEs in arbitrary dimensions (Zakharov-Shabat, Funct. Anal. Appl. '79)
Our example:

$$\begin{array}{l} \hat{L}_i := \partial_{t^i} + \lambda \partial_{z^i} + \vec{u}_i \cdot \nabla_{\vec{x}}, \qquad i = 1,2 \\ \nabla_{\vec{x}} = (\partial_{x^1}, .., \partial_{x^N}), \quad \vec{u}_i = (u_i^1, .., u_i^N) \end{array} \tag{7}$$

$$\hat{L}_1 \psi = \hat{L}_2 \psi = 0 \quad \Rightarrow \quad [\hat{L}_1, \hat{L}_2] = 0 : \tag{8}$$

First order quasi-linear PDEs in $4 + N$ dimensions:

$$\begin{array}{l} \vec{u_1}_{z_2} = \vec{u_2}_{z_1}, \\ \vec{u_1}_{t_2} - \vec{u_2}_{t_1} + (\vec{u}_2 \cdot \nabla_{\vec{x}})\, \vec{u}_1 - (\vec{u}_1 \cdot \nabla_{\vec{x}})\, \vec{u}_2 = \vec{0}. \end{array} \tag{9}$$

First potential $\vec{U}$:

$$\begin{array}{l} \vec{u}_i = \vec{U}_{z_i}, \qquad i = 1,2, \\ \vec{U}_{t_1 z_2} - \vec{U}_{t_2 z_1} + \left(\vec{U}_{z_1} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_2} - \left(\vec{U}_{z_2} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_1} = \vec{0}, \end{array} \tag{10}$$

Divergence-less (and Hamiltonian, if $N = 2$) reduction:

$$\nabla_{\vec{x}} \cdot \vec{U} = 0 \qquad \Rightarrow \nabla_{\vec{x}} \cdot \vec{u}_i = 0, i = 1,2 \tag{11}$$

Important subcase: $N = 2, \quad z_i = x_i, \quad i = 1, 2.$ (12)

change of notation: $t_1 = z,\ t_2 = t, \quad x_1 = x, \quad x_2 = y$ (13)

$$\begin{aligned} &\vec{U}_{tx} - \vec{U}_{zy} + \left(\vec{U}_y \cdot \nabla_{\vec{x}}\right) \vec{U}_x - \left(\vec{U}_x \cdot \nabla_{\vec{x}}\right) \vec{U}_y = \vec{0}, \\ &\vec{U} \in \mathbb{R}^2, \quad \vec{x} = (x, y), \quad \nabla_{\vec{x}} = (\partial_x, \partial_y), \end{aligned} \tag{14}$$

corresponding to the Lax pair:

$$\begin{aligned} \hat{L}_1 &= \partial_z + \lambda \partial_x + \vec{u}_1 \cdot \nabla_{\vec{x}}, \qquad \vec{u}_1 = \vec{U}_x, \\ \hat{L}_2 &= \partial_t + \lambda \partial_y + \vec{u}_2 \cdot \nabla_{\vec{x}}, \qquad \vec{u}_2 = \vec{U}_y. \end{aligned} \tag{15}$$

If $\nabla \cdot \vec{u}_i = 0, \ i = 1, 2$, the two vector fields are Hamiltonian:

$$\vec{u}_i = (H_{iy}, -H_{ix}), \quad i = 1, 2 \tag{16}$$

Existence of a second potential θ:

$$\begin{aligned} &(H_1, H_2) = \nabla \theta, \qquad \vec{U} = (\theta_y, -\theta_x), \\ &\vec{u}_1 = (\theta_{xy}, -\theta_{xx}), \qquad \vec{u}_2 = (\theta_{yy}, -\theta_{xy}). \end{aligned} \tag{17}$$

Then the Lax pair (15) and the system (14) can be written in Hamiltonian form with respect to the times z, t:

$$\psi_z = \{H_1 + \lambda y, \psi\}_{\vec{x}}, \qquad \psi_t = \{H_2 - \lambda x, \psi\}_{\vec{x}},$$

$$\theta_{tx} - \theta_{zy} + \{\theta_x, \theta_y\}_{\vec{x}} = \text{constant}, \qquad \text{heavenly equation.} \tag{18}$$

$$\{f, g\}_{\vec{x}} \equiv f_x g_y - f_y g_x, \qquad \text{Poisson bracket.} \tag{19}$$

Since the Lax pair is made of vector fields, Hamiltonian in the heavenly reduction:

1) The space of eigenfunctions is a ring: if $f_1,\ f_2$ are two eigenfunctions, then an arbitrary differentiable function $F(f_1, f_2)$ of them is also an eigenfunction.

2) In the heavenly (Hamiltonian) reduction, the space of eigenfunctions is also a Lie algebra, whose Lie bracket is the natural Poisson bracket: if $f_1,\ f_2$ are two eigenfunctions, then their Poisson bracket $\{f_1, f_2\}_{(x,y)}$ is also an eigenfunction.

Cauchy problem within the class of rapidly decreasing real potentials u_i^j:

$$\begin{array}{l} u_i^j \to \ 0, \quad (x^2+y^2+z^2) \to \infty, \\ u_i^j \in \mathbb{R}, \quad (x,y,z) \in \mathbb{R}^3, \quad t>0, \end{array} \tag{20}$$

interpreting t as time and the other three variables x, y, z as space variables.

If f is a solution of $\hat{L}_1 f = 0$, then

$$\begin{array}{l} f(\vec{x}, z, \lambda) \to f_{\pm}(\vec{\xi}, \lambda), \ \ z \to \pm\infty, \\ \vec{\xi} := \vec{x} - (\lambda, 0)z = (x - \lambda z, y); \end{array} \tag{21}$$

i.e., asymptotically, f is an arbitrary function of $(x-\lambda z)$, y and λ.

Jost eigenfunctions $\vec{\varphi}(\vec{x}, z, \lambda)$:

$$\vec{\varphi}(\vec{x}, z, \lambda) \equiv \begin{pmatrix} \varphi_1(\vec{x}, z, \lambda) \\ \varphi_2(\vec{x}, z, \lambda) \end{pmatrix} \rightarrow \begin{pmatrix} \xi \\ y \end{pmatrix} \equiv \vec{\xi}, \quad z \rightarrow -\infty. \tag{22}$$

Their integral equation:

$$\vec{\varphi}(\vec{x}, z, \lambda) + \int_{\mathbb{R}^3} d\vec{x}' dz' G_J(\vec{x} - \vec{x}', z - z'; \lambda) \left(\vec{u}_1(\vec{x}', z') \cdot \nabla_{\vec{x}'}\right) \vec{\varphi}(\vec{x}', z', \lambda) = \vec{\xi},$$

$$G_J(\vec{x}, z; \lambda) = \theta(z)\delta(x - \lambda z)\delta(y). \tag{23}$$

Analytic eigenfunctions $\vec{\psi}_\pm(\vec{x}, z, \lambda)$:

$$\vec{\psi}_\pm(\vec{x}, z, \lambda) + \int_{\mathbb{R}^3} d\vec{x}' dz' G_\pm(\vec{x} - \vec{x}', z - z'; \lambda) \left(\vec{u}_1(\vec{x}', z') \cdot \nabla_{\vec{x}'}\right) \vec{\psi}_\pm(\vec{x}', z', \lambda) = \vec{\xi},$$

$$G_\pm(\vec{x}, z; \lambda) = \pm \frac{\delta(y)}{2\pi i[x - (\lambda \pm i\epsilon)z]}. \tag{24}$$

$\vec{\psi}_+(\vec{x}, z, \lambda)$ and $\vec{\psi}_-(\vec{x}, z, \lambda)$ are analytic in the upper and lower halves of the complex λ - plane, with:

$$\vec{\psi}_\pm(\vec{x}, z, \lambda) = \vec{\xi} + \frac{\vec{Q}_\pm(\vec{x}, z)}{\lambda} + O(\lambda^{-2}). \quad |\lambda| >> 1,$$

$$\vec{Q}_\pm(\vec{x}, z) \equiv -\frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) dx' \vec{u}_1(x', y, z), \tag{25}$$

$$\vec{u}_1(\vec{x}, z) = -\vec{Q}_{\pm x}(\vec{x}, z).$$

Spectral data The $z = +\infty$ limit of $\vec{\varphi}$ defines the scattering vector $\vec{\sigma}$ of $\hat{L}_1$:

$$\lim_{z \to +\infty} \vec{\varphi}(\vec{x}, z; \lambda) \equiv \vec{\mathcal{S}}(\vec{\xi}, \lambda) = \vec{\xi} + \vec{\sigma}(\vec{\xi}, \lambda). \tag{26}$$

$$\textit{Direct Problem}: \quad \vec{u}_1(\vec{x}, z) \quad \to \quad \vec{\sigma}(\vec{\xi}, \lambda) \tag{27}$$

Linear limit: If $|\vec{u}_1| << 1$:

$$\sigma(\xi_1, \xi_2, \lambda) = -\int_{\mathbb{R}} u(\xi_1 + \lambda x, \xi_2, z) dz. \tag{28}$$

The ST is a nonlinear analogue of the Radon transform w.r.t. the 1^{st} and 3^{rd} variables!
The Jost solutions $\varphi_{1,2}$ and λ form a basis in the space of eigenfunctions of $\hat{L}_1$ (which is a ring). The representation of the analytic eigenfunctions $\vec{\psi}_\pm$ in terms of $\vec{\varphi}$ defines other spectral data $\vec{\chi}_\pm$:

$$\vec{\psi}_\pm(\vec{x}, z, \lambda) = \vec{\mathcal{K}}_\pm\left(\vec{\varphi}(\vec{x}, z, \lambda), \lambda\right) = \vec{\varphi}(\vec{x}, z, \lambda) + \vec{\chi}_\pm\left(\vec{\varphi}(\vec{x}, z, \lambda), \lambda\right), \tag{29}$$

The step: $\vec{\sigma}(\vec{\xi}, \lambda) \ \to \ \vec{\chi}_\pm(\vec{\xi}, \lambda)$:

$$\begin{aligned} &\tilde{\vec{\chi}}_+(\vec{\omega}, \lambda) + \theta(\omega_1)\left(\tilde{\vec{\sigma}}(\vec{\omega}, \lambda) + \int_{\mathbb{R}^2} d\vec{\eta}\ \tilde{\vec{\chi}}_+(\vec{\eta}, \lambda) Q(\vec{\eta}, \vec{\omega}, \lambda)\right) = \vec{0}, \\ &\tilde{\vec{\chi}}_-(\vec{\omega}, \lambda) + \theta(-\omega_1)\left(\tilde{\vec{\sigma}}(\vec{\omega}, \lambda) + \int_{\mathbb{R}^2} d\vec{\eta}\ \tilde{\vec{\chi}}_-(\vec{\eta}, \lambda) Q(\vec{\eta}, \vec{\omega}, \lambda)\right) = \vec{0} \end{aligned} \tag{30}$$

for the Fourier transforms:

$$\begin{aligned} &\tilde{\vec{\sigma}}(\vec{\omega}, \lambda) \equiv \int_{\mathbb{R}^2} d\vec{\xi} \vec{\sigma}(\vec{\xi}, \lambda) e^{-i\vec{\omega}\cdot\vec{\xi}}, \quad \tilde{\vec{\chi}}_\pm(\vec{\omega}, \lambda) \equiv \int_{\mathbb{R}^2} d\vec{\xi} \vec{\chi}_\pm(\vec{\xi}, \lambda) e^{-i\vec{\omega}\cdot\vec{\xi}} \\ &Q(\vec{\eta}, \vec{\omega}, \lambda) \equiv \int_{\mathbb{R}^2} \frac{d\vec{\xi}}{(2\pi)^2} e^{i(\vec{\eta}-\vec{\omega})\cdot\vec{\xi}} [e^{i\vec{\eta}\cdot\vec{\sigma}(\vec{\xi}, \lambda)} - 1]. \end{aligned} \tag{31}$$

Inverse Problem

An inverse problem can be constructed from equations $\vec{\psi}_\pm = \vec{\mathcal{K}}_\pm(\vec{\varphi}, \lambda) = \vec{\varphi} + \vec{\chi}_\pm(\vec{\varphi}, \lambda)$. Subtracting $\vec{\xi}$, applying the analyticity projectors $\hat{P}_+$ and $\hat{P}_-$:

$$\hat{P}_\pm \equiv \pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda \pm i\epsilon)}. \tag{32}$$

and adding up the resulting equations, one obtains the following nonlinear integral equation for the Jost eigenfunction $\vec{\varphi}$:

$$\begin{aligned} &\vec{\varphi}(\vec{x}, z, \lambda) + \tfrac{1}{2\pi i} \textstyle\int_{\mathbb{R}} \tfrac{d\lambda'}{\lambda' - (\lambda + i\epsilon)} \vec{\chi}_-(\vec{\varphi}(\vec{x}, z, \lambda'), \lambda') - \\ &\tfrac{1}{2\pi i} \textstyle\int_{\mathbb{R}} \tfrac{d\lambda'}{\lambda' - (\lambda - i\epsilon)} \vec{\chi}_+(\vec{\varphi}(\vec{x}, z, \lambda'), \lambda') = \vec{\xi}. \end{aligned} \tag{33}$$

Given the spectral data $\vec{\chi}_\pm$, one reconstructs the eigenfunction $\vec{\varphi}$ from (33), the analytic eigenfunctions from (29), and $\vec{u}_1$ from the λ large asymptotics. This inversion procedure was first introduced in [Manakov (KP1)].

t-evolution of the spectral data

As the potentials $\vec{u}_{1,2}$ evolve in time according to equation (14), the t-dependence of the spectral data $\vec{\sigma}$ and $\vec{\chi}_\pm$ is described by the equation:

$$\begin{aligned} \vec{\sigma}(\vec{\xi}, \lambda, t) &= \vec{\sigma}(\vec{\xi} - (0, \lambda)t, \lambda, 0), \\ \vec{\chi}_\pm(\vec{\xi}, \lambda, t) &= \vec{\chi}_\pm(\vec{\xi} - (0, \lambda)t, \lambda, 0). \end{aligned} \tag{34}$$

The heavenly reduction

In the heavenly (Hamiltonian) reduction,

the transformations $\vec{\xi} \to \vec{\mathcal{S}}(\vec{\xi},\lambda)$, $\vec{\xi} \to \vec{\mathcal{K}}_{\pm}(\vec{\xi},\lambda)$ are canonical:

$$\{\mathcal{S}_1,\mathcal{S}_2\}_{\vec{\xi}} = \{\mathcal{K}_{\pm 1},\mathcal{K}_{\pm 2}\}_{\vec{\xi}} = 1, \tag{35}$$

or, in terms of $\vec{\sigma}(\xi,y,\lambda)$ and $\vec{\chi}_{\pm}(\xi,y,\lambda)$:

$$\sigma_{1\xi} + \sigma_{2y} + \{\sigma_1,\sigma_2\}_{\vec{\xi}} = \chi_{\pm 1\xi} + \chi_{\pm 2y} + \{\chi_{\pm 1},\chi_{\pm 2}\}_{\vec{\xi}} = 0. \tag{36}$$

Other inverse problems

1. A nonlinear RH problem

Solving the algebraic system $(29)_-$ with respect to $\vec{\varphi}$: $\vec{\varphi} = L(\vec{\psi}_-, \lambda)$ (assuming local invertibility) and replacing this expression in the algebraic system $(29)_+$, one obtains the representation of the analytic eigenfunction $\vec{\psi}_+$ in terms of the analytic eigenfunction $\vec{\psi}_-$:

$$\vec{\psi}_+ = \vec{\mathcal{R}}(\vec{\psi}_-, \lambda) = \vec{\psi}_- + \vec{R}(\vec{\psi}_-, \lambda), \quad \lambda \in \mathbb{R}, \tag{37}$$

which defines a vector nonlinear RH problem on the real λ axis. The RH data $\vec{R}$ are therefore constructed from the data $\vec{\chi}_\pm$ by algebraic manipulation. Viceversa, given the RH data $\vec{R}$, one constructs the solutions $\vec{\psi}_\pm$ of the nonlinear RH problem (37) and, via the asymptotics (25), the potential $\vec{u}_1$.

As for the other spectral data, one can show that the t-dependence of $\vec{R}$ is described by $\vec{R}(\vec{\xi}, \lambda, t) = \vec{R}(\vec{\xi} - (0, \lambda)t, \lambda, 0)$, and the reality constraint takes the following form, for $\lambda \in \mathbb{R}$: $\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\bar{\vec{\zeta}}, \lambda)}, \lambda) = \vec{\zeta}, \ \forall \vec{\zeta}$. At last, the heavenly constraint reads $\{\mathcal{R}_1, \mathcal{R}_2\}_{\vec{\zeta}} = 1$, or, in terms of $\vec{R}(\vec{\zeta}, \lambda)$:

$$R_{1\zeta_1} + R_{2\zeta_2} + \{R_1, R_2\}_{\vec{\zeta}} = 0. \tag{38}$$

2. Linearization of the inverse problem via exponentiation

Define the new eigenfunctions:

$$\Phi(\vec{x},z,\lambda,\vec{\alpha}) \equiv e^{i\vec{\alpha}\cdot\vec{\varphi}(\vec{x},z,\lambda)}, \quad \Psi_{\pm}(\vec{x},z,\lambda,\vec{\alpha}) \equiv e^{i\vec{\alpha}\cdot\vec{\psi}_{\pm}(\vec{x},z,\lambda)}, \quad \vec{\alpha} \in \mathbb{R}^2. \tag{39}$$

From the scattering equation $\vec{\psi}_{\pm} = \vec{\mathcal{K}}_{\pm}(\vec{\varphi},\lambda)$, one gets the *linear* representations the analytic eigenfunctions $\Psi_{\pm}$ in terms of the Jost eigenfunction Φ:

$$\begin{aligned}
&\Psi_{\pm}(\vec{x},z,\lambda,\vec{\alpha}) = \Phi(\vec{x},z,\lambda,\vec{\alpha}) + \int_{\mathbb{R}^2} d\vec{\beta} K_{\pm}(\vec{\alpha},\vec{\beta},\lambda)\Phi(\vec{x},z,\lambda,\vec{\beta}),\\
&K_{\pm}(\vec{\alpha},\vec{\beta},\lambda) \equiv \int_{\mathbb{R}^2} \frac{d\vec{\xi}}{(2\pi)^2} e^{i(\vec{\alpha}-\vec{\beta})\cdot\vec{\xi}}[e^{i\vec{\alpha}\cdot\vec{\chi}_{\pm}(\vec{\xi},\lambda)} - 1].
\end{aligned} \tag{40}$$

and the linear integral equation of the inverse problem:

$$\begin{aligned}
&\Phi(\lambda,\vec{\alpha}) + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{d\lambda'}{\lambda'-(\lambda+i\epsilon)} \int_{\mathbb{R}^2} d\vec{\beta} K_{-}(\vec{\alpha},\vec{\beta},\lambda')\Phi(\lambda',\vec{\beta}) e^{i\alpha_1(\lambda'-\lambda)z} -\\
&-\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{d\lambda'}{\lambda'-(\lambda-i\epsilon)} \int_{\mathbb{R}^2} d\vec{\beta} K_{+}(\vec{\alpha},\vec{\beta},\lambda')\Phi(\lambda',\vec{\beta}) e^{i\alpha_1(\lambda'-\lambda)z} = e^{i\vec{\alpha}\cdot\vec{\xi}},
\end{aligned} \tag{41}$$

Reality constraints for $\lambda \in \mathbb{R}$:

$$\begin{aligned}
&\overline{\Phi(\vec{x},z,\lambda,\vec{\alpha})} = \Phi(\vec{x},z,\lambda,-\vec{\alpha}), \quad \overline{\Psi_{+}(\vec{x},z,\lambda,\vec{\alpha})} = \Psi_{-}(\vec{x},z,\lambda,-\vec{\alpha}),\\
&\overline{K_{+}(\vec{\alpha},\vec{\beta},\lambda)} = K_{-}(-\vec{\alpha},-\vec{\beta},\lambda),
\end{aligned} \tag{42}$$

t-evolution of $K_{\pm}$:

$$K_{\pm}(\vec{\alpha},\vec{\beta},\lambda,t) = K_{\pm}(\vec{\alpha},\vec{\beta},\lambda,0)e^{i\lambda(\alpha_2-\beta_2)t}. \tag{43}$$

RH - Dressing for heavenly

Consider the nonlinear RH problem on the real λ-axis:

$$
\begin{aligned}
&\vec{\pi}^{+} = \vec{\mathcal{R}}(\vec{\pi}^{-},\lambda), \quad \lambda \in \mathbb{R}, \\
&\vec{\pi}^{\pm}(\vec{x},z,\lambda) = \begin{pmatrix} x-\lambda z \\ y-\lambda t \end{pmatrix} + O(\lambda^{-1})
\end{aligned} \tag{44}
$$

for the functions $\vec{\pi}_{+}(\vec{x},z,t,\lambda)$ and $\vec{\pi}_{-}(\vec{x},z,t,\lambda)$, analytic respectively in the upper and lower halves of the complex plane λ, with asymptotics:

$$
\vec{\pi}_{\pm}(\vec{x},z,\lambda) = \begin{pmatrix} x-\lambda z \\ y-\lambda t \end{pmatrix} + \frac{\vec{Q}_{\pm}(\vec{x},z,t)}{\lambda} + O(\lambda^{-2}). \quad |\lambda| >> 1, \tag{45}
$$

If, in addition, the spectral data $\vec{R}(\vec{\zeta},\lambda)$, $\vec{\zeta} \in \mathbb{C}^2$, satisfy the following properties:

$$
\begin{aligned}
&\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\bar{\vec{\zeta}})}) = \vec{\zeta}, \quad \forall \vec{\zeta} \in \mathbb{C}^2, && \text{reality} \\
&R_{1\zeta_1} + R_{2\zeta_2} + \{\mathcal{R}_1,\mathcal{R}_2\}_{\vec{\zeta}} = 0, && \text{heav. constraint.}
\end{aligned} \tag{46}
$$

then the potentials $\vec{u}_{1,2}$ satisfy the heavenly equation (18b), where

$$
\begin{aligned}
&\vec{u}_1(\vec{x},z,t) = -\vec{Q}_{\pm x}, \quad \vec{u}_2(\vec{x},z,t) = -\vec{Q}_{\pm y}. \\
&\vec{Q}_{\pm} = \lim_{\lambda\to\infty} \lambda\Big(\vec{\pi}^{\pm} - \begin{pmatrix} x-\lambda z \\ y-\lambda t \end{pmatrix}\Big)
\end{aligned} \tag{47}
$$

The dKP system

$$u_{xt}+u_{yy}=-(uu_x)_x-v_xu_{xy}+v_yu_{xx}, \quad u,v\in\mathbb{R}, \quad x,y,t\in\mathbb{R},$$
$$v_{xt}+v_{yy}=-uv_{xx}-v_xv_{xy}+v_yv_{xx} \tag{48}$$

Reductions:

$$v=0: \; u_{xt}+u_{yy}+(uu_x)_x=0, \qquad u=u(x,y,t) \quad \text{dKP}$$
$$u=0: \; v_{xt}+v_{yy}+v_xv_{xy}-v_yv_{xx}=0, \; v=v(x,y,t) \text{Pavlov} \tag{49}$$

Lax pair formulation

$$\hat{L}_1\psi=0, \quad \hat{L}_2\psi=0, \quad \Rightarrow \quad [\hat{L}_1,\hat{L}_2]=0 \tag{50}$$

$$\hat{L}_1\equiv\partial_y+(\lambda+v_x)\partial_x-u_x\partial_\lambda,$$
$$\hat{L}_2\equiv\partial_t+(\lambda^2+\lambda v_x+u-v_y)\partial_x+(-pu_x+u_y)\partial_\lambda. \tag{51}$$

Setting $v=0$ in (54), one obtains the Hamiltonian formulation of the dKP equation (Zakharov 94):

$$\psi_y+\lambda\psi_x-u_x\psi_\lambda=\psi_y+\{H_1,\psi\}_{(\lambda,x)}=0,$$
$$\psi_t+(\lambda^2+u)\psi_x+(-\lambda u_x+u_y)\psi_\lambda=\psi_t+\{H_2,\psi\}_{(\lambda,x)}=0,$$

$$H_{1t}-H_{2y}+\{H_2,H_1\}_{(\lambda,x)}=0, \tag{52}$$

$$H_1=\frac{\lambda^2}{2}+u, \qquad H_2=\frac{\lambda^3}{3}+\lambda u-\partial_x^{-1}u_y, \tag{53}$$

$$\{f,g\}_{(\lambda,x)}\equiv f_\lambda g_x-f_xg_\lambda,$$

Elegant integration scheme (Krichever 94).

Main differences between dKP and heavenly:

$$\begin{aligned}&(\partial_y+\lambda\partial_x-u_x\partial_\lambda)\psi=0,\\&(\partial_t+(\lambda^2+u)\partial_x+(-\lambda u_x+u_y)\partial_\lambda)\psi=0\end{aligned} \tag{54}$$

i) The vector fields contain derivatives wrt the spectral parameter λ; ii) quadratic in the spectral parameter implying the following t-evolution of the spectral data $\vec{\mathcal{S}}$, $\vec{\mathcal{K}}_\pm$, $\vec{\mathcal{R}}$:

$$\begin{aligned}&\Sigma_1(\xi,\lambda,t)=t\big(\Sigma_2(\xi-\lambda^2t,\lambda,0)\big)^2+\Sigma_1(\xi-\lambda^2t,\lambda,0),\\&\Sigma_2(\xi,\lambda,t)=\Sigma_2(\xi-\lambda^2t,\lambda,0),\end{aligned} \tag{55}$$

where Σ_1 and Σ_2 are the two components of the vector $\vec{\Sigma}$, identifiable with $\vec{\mathcal{S}}$, $\vec{\mathcal{K}}_\pm$ and $\vec{\mathcal{R}}$.

The resonant character of the explicit t-dependence of the spectral data, absent in the heavenly case, is the spectral reason for the blow-up at finite time of the slope of the localized solution (the breaking) of dKP.

Riemann-Hilbert dressing for dKP

Consider the vector nonlinear Riemann problem on the real line:

$$\begin{aligned} &\vec{\pi}_{+}(\lambda) = \vec{\pi}^{-}(\lambda) + \vec{R}(\vec{\pi}^{+}(\lambda)), \quad \lambda \in \mathbb{R}, \\ &\vec{\pi}_{\pm}(\lambda) = \vec{\xi}(\lambda; x-2ut, y, t) + \vec{O}(\lambda^{-1}), \end{aligned} \tag{56}$$

where

$$\begin{aligned} &\vec{\xi}(\lambda; x-2ut, y, t) = \begin{pmatrix} -\lambda^2 t - \lambda y + x - 2ut \\ \lambda \end{pmatrix}, \\ &u = \lim_{\lambda\to\infty} \left(\lambda(\pi_2^{\pm}(\lambda) - \lambda\right), \end{aligned} \tag{57}$$

and the spectral data $\vec{R}(\vec{\zeta}) = (R_1(\zeta_1,\zeta_2), R_2(\zeta_1,\zeta_2)) \in \mathbb{C}^2$, $\vec{\zeta} \in \mathbb{C}^2$, satisfy the following properties:

$$\begin{aligned} &\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\overline{\vec{\zeta}})}) = \vec{\zeta}, \quad \forall \vec{\zeta} \in \mathbb{C}^2, && \text{reality} \\ &R_{1\zeta_1} + R_{2\zeta_2} + \{R_1, R_2\}_{\vec{\zeta}} = 0, && \text{dKP constraint.} \end{aligned} \tag{58}$$

Then

$$u = F\left(x - 2ut, y, t\right) \in \mathbb{R} \tag{59}$$

is solution of the dKP equation, where

$$F\left(\xi, y, t\right) = -\int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R_2\Big(\pi_1^{-}(\lambda; \xi, y, t), \pi_2^{-}(\lambda; \xi, y, t)\Big). \tag{60}$$

The solution of this Riemann problem, depends parametrically on $(x - 2ut, y, t)$ through the normalization of the RH problem. The inverse formula is an implicit equation for the dKP solution, similar to the solution of the 1+1 dimensional Hopf equation $\Rightarrow$ localized solutions of dKP are expected to break at finite time.

Longtime behavior of solutions of dKP

Let $t >> 1$ and

$$x = \tilde{x} + v_1 t, \quad y = v_2 t, \\ \tilde{x} - 2ut, v_1, v_2 = O(1), \quad v_2 \neq 0, \quad t >> 1. \tag{61}$$

On the parabola

$$x = \tilde{x} - \frac{y^2}{4t} \quad (v_1 = -\frac{v_2^2}{4}), \tag{62}$$

the longtime behaviour of the solution of the dKP equation is given by

$$u = \frac{1}{\sqrt{t}} F_\infty\Big(x - 2ut + \frac{y^2}{4t}, \frac{y}{2t}\Big) + o\left(\frac{1}{\sqrt{t}}\right), \\ F_\infty(\xi, \eta) = -\frac{1}{2\pi i} \int_{\mathbb{R}} d\mu R_2\Big(\xi + \mu^2 + a_1(\mu; \xi, \eta), \eta + a_2(\mu; \xi, \eta)\Big), \tag{63}$$

where $a_j(\mu : \xi, \eta)$ are associated with the following "asymptotic" vector nonlinear Riemann problem on the real axis:

$$\vec{A}^{+}(\mu; \xi, \eta) = \vec{A}^{-}(\mu; \xi, \eta) + \vec{R}(\vec{A}^{-}(\mu; \xi, \eta)), \quad \mu \in \mathbb{R}, \\ \vec{A}^{\pm}(\mu; \xi, \eta) = \begin{pmatrix} \xi + \mu^2 \\ \eta \end{pmatrix} + \vec{a}(\mu; \xi, \eta), \quad |\mu| >> 1, \\ \vec{a}(\mu; \xi, \eta) = \vec{O}(\mu^{-1}). \tag{64}$$

Outside the parabola, the solution decays faster.
Also asymptotically, the solution u depends parametrically on $(x - 2ut)$: (small) localized solutions will break in the longtime regime.

Asymptotic breaking of solutions

Equation (63) defines a nonlinear functional equation for the asymptotics of the dKP solution u.

Let $U(x, y, t)$ be the exact solution of the functional equation (63); i.e.:

$$U(x, y, t) = \frac{1}{\sqrt{t}} G\Big(x - 2\,U\,t + \frac{y^2}{4t}, \frac{y}{2t}\Big), \tag{65}$$

where G is a largely arbitrary differentiable function of two arguments. It is easy to verify that U is the general solution of the quasilinear PDE in $2+1$ dimensions:

$$U_t + \frac{y}{t} U_y - \frac{y^2}{4t^2} U_x + \frac{U}{2t} + U U_x = 0. \tag{66}$$

Its implicit solution (65) suggests to introduce the convenient variables:

$$\begin{aligned} &V = \sqrt{t}\, U, \\ &\tilde{x} = x + \tfrac{y^2}{4t}, \quad \tilde{y} = \tfrac{y}{2t}, \quad \tilde{t} = 2\sqrt{t}, \end{aligned} \tag{67}$$

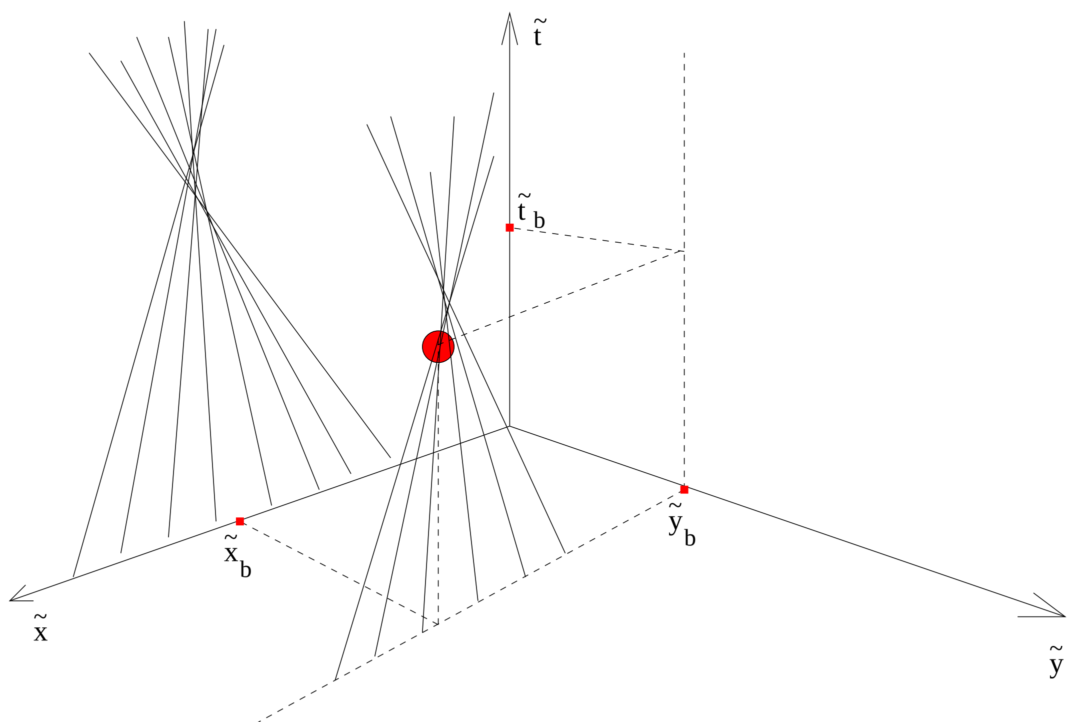
transforming the PDE (66) into the $1+1$ dimensional Hopf equation:

$$V_{\tilde{t}} + V V_{\tilde{x}} = 0. \tag{68}$$

The longtime behavior of the dKP solutions is reduced to the study of the evolution of a two-dimensional localized wave under the $1+1$ dimensional Hopf equation (68).
Its solution is defined implicitely by the equations

$$\begin{aligned} V &= G(\xi, \tilde{y}), \\ \tilde{x} &= \xi + G(\xi, \tilde{y})\tilde{t}, \end{aligned} \tag{69}$$

describing a 2-parameter family (the parameters being $\xi, \tilde{y}$) of straight line characteristics.
On each $\tilde{y} =$ const plane, we have a 1-parameter family of intersecting straight lines. The first breaking will occur at a certain time, in a specific point of the (x, y) plane, and all the characteristics of this breaking depend on the initial condition $u(x, y, 0)$ through the Riemann-Hilbert spectral data.



2D wave breaking according to the Hopf equation

One solves (70b)

$$\begin{aligned} V &= G(\xi,\tilde{y}), \\ \tilde{x} &= \xi + G(\xi,\tilde{y})\tilde{t}, \end{aligned} \tag{70}$$

with respect to the parameter ξ, obtaining $\xi(\tilde{x},\tilde{y},\tilde{t})$, and replaces it into (70a), to obtain the solution $V = G(\xi(\tilde{x},\tilde{y},\tilde{t}),\tilde{y})$. The inversion of equation (70b) is possible iff its ξ-derivative is different from zero.

The two - dimensional singularity manifold (SM):

$$\mathcal{S}(\xi,\tilde{y},t) \equiv 1 + G_{\xi}(\xi,\tilde{y})\tilde{t} = 0 \quad \Rightarrow \quad \tilde{t} = -\frac{1}{G_{\xi}(\xi,\tilde{y})}. \tag{71}$$

Since

$$\nabla_{(\tilde{x},\tilde{y})}V = \frac{\nabla_{(\xi,\tilde{y})}G(\xi,\tilde{y})}{1+G_{\xi}(\xi,\tilde{y})\tilde{t}}, \tag{72}$$

the gradient catastrophe takes place on the SM

The first breaking time $\tilde{t}_b$, and the corresponding characteristic parameters $\vec{\xi_b} = (\xi_b, \tilde{y}_b)$ are defined by

$$\tilde{t}_b = -\frac{1}{G_{\xi}(\vec{\xi_b})} = \text{global min}\left(-\frac{1}{G_{\xi}(\xi,\tilde{y})}\right) > 0, \tag{73}$$

and characterized by the equations:

$$\begin{aligned} &G_{\xi}(\vec{\xi_b}) < 0, \qquad && G_{\xi\xi}(\vec{\xi_b}) = G_{\xi\tilde{y}}(\vec{\xi_b}) = 0, \\ &G_{\xi\xi\xi}(\vec{\xi_b}) > 0, \qquad && \alpha \equiv G_{\xi\xi\xi}(\vec{\xi_b})G_{\xi\tilde{y}\tilde{y}}(\vec{\xi_b}) - G^2_{\xi\xi\tilde{y}}(\vec{\xi_b}) > 0. \end{aligned} \tag{74}$$

The breaking point $\vec{\tilde{x}}_b = (\tilde{x}_b, \tilde{y}_b)$:

$$\tilde{x}_b = \xi_b + G(\vec{\xi_b})\tilde{t}_b. \tag{75}$$

Now we evaluate equations (70b) and (71) near breaking, in the regime:

$$\tilde{x} = \tilde{x}_b + \tilde{x}', \quad \tilde{y} = \tilde{y}_b + \tilde{y}', \quad \tilde{t} = \tilde{t}_b + \tilde{t}', \quad \xi = \xi_b + \xi', \tag{76}$$

where $\tilde{x}', \tilde{y}', \tilde{t}', \xi'$ are small. At the leading order, we get a cubic equation in ξ':

$$\xi'^3 + a(\tilde{y}')\xi'^2 + b(\tilde{y}', \tilde{t}')\xi' - \gamma X(\tilde{x}', \tilde{y}', \tilde{t}') = 0, \tag{77}$$

where

$$\begin{gathered} a(\tilde{y}') = \tfrac{3G_{\xi\xi\tilde{y}}}{G_{\xi\xi\xi}}\tilde{y}', \qquad b(\tilde{y}', \tilde{t}') = \tfrac{3}{G_{\xi\xi\xi}}\left[G_\xi \epsilon + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2\right], \\ X(\tilde{x}', \tilde{y}', \tilde{t}') = \tilde{x}' - G(\xi_b, \tilde{y}_b + \tilde{y}')\tilde{t}' - \left[G(\xi_b, \tilde{y}_b + \tilde{y}') - G\right]\tilde{t}_b \sim \\ \tilde{x}' + \tfrac{G_{\tilde{y}}}{G_\xi}\tilde{y}' - G\tilde{t}' + \tfrac{G_{\tilde{y}\tilde{y}}}{2G_\xi}\tilde{y}'^2 - G_{\tilde{y}}\tilde{y}'\tilde{t}' + \tfrac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_\xi}\tilde{y}'^3, \qquad \gamma = \tfrac{6|G_\xi|}{G_{\xi\xi\xi}}, \end{gathered} \tag{78}$$

with the small parameter

$$\epsilon \equiv 2\frac{\tilde{t} - \tilde{t}_b}{\tilde{t}_b}, \tag{79}$$

corresponding to the maximal balance:

$$|\xi'|, |\tilde{y}'| = O(|\epsilon|^{1/2}), \quad |X| = O(|\epsilon|^{3/2}). \tag{80}$$

The three roots of the cubic are given by the well-known Cardano's formula:

$$\xi'_0\,(\tilde{x}',\tilde{y}',\tilde{t}') = -\tfrac{a}{3} + (A_+)^{\frac{1}{3}} + (A_-)^{\frac{1}{3}},$$
$$\xi'_\pm\,(x',y',t') = -\tfrac{a}{3} - \tfrac{1}{2}\left((A_+)^{\frac{1}{3}} + (A_-)^{\frac{1}{3}}\right) \pm \tfrac{\sqrt{3}}{2} i \left((A_+)^{\frac{1}{3}} - (A_-)^{\frac{1}{3}}\right), \qquad (81)$$

where

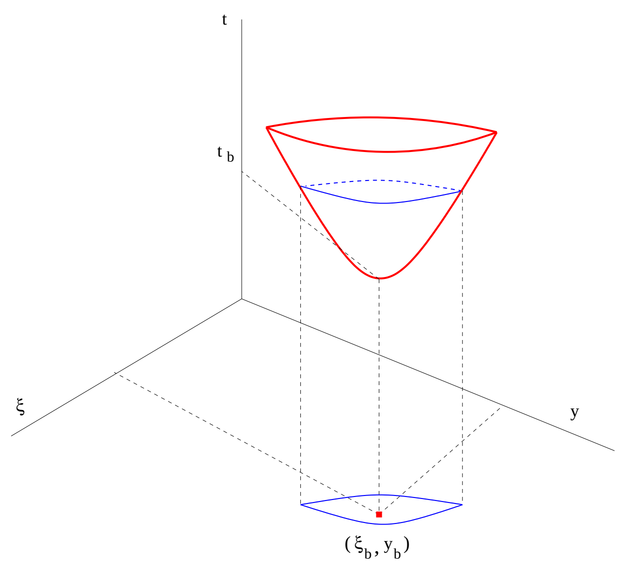
$$A_\pm = R \pm \sqrt{\Delta}, \qquad (82)$$
$$\Delta = R^2 + Q^3, \qquad \textit{discriminant}$$

with

$$Q(\tilde{y}',\tilde{t}') = \tfrac{3b-a^2}{9} = -\tfrac{|G_\xi|}{G_{\xi\xi\xi}}\epsilon + \tfrac{\alpha}{G^2_{\xi\xi\xi}}\tilde{y}'^2, \qquad (83)$$
$$R(\tilde{x}',\tilde{y}',\tilde{t}') = \tfrac{\gamma}{2} X(\tilde{x}',\tilde{y}',\tilde{t}') + \tfrac{ab}{18} + \tfrac{a}{3} Q(\tilde{y}',\tilde{t}').$$

At the same order;

$$\mathcal{S}(\xi,\tilde{y},t) = G_\xi \tilde{t}' + \tfrac{1}{2}[G_{\xi\xi\xi}\xi'^2 + 2G_{\xi\xi\tilde{y}}\xi'\tilde{y}' + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2]\tilde{t}_b. \qquad (84)$$



Known ξ' as function of $(\tilde{x}, \tilde{y}, \tilde{t})$ solving the cubic, the solution V of the Hopf equation and its gradient are then approximated, near breaking, by the formulae:

$$\begin{aligned} &V(\tilde{x}, \tilde{y}, \tilde{t}) \sim G(\xi_b + \xi', \tilde{y}_b + \tilde{y}'), \\ &\nabla_{(\tilde{x},\tilde{y})} V \sim \frac{\nabla_{(\xi',\tilde{y}')} G(\xi_b+\xi', \tilde{y}_b+\tilde{y}')}{G_\xi \tilde{t}' + \frac{1}{2}[G_{\xi\xi\xi}{\xi'}^2 + 2G_{\xi\xi\tilde{y}}\xi'\tilde{y}' + G_{\xi\tilde{y}\tilde{y}}{\tilde{y}'}^2]\tilde{t}_b}. \end{aligned} \tag{85}$$

Another distinguished point: the inflection point $\vec{\tilde{x}}_f$:

$$\vec{\tilde{x}}_f = (\tilde{x}_f(\tilde{t}'), \tilde{y}_b), \qquad \tilde{x}_f(\tilde{t}') = \tilde{x}_b + G\tilde{t}' \tag{86}$$

at which

$$\begin{aligned} &R = X = \tilde{y}' = a = \xi' = \xi'_{\tilde{x}\tilde{x}} \equiv \xi'_{\tilde{x}\tilde{y}} = 0, \\ &V = G, \quad \nabla_{(\tilde{x},\tilde{y})} V = \frac{1}{\tilde{t}'}\left(1, \frac{G_{\tilde{y}}}{G_\xi}\right), \quad V_{\tilde{x}\tilde{x}} = V_{\tilde{x}\tilde{y}} = 0. \end{aligned} \tag{87}$$

Before breaking. If $\tilde{t} < \tilde{t}_b$ ($\tilde{t}' < 0$), the discriminant $\Delta = R^2 + Q^3$ is strictly positive and only the root ξ'_0 is real. $\Rightarrow$ the real solution of the Hopf equ. is single valued and described by Cardano's formula. In addition, $\mathcal{S} > 0$ and $\nabla_{(\tilde{x},\tilde{y})} V$ is finite $\forall\ \tilde{x}, \tilde{y}$.

More explicit solution in the narrower strip around the inflection point:

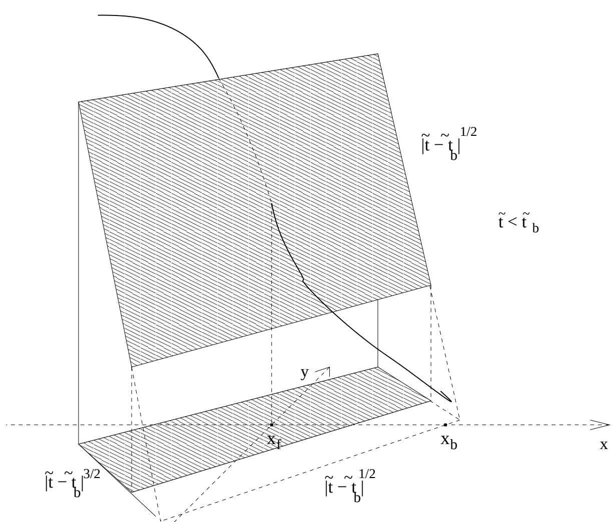
$$
\begin{aligned}
&|\tilde{y}'| = O(\epsilon^q), \quad |X(\tilde{x}', \tilde{y}', \tilde{t}')| = \vec{\tilde{k}} \cdot (\vec{\tilde{x}} - \vec{\tilde{x}}_f(\tilde{t}')) = O(\epsilon^{p+1}), \\
&\max\left(\frac{p+1}{2}, p\right) < q < p+1, \quad p > \frac{1}{2}, \\
&\vec{\tilde{k}} = \left(1, \frac{G_{\tilde{y}}}{G_\xi}\right), \quad \text{breaking direction}
\end{aligned}
\tag{88}
$$

The solution exhibits a universal behaviour, coinciding with the exact similarity solution of the Hopf equation:

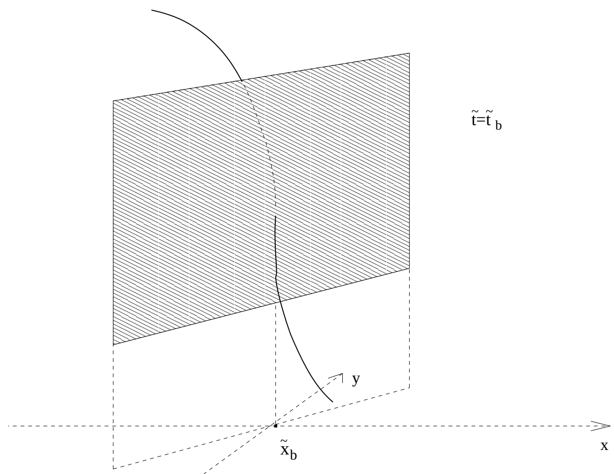
$$
\begin{aligned}
&V \sim \frac{\tilde{x} - \tilde{x}_b + (G_{\tilde{y}}/G_\xi)(\tilde{y} - \tilde{y}_b)}{\tilde{t} - \tilde{t}_b}, \\
&\nabla_{(\tilde{x},\tilde{y})} V \sim \frac{\nabla_{(\xi,\tilde{y})} G}{G_\xi t' + \frac{G_{\xi\xi\xi}}{2|G_\xi|}\left(\frac{\tilde{x}' + (G_{\tilde{y}}/G_\xi)\tilde{y}' - G t'}{\tilde{t}'} + \frac{G_{\xi\xi\tilde{y}}}{G_{\xi\xi\xi}}\tilde{y}'\right)^2 + \frac{\alpha}{2|G_\xi| G_{\xi\xi\xi}}\tilde{y}'^2},
\end{aligned}
\tag{89}
$$

Therefore

$$
\begin{aligned}
&\nabla_{(\tilde{x},\tilde{y})} V \sim \frac{1}{\tilde{t}'}\left(1, \frac{G_{\tilde{y}}}{G_\xi}\right), \quad \textit{in the narrow strip}, \\
&\nabla_{(\tilde{x},\tilde{y})} V = O(1), \qquad |X| = O(|\epsilon|)
\end{aligned}
\tag{90}
$$



At breaking: $\tilde{t} \uparrow \tilde{t}_b$, the inflection point becomes the breaking point: $\vec{x}_f \longrightarrow (\tilde{x}_b, \tilde{y}_b)$, the above tangent plane becomes vertical, with equation $x' - G_{\tilde{y}}(\tilde{\xi}_b, \tilde{y}_b)y' = 0$, the above strip reduces to the breaking point $(\tilde{x}_b, \tilde{y}_b)$.

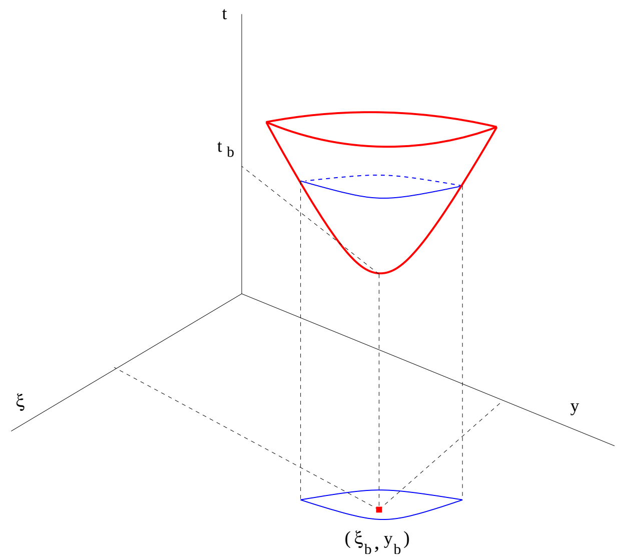


$$V \sim G\left(\xi_b + \sqrt[3]{\gamma(\tilde{x} - \tilde{x}_b)}, \tilde{y}\right) \quad \Rightarrow \quad V_{\tilde{x}} \sim \frac{\sqrt[3]{\gamma}}{3} \frac{G_\xi(\vec{\xi}_b)}{\sqrt[3]{(\tilde{x} - \tilde{x}_b)^2}}. \quad \tilde{y} = \tilde{y}_b,$$
$$V \sim G\left(\xi_b - \sqrt[3]{\frac{6G_{\tilde{y}}}{G_{\xi\xi\xi}}(\tilde{y} - \tilde{y}_b)}, \tilde{y}\right) \Rightarrow V_{\tilde{y}} \sim -\sqrt[3]{\frac{2G_{\tilde{y}}}{3G_{\xi\xi\xi}}} \frac{G_\xi(\vec{\xi}_b)}{\sqrt[3]{(\tilde{y} - \tilde{y}_b)^2}}. \ \tilde{x} = \tilde{x}_b \tag{91}$$

After breaking. If $\tilde{t} > \tilde{t}_b$ ($\tilde{t}' > 0$), the SM equation $\mathcal{S} = 0$:

$$G_{\xi\xi\xi}\xi'^2 + 2G_{\xi\xi\tilde{y}}\xi'\tilde{y}' + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2 = |G_\xi|\epsilon \qquad (92)$$

describes an elliptic paraboloid in the $(\xi, \tilde{y}, \tilde{t})$ space, with minimum at the point $(\vec{\xi}_b, \tilde{t}_b)$

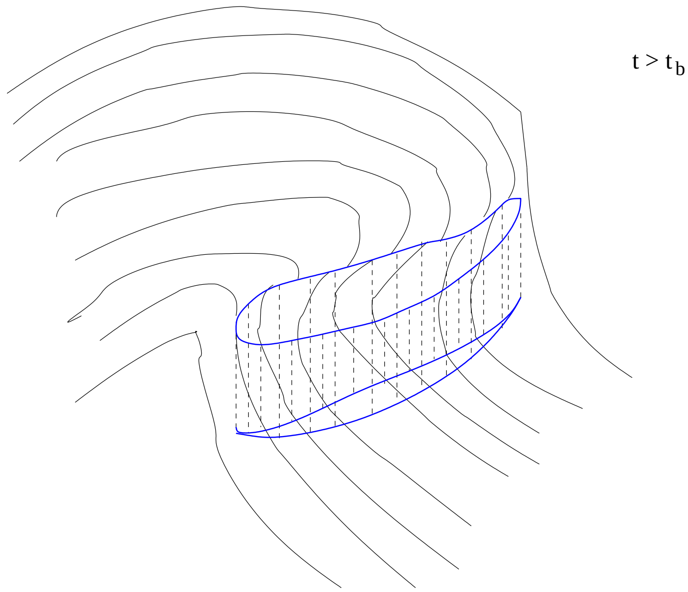


Eliminating ξ' from equations (92) and (77), one obtains the SM equation in space-time coordinates:

$$\Big[3|G_\xi|G^2_{\xi\xi\xi}\left(\tilde{x}' + \tfrac{G_{\tilde{y}}}{G_\xi}\tilde{y}' - G\tilde{t}' + \tfrac{G_{\tilde{y}\tilde{y}}}{2G_\xi}\tilde{y}'^2 - G_{\tilde{y}}\tilde{y}'\tilde{t}' + \tfrac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_\xi}\tilde{y}'^3\right) + \tfrac{G_{\xi\xi\xi}G_{\xi\xi\tilde{y}}G_{\xi\tilde{y}\tilde{y}}}{2}\left(\tfrac{G_\xi}{G_{\xi\tilde{y}\tilde{y}}}\epsilon + y'^2\right)\tilde{y}' - \alpha G_{\xi\xi\tilde{y}}\left(\tfrac{|G_\xi|G_{\xi\xi\xi}}{\alpha}\epsilon - \tilde{y}'^2\right)\tilde{y}'\Big]^2 = \alpha^3\left(\tfrac{|G_\xi|G_{\xi\xi\xi}}{\alpha}\epsilon - \tilde{y}'^2\right)^3, \qquad \Delta = 0 \ \textit{condition} \qquad (93)$$

It describes a closed caustic of the $(\tilde{x}, \tilde{y})$ plane possessing two cusps

$$\vec{\tilde{x}}^{\pm}_c(\tilde{t}') \sim \vec{\tilde{x}}_b \mp \sqrt{\frac{|G_\xi|G_{\xi\xi\xi}\epsilon}{\alpha}}\left(\frac{G_{\tilde{y}}}{G_\xi}, 1\right). \qquad (94)$$

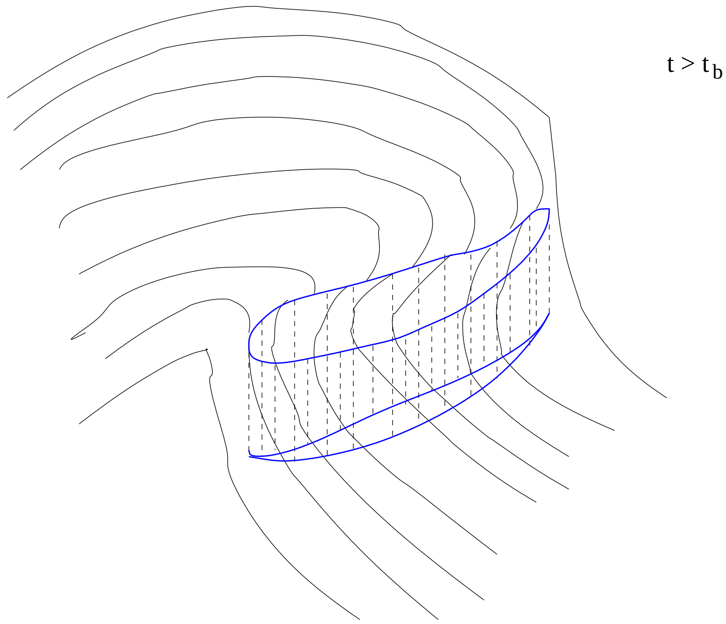


On the caustic $\Delta = 0$, the cubic has three real solutions, but two of the branches coincide and their slopes are ∞. At the cusps, all the three branches coincide . Inside the caustic, $\Delta < 0$ and the solution is three-valued (this multivalued region has to be replaced by a proper shock layer, whose features depend on the wanted regularization). Outside the caustic, $\Delta > 0$ and the solution is single valued.

In addition:

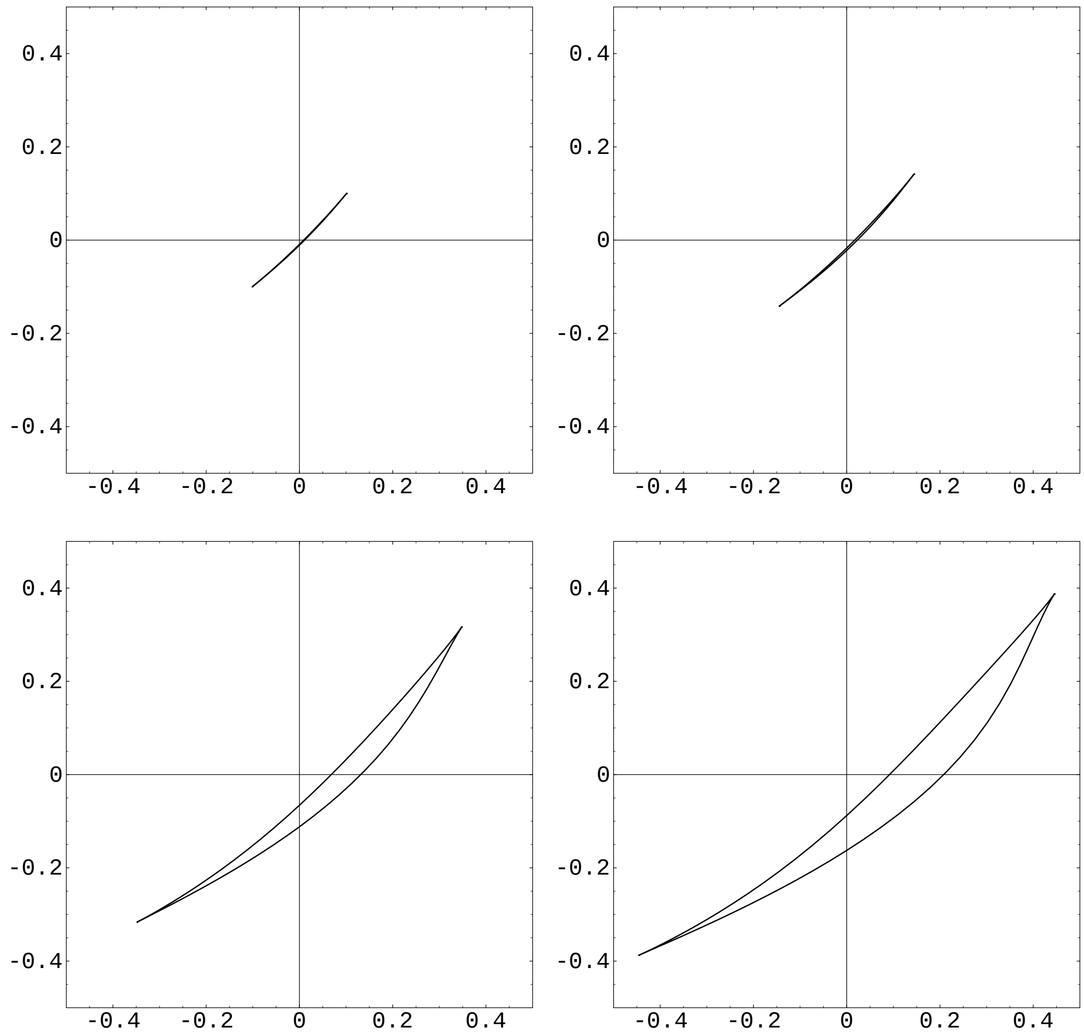
The caustic is the boundary of a narrow region of thickness $O(\epsilon^{3/2})$ in the longitudinal direction, and of thickness $O(\epsilon^{1/2})$ in the transversal direction $\Rightarrow$ the caustics develops, at the breaking point, with ∞ speed in the transversal direction and with 0 speed in the longitudinal direction

That's way, when we watch a 2D water wave breaking, it appears that it breaks on a transversal line and not at a point ....



The cage made by the vertical planes delimiting the caustic.

Four snapshots describing the evolution of such caustic immediately after breaking:



The similarity solution before breaking, the vertical inflection at breaking, and the caustic after breaking make clear the universal character of the gradient catastrophe of two-dimensional waves, for the Hopf and dKP equations.

Longtime breaking of dKP waves. Inverting the transformation

$$\begin{array}{l} V=\sqrt{t}\,U, \\ \tilde{x}=x+\frac{y^2}{4t}, \quad \tilde{y}=\frac{y}{2t}, \quad \tilde{t}=2\sqrt{t}, \end{array} \tag{95}$$

one describes the longtime breaking of dKP solutions. Now

$$\begin{array}{c} U(x,y,t)=\frac{1}{\sqrt{t}}G(\xi,\tilde{y}), \\ \xi=x+\frac{y^2}{4t}-2\sqrt{t}G(\xi,\tilde{y}), \quad \tilde{y}=\frac{y}{2t} \end{array} \tag{96}$$

and

$$\nabla_{(x,y)}U=\frac{1}{\sqrt{t}}\frac{\left(G_\xi(\xi,\tilde{y}),\frac{y}{2t}G_\xi(\xi,\tilde{y})+\frac{1}{2t}G_{\tilde{y}}(\xi,\tilde{y})\right)}{1+2\sqrt{t}G(\xi,\tilde{y})}. \tag{97}$$

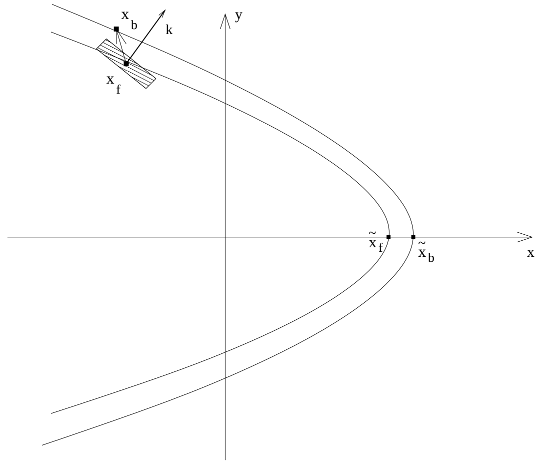
Let $\vec{\xi}_b=(\xi_b,\tilde{y}_b)$ be the breaking parameters and $(\tilde{x}_b,\tilde{y}_b,\tilde{t}_b)$ be the breaking point associated with the Hopf solution, then for a longtime solution $U(x,y,t)$ of dKP the breaking point $\vec{x}_b=(x_b,y_b)$ and the inflection point $\vec{x}_f(t)=(x_f(t),y_f(t))$ read:

$$\begin{array}{l} t_b=\left(\frac{\tilde{t}_b}{2}\right)^2=(2G_\xi)^{-2}, \\ \vec{x}_b=(x_b,y_b): \qquad x_b=\tilde{x}_b-\tilde{y}_b^2t_b, \qquad y_b=2\tilde{y}_bt_b; \end{array} \tag{98}$$

$$\begin{array}{l} x_f(t)=x_b+2G(\sqrt{t}-\sqrt{t_b})-\tilde{y}_b^2(t-t_b)\sim x_b+ \\ \left[2|G_\xi|G(1-\epsilon/4)-\tilde{y}_b^2\right](t-t_b), \\ y_f(t)=y_b+2\tilde{y}_b(t-t_b), \end{array} \tag{99}$$

where the small parameter ϵ reads

$$\epsilon=\frac{t-t_b}{t_b}. \tag{100}$$

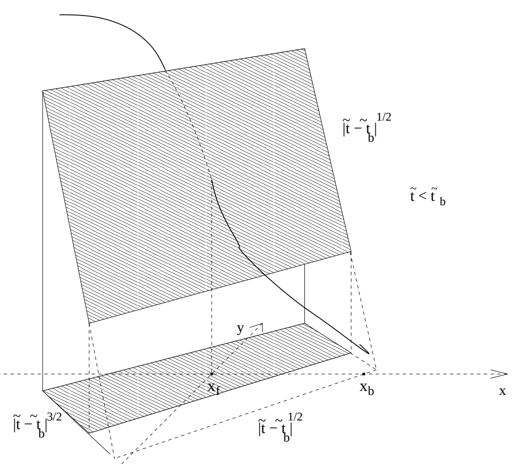


$\vec{x}_b$ and $\vec{x}_f(t)$ are the intersections of the parabolas $x + y^2/4t_b = \tilde{x}_b$ and $x + y^2/4t = \tilde{x}_f = \tilde{x}_b + 2G(\sqrt{t} - \sqrt{t_b})$ with the straight lines $y = y_b$ and $y = y_f(t)$.

Before breaking, in the narrow strip of the figure, U is approximated by the exact similarity solution of equation (66):

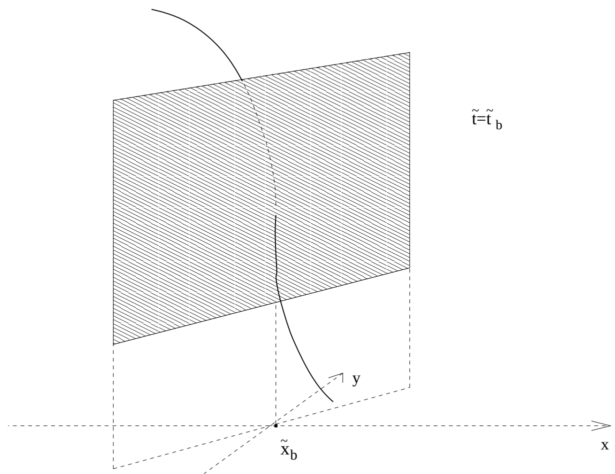
$$U \sim \frac{x + \frac{y^2}{4t} - \tilde{x}_b + \frac{G_\eta}{G_\xi}\left(\frac{y}{2t} - \tilde{y}_b\right)}{2\sqrt{t}(\sqrt{t} - \sqrt{t_b})} \sim \frac{x - x_b + \eta_b^2(t - t_b) + (\eta_b + 2G_\xi G_\eta)[y - y_b - 2\eta_b(t - t_b)]}{t - t_b} \tag{101}$$

describing the plane tangent to the wave at $\vec{x}_f$,



$$\nabla_{(x,y)}U = \frac{1}{\sqrt{t}}\frac{\left(G_\xi, \frac{y}{2t}G_\xi + \frac{1}{2t}G_{\tilde{y}}\right)}{\frac{G_\xi}{\sqrt{t_b}}t' + \frac{G_{\xi\xi\xi}}{2|G_\xi|}\left[\left(\xi' + \frac{G_{\xi\xi\tilde{y}}}{G_{\xi\xi\xi}}\tilde{y}'\right)^2 + \frac{\alpha}{G^2_{\xi\xi\xi}}\tilde{y}'^2\right]}, \quad (102)$$

At breaking, the plane tangent to the wave becomes vertical,



$$U \sim \frac{1}{\sqrt{t_b}}G\left(\xi_b + \sqrt[3]{\gamma(x-x_b)}, \tilde{y}_b\right) \;\Rightarrow\; U_x \sim \frac{\sqrt[3]{\gamma}}{3\sqrt{t_b}}\frac{G_\xi}{\sqrt[3]{(x-x_b)^2}}.\; y = y_b$$
$$U \sim \frac{1}{\sqrt{t_b}}G\left(\xi_b + \sqrt[3]{2\gamma G_\xi G_{\tilde{y}}(y-y_b)}, \frac{y}{2t_b}\right) \;\Rightarrow$$
$$U_y \sim \frac{\sqrt[3]{2\gamma G_\xi G_\eta}}{3\sqrt{t_b}}\frac{G_\xi}{\sqrt[3]{(y-y_b)^2}}, \quad x - x_b + \eta_b(y-y_b) + \frac{(y-y_b)^2}{4t_b} = 0 \quad (103)$$

After breaking $t > t_b$, the intersection of the SM with any t - constant plane, defines an ellipse in the (ξ, η) -

plane corresponding to the following caustic in the (x,y) - plane, defined, as in (93), by

$$\Big[3|G_\xi|G_{\xi\xi\xi}^2X+\tfrac{G_{\xi\xi\xi}G_{\xi\xi\tilde{y}}G_{\xi\tilde{y}\tilde{y}}}{2}\left(\tfrac{G_\xi}{G_{\xi\tilde{y}\tilde{y}}}\epsilon+\tilde{y}'^2\right)\tilde{y}'- \\ \alpha G_{\xi\xi\tilde{y}}\left(\tfrac{|G_\xi|G_{\xi\xi\xi}}{\alpha}\epsilon-\tilde{y}'^2\right)\tilde{y}'\Big]^2=\alpha^3\Big(\tfrac{|G_\xi|G_{\xi\xi\xi}}{\alpha}\epsilon-\tilde{y}'^2\Big)^3, \tag{104}$$

where now

$$\tilde{y}'=\tilde{y}_b\tfrac{y-y_f(t)}{y_b}, \\ X=x-x_b-[2G|G_\xi|(1-\tfrac{1}{4}\epsilon)-\tilde{y}_b^2(1-\epsilon)](t-t_b)+ \\ \left[(\tilde{y}_b+2G_\xi G_{\tilde{y}})(y-y_f(t))+G_\xi^2(y-y_b)^2\right](1-\epsilon)+\tfrac{G_{\tilde{y}\tilde{y}}}{2G_\xi}\tilde{y}_b^2\left(\tfrac{y-y_f(t)}{y_b}\right)^2 - \\ 2|G_\xi|G_{\tilde{y}}\tilde{y}_b(t-t_b)\tfrac{y-y_f(t)}{y_b}+\tfrac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_\xi}\tilde{y}_b^3\left(\tfrac{y-y_f(t)}{y_b}\right)^3. \tag{105}$$

The caustic exhibits two cusps at the points

$$\vec{x}^{\pm}(t)\sim\vec{x}_b\mp\sqrt{2\frac{G_{\xi\xi\xi}}{\alpha}(t-t_b)}.\left(\tilde{y}_b+2G_\xi G_\eta,-1\right) \tag{106}$$

In addition, if $\left(x^{\pm}(y),y\right)$ are the two intersection points of the caustic with the line $y=$ const., we have

$$x^+(y)-x^-(y)=\frac{2\alpha^{3/2}}{3|G_\xi|G_{\xi\xi\xi}}\Big(\frac{|G_\xi|G_{\xi\xi\xi}}{\alpha}\epsilon-\left(\frac{y-y_f}{2t_b}\right)^2\Big)^{3/2}=O(|\epsilon|^{3/2}); \tag{107}$$

therefore the caustic is the boundary of a narrow region of thickness $O(|\epsilon|^{3/2})$ in the longitudinal direction, and of thickness $O(\sqrt{2\frac{G_{\xi\xi\xi}}{\alpha}(t-t_b)})$ in the transversal direction.

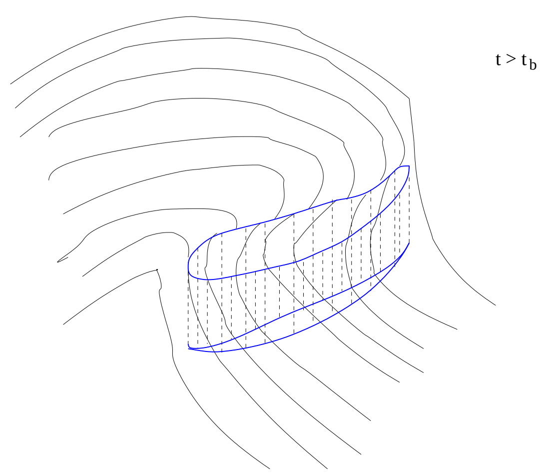
On it, the discriminant Δ of the cubic is zero and, away from the cusps, the solution of equation (65) is three valued, two of the branches coincide:

$$U_0 = \frac{1}{\sqrt{t}} G(\xi_b + \xi_0', \tilde{y}_b + \tilde{y}'), \quad U_+ = U_- = \frac{1}{\sqrt{t}} G(\xi_b + \xi_+', \tilde{y}_b + \tilde{y}') \tag{108}$$

and the slopes of the coincident solutions are ∞. At the two cusps, characterized by the condition $Q = R = 0$, the three real roots of the cubic coincide and the corresponding three solutions of equation (65) coincide as well:

$$U_0 = U_+ = U_- = \tfrac{1}{\sqrt{t}} G\Big(\xi_b \pm 2 \tfrac{G_{\xi\xi\eta}|G_\xi|^{3/2}}{G_{\xi\xi\xi}} \sqrt{\tfrac{G_{\xi\xi\xi}}{\alpha}(t - t_b)}, \tilde{y}_b \mp 2|G_\xi|^{3/2} \sqrt{\tfrac{G_{\xi\xi\xi}}{\alpha}(t - t_b)}\Big). \tag{109}$$

Inside the caustic, the discriminant Δ is strictly negative, the cubic admits three different real roots and the solution of equation (65) is three-valued. Outside, $\Delta > 0$ and the solution of the equation (65) is single valued.



The cage made by the vertical planes delimiting the caustic.

The formulae of this section describe, after replacing U by u, the longtime breaking of the dKP solutions u if, for instance, the dKP initial data $u_0(x,y) = u(x,y,0)$ are small. For small initial data, the inverse spectral transform for dKP simplifies enormously. The RH spectral data are expressed in terms of the initial data as follows:

$$R_2(\zeta_1,\zeta_2) \sim \frac{1}{\pi i}\int\limits_{\mathbb{R}^2} \frac{d\xi' dy}{\zeta_1 - \xi'} u_{0\xi'}(\xi' + \zeta_2 y, y), \qquad (110)$$

and function G, appearing in all formulas of this section, is also given explicitly in terms of u_0:

$$G(\xi,\eta) \sim \frac{1}{2\pi^2}\int\limits_{\mathbb{R}^3} \frac{d\xi' d\mu dy}{\xi-\mu^2-\xi'} u_{0\xi'}(\xi'-\eta y, y). \tag{111}$$

Summarizing, small and localized initial data evolving according to the dKP equation break in the longtime regime; the similarity solution before breaking, the vertical inflection at breaking, and the caustic after breaking make clear the universal character of such a gradient catastrophe.

Analogous considerations can be made in the case of a not small initial datum; in this case the solution breaks at finite time, but the main features of the phenomenon are the same.