Bi-hamiltonian descriptions for composite quantum systems

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GALLIPOLI JUNE 2008

A large number of nonlinear evolution equations are integrable systems.

In almost all cases, the integrable systems admit more than one Hamiltonian description and are often called bi-Hamiltonian systems.

If we adopt Dirac's standpoint:

"CM must be a limiting case of QM. We should thus expect to find that important concepts in CM correspond to important concepts in QM and, from an understanding of the general nature of the analogy between CM and QM, we may hope to get laws and theorems in QM appearing as simple generalizations of well known results in CM."

It is possible to exhibit the analogue of alternative Hamiltonian descriptions in the quantum framework.

Which alternative quantum description survive when we consider composite systems and interactions among them?

Completely integrable classical systems are bi-Hamiltonian systems:

$$\frac{df}{dt} = \{H_1, f\}_1 = \{H_2, f\}_2$$

and, in general,

$$\lambda_1\{.,.\}_1 + \lambda_2\{.,.\}_2$$

is a Poisson structure, i.e. it satisfies the Jacobi identity.

Poisson tensor:

$$\{\xi_j, \xi_k\} = \Lambda_{jk} \quad , \quad \Lambda = \Lambda_{jk} \frac{\partial}{\partial \xi_j} \wedge \frac{\partial}{\partial \xi_k} \quad .$$

Given a dynamical system associated with a vector field Γ on a manifold M, defining the equations of motion

$$\frac{df}{dt} = L_\Gamma f \quad ,$$

how do we find alternative Hamiltonian descriptions, if any? We have to solve the equation for the Poisson tensor

$$L_\Gamma \Lambda = 0 \quad .$$

Out of the alternative solutions we search for compatible Poisson tensors.

Summarizing:

Given

$$\Gamma = \Gamma_j \frac{\partial}{\partial \xi_j}$$

we search for pairs (Λ, H) such that

$$\Gamma_j = \Lambda_{jk} \frac{\partial H}{\partial \xi_k}$$

with the condition (Jacobi identity):

$$\Lambda_{jk} \partial_k \Lambda_{lm} + \Lambda_{lk} \partial_k \Lambda_{mj} + \Lambda_{mk} \partial_k \Lambda_{jl} = 0 \ .$$

To have an idea on how to search for alternative descriptions for quantum systems it is convenient to consider Weyl approach to quantization because in this approach the symplectic structure plays a well identified role.

Weyl Systems

Given a symplectic vector space (E,ω), a Weyl system is defined to be a strongly continuous map from E to unitary transformations on some Hilbert space H :

$$W : \mathrm{E} \to U(\mathrm{H})$$

with

$$W(e_1)W(e_2)W^\dagger(e_1)W^\dagger(e_2) =$$

$$= e^{\frac{i}{\hbar}\omega(e_1,e_2)}$$

A Weyl system defines a projective unitary representation of the vector group E .

Consider a Lagrangian subspace L, and an associated isomorphism E$\rightleftharpoons L \oplus L^* = T^*L$.

On L we consider square integrable functions with respect to a Lebesgue measure on L invariant under translations.

Define e $= (\alpha, x)$ and set

$$(W(0,x)\Psi)(y) = \Psi(x+y)$$

$$(W(\alpha,0)\Psi)(y) = e^{i\alpha(y)}\Psi(x)$$

$$x, y \in L\ , \alpha \in L^*, \Psi \in L^2(L, d^n\xi)$$

The strong continuity allows to use Stone's theorem to get infinitesimal generators

$$W(e) = e^{iR(e)} \quad \forall e \in \mathsf{E}\,.$$

We select a complex structure

$$J : \mathsf{E} \to \mathsf{E}\ ,\ J^2 = -1\ .$$

With this complex structure on E we construct an inner product on E as

$$\langle e_1, e_2 \rangle = \omega(Je_1, e_2) - i\omega(e_1, e_2)$$

The Weyl map allows to associate automorphisms on the space of operators with elements of the symplectic linear group acting on (E, ω) , by setting

$$\nu_S(W(e)) \ = W(S(\mathsf{E})) \ =$$

$$= U_S^\dagger W(e) U_S.$$

The Weyl map can be extended to functions: first define the Fourier transform

of $f \in \mathsf{F}(\mathsf{E})$

$$f(q,p) = \int \widetilde{f}(\alpha, x) e^{i(\alpha q + xp)} d\alpha dx$$

and then associate with it

$$\widehat{A}_f = \int \widetilde{f}(\alpha, x) e^{i(\alpha \widehat{Q} + x\widehat{P})} d\alpha dx$$

Vice versa, with any operator A acting on H we associate a function f_A on E by setting

$$f_A(e) = TrAW(e) \ ,$$

this map is called the Wigner map. A new product of functions is introduced by setting

$$(f_A * f_B)(e) = TrABW(e)$$

and we find an associative product on F(E) which is not commutative.

The dynamics on F(E) can be written in terms of this non-commutative product as

$$i\hbar \frac{df_A}{dt} = f_H * f_A - f_A * f_H$$

In this approach

$$\lim_{\hbar \to 0} -\frac{i}{\hbar}(f_A * f_B - f_B * f_A) = \{f_A, f_B\}.$$

Hamiltonian Quantum Mechanics($\hbar = 1$)

Recasting the classical scheme: Γ a linear v.f. on the space of states M -a (***complex***) Hilbert space.

$$\Gamma(\psi) = (\psi; -iH\psi)$$

where H is a linear operator on M determines the Schroedinger equation:

$$\frac{d}{dt}\psi = -iH\psi$$

To look for alternative Hermitian structures we have to solve for Hermitian forms h s.t.

$$L_\Gamma h = 0$$

or in integrated form : $\forall$t

$$\Phi_t^* h = h \Leftrightarrow \Phi_t^\dagger \Phi_t = \Phi_t \Phi_t^\dagger = 1$$

The same flow Φ_t may result unitary w.r.t. several different h's. In this case it may be necessary to use different simbols for the adjoint operator:

$$\dagger_1, \dagger_2 \quad \text{w.r.t. } h_1, h_2 \quad .$$

Note that any h on M defines an Euclidean metric g, a symplectic form ω and a complex structure J on the realification of the complex space M:

$$h(.,.) \;=: g(.,.) \;+ ig(J.,.).$$

The imaginary part of h is a ***symplectic structure*** ω on the real v.s. M :

$$\omega(.,.) \;:= g(J.,.)$$

so any two of the previous structures determine the third one, and

$$L_\Gamma h = 0 \Leftrightarrow \begin{array}{l} L_\Gamma g = 0 \\ L_\Gamma J = 0 \\ L_\Gamma \omega = 0 \end{array} .$$

Bi-unitary descriptions in H

Now we consider on H two different Hermitian structures, coming from two admissible triples (g_1, J_1, ω_1) and (g_2, J_2, ω_2) with the same complex structure: $J_1 = J_2 = J$. $J_1 = J_2$ implies that the corresponding Hermitian structures are compatible.

Denoting with $h_1(.,.)$ and $h_2(.,.)$ the Hermitian structures given on H, we search for the group that leaves both h_1 and h_2 invariant, that is the bi-unitary group.

By the Riesz's theorem, a bounded, positive operator η may be defined, which is self-adjoint with respect to both h_1 and h_2

$$h_2(x, y) = h_1(\eta x, y) \quad \forall x, y \in \mathrm{H}.$$

Moreover, any bi-unitary operator U must commute with η

$$U^\dagger \eta U = \eta, \;\; [\eta, U] = 0.$$

Therefore the group of bi-unitary operators belongs to the commutant η' of the operator η.

When H is finite-dimensional, η is diagonalizable and the two Hermitian structures result proportional in each eigenspace of η *via* the eigenvalue. Then the group of bi-unitary transformations is

$$U(n_1) \times U(n_2) \times \ldots \times U(n_k) \quad n_1 + n_2 + \ldots + n_k = n,$$

where n_l denotes the degeneracy of the l-th eigenvalue of η.

The following statement holds:

Two Hermitian forms are in generic position if and only if their connecting operator η *is cyclic which in turn implies that the eigenvalues of* η *are nondegenerate.*

Then, if h_1 and h_2 are in generic position the group of bi-unitary transformations becomes

$$\underbrace{U(1) \times U(1) \times \ldots \times U(1)}_{n \text{ factors}}.$$

This means that η generates a complete set of observables.

Bi-hamiltonian composite quantum systems

Let us consider on H^{nm} a bi-unitary composite quantum system which dynamics is described by the Hamiltonian H

$$H = H^\dagger, \quad [H, \eta] = 0, \quad \eta = \eta^\dagger > 0$$

Then,

$$h_1(.,.) \text{ and } h_\eta(.,.) = h_1(\eta.,.),$$

are invariant under the dynamics generated by H.

The equations of motion with respect to h_1 and h_η are given respectively by,

$$\frac{d}{dt}\rho(t) = -i[H, \rho], \quad \rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}, \quad |\psi\rangle \in \mathrm{H}^{nm}$$

and

$$\frac{d}{dt}\widetilde{\rho}(t) = -i[H, \widetilde{\rho}], \quad \widetilde{\rho} = \frac{|\psi\rangle\langle\psi|\eta}{\langle\psi|\eta|\psi\rangle}, \quad |\psi\rangle \in \mathrm{H}^{nm}.$$

The full set of canonical transformations on phase space that maps every Hamiltonian system into an Hamiltonian one for quantum systems are represented by unitary group transformations which preserve the alternative Hermitian structure. Then, such groups are given by

$$U(nm, \mathbf{C}, h_1) \equiv U(nm, \mathbf{C}) \text{ and } U(nm, \mathbf{C}, h_\eta).$$

Having in mind component systems, and recalling that

$$U(n, \mathbf{C}) \otimes U(m, \mathbf{C}) \subset U(nm, \mathbf{C}),$$

the following proposition give us a necessary and sufficient condition such that the Hilbert spaces H^n and H^m associated with the component systems are provided of suitable alternative Hermitian structures h_ξ and h_ζ such that

$$U(n, \mathbf{C}, h_\xi) \otimes U(m, \mathbf{C}, h_\zeta) \subset U(nm, \mathbf{C}, h_\eta).$$

Proposition. *The group* $U(nm, \mathbf{C}, h_\eta)$ *contain as its subgroup the tensor product*

$$U(n, \mathbf{C}, h_\xi) \otimes U(m, \mathbf{C}, h_\zeta),$$

if and only if $\eta = \xi \otimes \zeta$.

A direct consequence of the proposition is that the tensor product of two observables in (H^n, h_ξ) and (H^m, h_ζ), is an observable in (H^{nm}, h_η), if and only if $\eta = \xi \otimes \zeta$.

In particular, if $\eta \neq \xi \otimes \zeta$, to the set of prolongation observables $\{O_\xi \otimes \mathbf{1}^m, \mathbf{1}^n \otimes O_\zeta\}$, belongs some element which cannot be observable.

Partial traces

Let be given the Hamiltonian

$$H = H_A \otimes \mathbf{1}^m + \mathbf{1}^n \otimes H_B + V_{int},$$

where

$$H_A = \sum_{i=1}^{n} a_i |u_i\rangle\langle u_i|, \quad a_i \in \mathrm{R},$$

$$\xi = \sum_{i=1}^{n} r_i |u_i\rangle\langle u_i|, \quad r_i > 0,$$

$$H_B = \sum_{j=1}^{m} b_j |v_j\rangle\langle v_j|, \quad b_j \in \mathrm{R},$$

$$\zeta = \sum_{j=1}^{m} s_j |v_j\rangle\langle v_j|, \quad s_j > 0.$$

Then, on H^{nm},

$$|\alpha\rangle \frac{1}{\langle\alpha|\eta|\alpha\rangle} = \frac{1}{\langle\alpha|\eta|\alpha\rangle} \sum_{i,j} |u_i\rangle \otimes |v_j\rangle(\langle v_j| \otimes \langle u_i|)|\alpha\rangle,$$

$$\widetilde{\rho}^{AB} = \frac{|\alpha\rangle\langle\alpha|\eta}{\langle\alpha|\eta|\alpha\rangle}$$

and the partial trace reads

$$\widetilde{\rho}_A = \mathrm{Tr}_B \widetilde{\rho}^{AB} = \sum_{j=1}^{m} \langle v_j|\widetilde{\rho}^{AB}|v_j\rangle,$$

$$\widetilde{\rho}_B = \mathrm{Tr}_A \widetilde{\rho}^{AB} = \sum_{i=1}^{n} \langle u_i|\widetilde{\rho}^{AB}|u_i\rangle.$$

Example

The system we consider is composed of two qubits A and B.

The Hamiltonian and the evolution operator of the overall system are

$$H = \sigma_3^A \otimes \mathbf{1}^B + \mathbf{1}^A \otimes \sigma_3^B + V_{int}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U = \cos t \mathbf{1}^A \otimes \mathbf{1}^B - i \sin t \sigma_3^A \otimes \sigma_3^B =$$

$$\begin{pmatrix} e^{-it} & 0 & 0 & 0 \\ 0 & e^{it} & 0 & 0 \\ 0 & 0 & e^{it} & 0 \\ 0 & 0 & 0 & e^{-it} \end{pmatrix}.$$

The initial state,

$$\rho^A(0) \otimes \rho^B(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

Then, we obtain by partial traces the final states

$$\rho^A(t) = \mathrm{Tr}_B \rho^{AB} = \frac{1}{2}\begin{pmatrix} 1 & \cos 2t \\ \cos 2t & 1 \end{pmatrix},$$

$$\rho^B(t) = \mathrm{Tr}_A \rho^{AB} = \frac{1}{2}\begin{pmatrix} 1 & -i\cos 2t \\ i\cos 2t & 1 \end{pmatrix}.$$

At the time $t_{bell} = \pi/4$, the overall state $\rho^{AB}(t = t_{bell})$ is equivalent to a Bell state $\frac{1}{\sqrt{2}}(|00> + |11>)$.

The von Neumann entropy give an entanglement measure:

$$\begin{aligned} S(\rho^A(t)) &= -\mathrm{Tr}(\rho^A \log \rho^A) = \\ & - (\sin^2 t)\log(\sin^2 t) - (\cos^2 t)\log(\cos^2 t). \end{aligned}$$

Let us consider in the Hilbert space associated with the composite system an alternative scalar product connected with the fiducial one by means of

$$\eta = \xi \otimes \zeta = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix} \otimes \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix}.$$

Then,

$$[\sigma_3^A, \xi] = 0, \;\; [\sigma_3^B, \zeta] = 0,$$

$$[H, \eta] = 0, \quad U^\dagger \eta U = \eta.$$

The initial density matrix reads

$$\widetilde{\rho}^{AB}(0) = \frac{(\rho^A(0) \otimes \rho^B(0))\eta}{\mathrm{Tr}(\rho^A(0) \otimes \rho^B(0))\eta}.$$

At time t we get

$$\widetilde{\rho}^{AB}(t) = U(t)\widetilde{\rho}^{AB}(0)U(t)^\dagger$$

and the reduced density matrices are given by

$$\widetilde{\rho}^A(t) = \mathrm{Tr}_B \widetilde{\rho}^{AB} =$$

$$\frac{1}{\xi_1+\xi_2} \begin{pmatrix} \xi_1 & \xi_2 \cos 2t \\ \xi_1 \cos 2t & \xi_2 \end{pmatrix},$$

$$\widetilde{\rho}^B(t) = \mathrm{Tr}_A \widetilde{\rho}^{AB} =$$

$$\frac{1}{\zeta_1+\zeta_2}\begin{pmatrix} \zeta_1 & -i\zeta_2\cos 2t \\ i\zeta_1\cos 2t & \zeta_2 \end{pmatrix}.$$

The eigenvalues of $\widetilde{\rho}^A(t)$ are:

$$r_\pm = \frac{1}{2}\left(1 \pm \frac{\sqrt{\xi_1^2+\xi_2^2+2\xi_1\xi_2\cos 4t}}{\xi_1+\xi_2} \right).$$

The von Neumann entropy reads,

$$\begin{aligned} S(\widetilde{\rho}^A(t)) &= -\mathrm{Tr}(\widetilde{\rho}^A \log \widetilde{\rho}^A) = \\ & - r_+ \log r_+ - r_- \log r_-. \end{aligned}$$

The entropy depends on the scalar product, in fact,

$$S(\rho^A(t)) \neq S(\widetilde{\rho}^A(t)).$$

Then, the entanglement measure strongly depends on the alternative hermitian structure.

Let now be

$$\eta = \mathrm{diag}(1,2,3,4) \neq \xi \otimes \zeta.$$

Then,

$$[H,\eta] = 0, \quad U^{\dagger}\eta U = \eta.$$

The initial density matrix reads

$$\widetilde{\rho}^{AB}(0) = \frac{(\rho^{A}(0) \otimes \rho^{B}(0))\eta}{\mathrm{Tr}(\rho^{A}(0) \otimes \rho^{B}(0))\eta}.$$

At time t we get

$$\widetilde{\rho}^{AB}(t) = U(t)\widetilde{\rho}^{AB}(0)U(t)^{\dagger}.$$

At time t the partial trace reads

$$\widetilde{\rho}^{A} = \mathrm{Tr}_{B}\widetilde{\rho}^{AB} =$$

$$\begin{pmatrix} \frac{1}{2} & -i\sqrt{2}\,e^{-2it} - i\frac{2\sqrt{3}}{3}e^{2it} \\ i\frac{\sqrt{2}}{2}e^{2it} + i\frac{\sqrt{3}}{2}e^{-2it} & \frac{1}{2} \end{pmatrix}$$

and its eigenvalues are

$$\frac{1}{6}\left(3 \pm \sqrt{6}\,e^{-4it}\sqrt{3\sqrt{6}\,e^{4it} + 12e^{8it} + 2\sqrt{6}\,e^{12it}}\right) \notin \mathbf{R}.$$

Conclusion

We have analysed tho what extent the bi-Hamiltonian quantum descriptions of composite systems survive.

We have shown that if (and only if) the alternative hermitian structure is connected with the fiducial ones by means of a positive operator η such that

$$\eta = \xi \otimes \zeta,$$

the projection on the component spaces can be performed via the alternative partial trace operation. Moreover, we have shown that the entanglement measure strongly depends on the alternative hermitian structure.

On the contrary, if

$$\eta \neq \xi \otimes \zeta$$

the alternative description for subsystems cannot be identifyied.

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