Integrable systems in (1+1) dimensions: *local* **symmetries vs.** *nonlocal* **recursion operators**

Artur Sergyeyev

Introduction

Integrable hierarchies in (1+1) dimensions usually **do not involve nonlocalities:**
if we have a **local** (=no integral terms) system

$$\partial\vec{u}/\partial t = \vec{F}(x,t,\vec{u},\partial\vec{u}/\partial x,\dots,\partial^n\vec{u}/\partial x^n)$$

then the higher flows that commute with our system and among themselves usually are *local too*, i.e., they have a similar form

$$\partial\vec{u}/\partial t_k = \vec{F}_k(x,t,\vec{u},\partial\vec{u}/\partial x,\dots,\partial^{n_k}\vec{u}/\partial x^{n_k}).$$

But how can we *prove* the existence of **infinitely many <u>local</u>** flows and construct them?

Commuting flows and symmetries

Consider a system $$\frac{\partial \vec{u}}{\partial t} = \vec{F}\left(x, t, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \ldots, \frac{\partial^n \vec{u}}{\partial x^n}\right) \quad (1)$$

The flow $$\frac{\partial \vec{u}}{\partial z} = \vec{G}\left(x, t, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \ldots, \frac{\partial^k \vec{u}}{\partial x^k}\right) \quad (2)$$

commutes with (1) if $$\frac{\partial^2 \vec{u}}{\partial z \partial t} = \frac{\partial^2 \vec{u}}{\partial z \partial t} \quad (3)$$

If (3) holds then $\vec{G}$ is a **symmetry** of (1).

How can we *prove* the existence of **infinitely many** *local* commuting flows and construct them?

Infinitely many local commuting flows

Solution 1. Higher Lax or zero-curvature reps.

Problems:

(i) what if a Lax pair or a ZCR are *not known*?

(e.g. new integrable systems)

(ii) what if a Lax pair or a ZCR *does not exist*?

(C-integrable systems like the Burgers equation)

Infinitely many local commuting flows II

Solution 2. Recursion operators.

Problems:

(i) recursion operator must be known (*not really a problem*, usually we know the RO itself or the Lax pair or a ZCR and can construct a RO from them)

(ii) recursion operators usually are nonlocal (involve integral operators), so **how can we prove that it produces local symmetries (=local higher flows)?** (under certain restrictions on ROs we *can*)

What is a recursion operator (informally)?

Consider a system $\frac{\partial \vec{u}}{\partial t} = \vec{F}\left(x, t, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \ldots, \frac{\partial^n \vec{u}}{\partial x^n}\right)$ (1)

R is a **recursion operator** for (1) if for *any* flow

$\frac{\partial \vec{u}}{\partial z} = \vec{G}\left(x, t, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \ldots, \frac{\partial^k \vec{u}}{\partial x^k}\right)$ that commutes with (1)

the new flow $\frac{\partial \vec{u}}{\partial \tau} = R\left(\vec{G}\right)$ *also commutes* with (1):

$$\frac{\partial^2 \vec{u}}{\partial z \partial t} = \frac{\partial^2 \vec{u}}{\partial t \partial z} \Rightarrow \frac{\partial^2 \vec{u}}{\partial \tau \partial t} = \frac{\partial^2 \vec{u}}{\partial t \partial \tau}.$$

What is a recursion operator – continued

If R is a RO for an evolution system

$$\frac{\partial \vec{u}}{\partial t} = \vec{F}\left(x, t, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \ldots, \frac{\partial^n \vec{u}}{\partial x^n}\right)$$

then it must satisfy $\left[D_t - \vec{F}', R\right] = 0,$ **(RO)**

where D_t is the total time derivative (i.e., the derivative taking (1) into account),

$$\vec{F}' = \sum_{i=0}^{n} \frac{\partial \vec{F}}{\partial \vec{u}_i} D^i$$

is the operator of *directional* (Frechet) derivative of $\vec{F}$, $D \equiv D_x$ is the total x-derivative, and $\vec{u}_i = \partial^i \vec{u} / \partial \vec{x}^i$.

Example: NLS ($\vec{u} = (\psi, \bar{\psi})^T$)

$$i\partial\psi/\partial t = -\partial^2\psi/\partial x^2 + \psi\bar{\psi}\psi,$$
$$-i\partial\bar{\psi}/\partial t = -\partial^2\bar{\psi}/\partial x^2 + \bar{\psi}\psi\bar{\psi}$$

and the recursion operator has the form

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D + \begin{pmatrix} -\psi \\ \bar{\psi} \end{pmatrix} \otimes D^{-1} \circ (\bar{\psi}, \psi),$$

i.e., $R\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} D(F) \\ -D(G) \end{pmatrix} + (D^{-1}(\bar{\psi}F + \psi G))\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

E.g. $R\begin{pmatrix} \psi_t \\ \bar{\psi}_t \end{pmatrix} = i\begin{pmatrix} \psi_{xxx} - 3\psi\bar{\psi}\psi_x \\ \bar{\psi}_{xxx} - 3\bar{\psi}\psi\bar{\psi}_x \end{pmatrix}$ is local.

But are $R^n\begin{pmatrix} \psi_t \\ \bar{\psi}_t \end{pmatrix}, n = 2, 3, \ldots,$ local too?

Example: NLS (nonlocal symmetries)

Consider the scaling symmetry

$$\begin{pmatrix} 2t\psi_t + x\psi_x + \psi \\ 2t\bar{\psi}_t + x\bar{\psi}_x + \bar{\psi} \end{pmatrix}$$

It is easily seen that acting by R on this symmetry yields

$$R\begin{pmatrix} 2t\psi_t + x\psi_x + \psi \\ 2t\bar{\psi}_t + x\bar{\psi}_x + \bar{\psi} \end{pmatrix} = \text{local terms} + \begin{pmatrix} -\psi \\ \bar{\psi} \end{pmatrix} w,$$

where $w = D^{-1}(\psi\bar{\psi})$ is a nonlocal variable.

Known results

Q: When a *nonlocal* recursion operator R produces an infinite hierarchy of *local* symmetries?

P.J. Olver 1988: $R = P_1 P_2^{-1}$, P_i are compatible Hamiltonian structures that satisfy some fairly restrictive requirements

V.E. Adler 1991: R originates from a zero-curvature representation of a special form

J.A. Sanders and J.-P. Wang 1998: R is hereditary, has a scaling symmetry and satisfies some further restrictions

Known limitations

- Not all recursion operators can be written as ratio of compatible Hamiltonian operators (C-*integrable systems*)

- ZCR is not of Adler's type or is not known yet (*e.g. many new integrable systems*) or does not exist at all (C-*integrable systems*)

- Some recursion operators have no scaling symmetry (*e.g. the Krichever–Novikov equation*)

Our goal

To find sufficient conditions for a *nonlocal* recursion operator to produce an infinite hierarchy of *local* symmetries **without appealing to**

- a specific form of the ZCR or Lax pair the RO originates from

- representation of the RO as ratio of compatible Hamiltonian operators

- existence of scaling symmetry

Basic definitions

Treat $x, t, \vec{u}, \vec{u}_1, \ldots$ as independent quantities.

A (scalar,vector, or matrix) function $f(x, t, \vec{u}, \vec{u}_1, \ldots)$ is **local** if $\exists k \in \mathbb{N} : \partial f / \partial \vec{u}_j = 0$ for all $j > k$.

$\mathcal{A} = \{f | f$ is local, scalar and locally analytic$\}$.

Notation: $\vec{u}_k = (u_k^1, \ldots, u_k^n)^T$, $\vec{u}_0 \equiv \vec{u}$.

The derivation of $\mathcal{A}$ (**total** x**-derivative**)

$$D = \partial/\partial x + \sum_{i=0}^{\infty} \vec{u}_{i+1} \partial/\partial \vec{u}_i$$

turns $\mathcal{A}$ into a differential algebra.

Variational and directional derivative

Variational derivative: $\frac{\delta}{\delta \vec{u}} \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} (-D)^i \frac{\partial}{\partial \vec{u}_i}$

Theorem: $f \in \mathcal{A}$ is in ImD iff $\delta f / \delta \vec{u} = 0$.

Directional derivative : $f' \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \frac{\partial f}{\partial \vec{u}_i} D^i$

for any local f (scalar, vector, or matrix).

Evolutionary vector fields

Let $\mathcal{V}$ be the Lie algebra of n-component vectors with entries from $\mathcal{A}$ (*evolutionary vector fields*) with the commutator

$$[\vec{K},\vec{H}] := \vec{H}'[\vec{K}] - \vec{K}'[\vec{H}],$$

and $\mathcal{V}^*$ be the (formal) dual of $\mathcal{V}$ (i.e., the elements of $\mathcal{V}^*$ are n-component columns with the entries from $\mathcal{A}$).

N.B.: n is the number of components of $\vec{u}$.

Evolution systems

Fix $\vec{F} \in \mathcal{V}$. Consider a dynamical system ($\vec{u}_0 = \vec{u}$)

$$\frac{\partial \vec{u}_i}{\partial t} = D^i(\vec{F}), \quad i = 0, 1, 2, \ldots, \tag{4}$$

and an associated derivations of $\mathcal{A}$:

$$D_{\vec{F}} = \sum_{i=0}^{\infty} D^i(\vec{F}) \frac{\partial}{\partial \vec{u}_i}, \quad \text{and} \quad D_t = \frac{\partial}{\partial t} + D_{\vec{F}}.$$

Eq.(4) is the proper equivalent, in our new setting, of the evolution system we started with, i.e.,

$$\frac{\partial \vec{u}}{\partial t} = \vec{F}\left(x, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \ldots, \frac{\partial^n \vec{u}}{\partial x^n}\right). \tag{1}$$

Lie derivative and hereditary operators

$$\forall \vec{Q} \in \mathcal{V} \;\; L_{\vec{Q}}(R) \stackrel{\text{def}}{=} [D_{\vec{Q}} - \vec{Q}', R] \equiv D_{\vec{Q}}(R) - [\vec{Q}', R].$$

Interpretation: $L_{\vec{Q}}(R) = 0$ iff R is a **recursion operator** for the system

$$\frac{\partial \vec{u}}{\partial \tau} = \vec{Q} \qquad \textbf{(important: } \tau \neq t\textbf{)}$$

Let $\mathcal{L} \subset \mathcal{V}$. An operator R is **hereditary on** $\mathcal{L}$ if

$$\forall \vec{Q} \in \mathcal{L} \quad L_{R(\vec{Q})}(R) = R \circ L_{\vec{Q}}(R)$$

If $\mathcal{L} = \mathcal{V}$ then R is said to be just *hereditary*. (Fuchssteiner & Fokas, Gelfand & Dorfman)

Weakly nonlocal operators

R is **weakly nonlocal** if it can be written as

$$R = \sum_{i=0}^{r} a_i D^i + \sum_{\alpha=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \vec{\gamma}_\alpha, \qquad (5)$$

where $a_i \in gl_n(\mathcal{A})$, $\vec{G}_\alpha \in \mathcal{V}$, $\vec{\gamma}_\alpha \in \mathcal{V}^*$, and $r \geq 0$.

Then for an $\vec{K} \in \mathcal{V}$ we have

$$R(\vec{K}) = \sum_{i=0}^{r} a_i D^i(\vec{K}) + \sum_{\alpha=1}^{p} (D^{-1}(\vec{\gamma}_\alpha \cdot \vec{K}))\vec{G}_\alpha$$

"$\cdot$" stands for the standard pairing of $\mathcal{V}^*$ and $\mathcal{V}$.

What is $[D_{\vec{Q}}, R]$ for a weakly nonlocal R?

Let $R = \sum\limits_{i=0}^{r} a_i D^i + \sum\limits_{\alpha=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \vec{\gamma}_\alpha,$

where $a_i \in gl_n(\mathcal{A})$, $\vec{G}_\alpha \in \mathcal{V}$, $\vec{\gamma}_\alpha \in \mathcal{V}^*$, and $r \geq 0$.

$$[D_{\vec{Q}}, R] = D_{\vec{Q}}(R) \stackrel{\text{def}}{=} \sum_{i=0}^{r} D_{\vec{Q}}(a_i) D^i$$
$$+ \sum_{\alpha=1}^{p} \left(D_{\vec{Q}}(\vec{G}_\alpha) \otimes D^{-1} \circ \vec{\gamma}_\alpha + \vec{G}_\alpha \otimes D^{-1} \circ D_{\vec{Q}}(\vec{\gamma}_\alpha) \right).$$

N.B. $D_t(R)$ is defined in a similar fashion.

Formal series, or what is $[\vec{F}', R]$?

$$gl_p(\mathcal{A})[\![D^{-1}]\!] = \left\{ \sum_{j=-\infty}^{q} b_j D^j, b_j \in gl_p(\mathcal{A}) \right\}$$

$GL_p(\mathcal{A})[\![D^{-1}]\!]$ is a (noncommutative) algebra w.r.t. $\circ$

$$aD^i \circ bD^j \stackrel{\text{def}}{=} \sum_{q=0}^{\infty} \frac{i(i-1)\cdots(i-q+1)}{q!} aD^q(b)D^{i+j-q}.$$

(extended by linearity to the whole of $gl_p(\mathcal{A})[\![D^{-1}]\!]$).

The commutator

$$[A, B] = A \circ B - B \circ A$$

turns $gl_p(\mathcal{A})[\![D^{-1}]\!]$ into a Lie algebra.

Our R (5) can be shown to belong to $gl_p(\mathcal{A})[\![D^{-1}]\!]$.

Weakly normal operators

A weakly nonlocal operator R of the form (5),

$$R = \sum_{i=0}^{r} a_i D^i + \sum_{\alpha=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \vec{\gamma}_\alpha,$$

where $a_i \in gl_n(\mathcal{A})$, $\vec{G}_\alpha \in \mathcal{V}$, $\vec{\gamma}_\alpha \in \mathcal{V}^*$, and $r \geq 0$, is **weakly normal** if

- $\vec{G}_\alpha$ are linearly independent over the field $\mathbb{T}$ of analytic functions of t
- $L_{\vec{G}_\alpha}(R) = 0$, $\alpha = 1, \ldots, p$
- $\forall \alpha = 1, \ldots, p \quad \exists \rho_\alpha \in \mathcal{A} : \vec{\gamma}_\alpha = \dfrac{\delta \rho_\alpha}{\delta \vec{u}}$

Main result

Theorem 1 *Let R be a weakly nonlocal and weakly normal operator of the form*

$$R = \sum_{i=0}^{r} a_i D^i + \sum_{\alpha=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \frac{\delta \rho_\alpha}{\delta \vec{u}},$$

where $a_i \in gl_n(\mathcal{A})$, $\vec{G}_\alpha \in \mathcal{V}$, $\rho_\alpha \in \mathcal{A}$, and $r \geq 0$.

Further assume that $\vec{Q} \in \mathcal{V}$ and R are such that

i) R is hereditary on span$(R^i(\vec{Q}), i = 0, 1, 2, \ldots)$,

ii) $L_{\vec{Q}}(R) = 0$ (i.e. R is an RO for $\vec{u}_\tau = \vec{Q}, \tau \neq t$)

iii) $R(\vec{Q})$ is local.

Then $\vec{Q}_j = R^j(\vec{Q})$ are local and commute for all $j = 0, 1, 2, \ldots$.

Example 1: generalized potential mKdV equation

Consider the recursion operator

$$R = D^2 + 2au_1^2 + \frac{4}{3}bu_1 + c - \frac{2}{3}(3au_1 + b)D^{-1} \circ u_2$$

of generalized potential modified KdV equation

$$u_t = u_3 + au_1^3 + bu_1^2 + cu_1,$$

where a, b, c are arbitrary constants. R meets the requirements of Theorem 1 for $\vec{Q} = u_1$, so all $\vec{Q}_j = R^j(\vec{Q})$, $j = 1, 2, \ldots$, are local and commute.

Example 2: generalized Harry Dym equation

Let $a, b = \text{const}$. The operator

$$R = \exp(-3(a+b)t)u^3 D^3 \circ u \circ D^{-1} \circ \frac{\exp((a+b)t)}{u^2}$$
$$= \exp(-2(a+b)t)(u^2 D^2 - uu_1 D + uu_2)$$
$$+ \exp(-3(a+b)t)u^3 u_3 D^{-1} \circ \exp((a+b)t)/u^2$$

is a hereditary recursion operator for

$$u_t = u^3 u_3 + axu_1 + bu.$$

Again, the requirements of Theorem 1 are met for $\vec{Q} = \exp(-3(a+b)t)u^3 u_3$, so all $\vec{Q}_j = R^j(\vec{Q})$, $j = 1, 2, \ldots$, are local and commute.

Example 3: Meshkov system $(n=2, u^1=u, u^2=v)$

A.G. Meshkov, *Fund. Prikl. Mat.* **12** (2006), No.7, 141–161:

$$
\begin{aligned}
u_t &= D\Big(u_2 + hv_1 + a(uv^2 - u^3)/2 \\
&\quad + buv/2 + k_1(3u^2 - v^2) + k_2 v\Big), \\
v_t &= D\Big(u_1 h + a(-u^2 v + v^3)/2 \\
&\quad + b(-u^2 + 3v^2)/4 + 2k_1 uv - k_2 u + k_3 v\Big),
\end{aligned}
$$

where $a, b, c, k_1, k_2, k_3 = \text{const}$,

$h = (av^2 + bv + c)^{1/2}$.

Example 3: Meshkov system II

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D_x^2 + \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} D_x + \begin{pmatrix} 4k_1 u - au^2 & -p_1 \\ p_2 & av^2 + bv + k_3 \end{pmatrix}$$

$$+a \begin{pmatrix} u_x \\ v_x \end{pmatrix} \otimes D_x^{-1} \circ (u, v) + (b/2) \begin{pmatrix} u_x \\ v_x \end{pmatrix} \otimes D_x^{-1} \circ (0, 1) \,.$$

where $p_1 = 2k_1 v - auv - k_2 - bu/2 - (2av + b) v_1/(2h)$,

$p_2 = 2k_1 v - auv - k_2 - bu/2 + (2av + b) v_1/(2h)$,

is a hereditary recursion operator for the Meshkov system.

Example 3: Meshkov system III

Again, the requirements of Theorem 1 are met for

$$\vec{Q} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix},$$

so all $\vec{Q}_j = R^j(\vec{Q})$, $j = 1, 2, \ldots$, are local and commute.

Formal adjoint

Let $$R = \sum_{i=0}^{r} a_i D^i + \sum_{\alpha=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \vec{\gamma}_\alpha,$$

where $a_i \in gl_n(\mathcal{A})$, $\vec{G}_\alpha \in \mathcal{V}$, $\vec{\gamma}_\alpha \in \mathcal{V}^*$, and $r \geq 0$.

The *(formal) adjoint* of R is

$$R^\dagger \overset{\text{def}}{=} \sum_{i=0}^{r} (-D)^i \circ a_i^T - \sum_{\alpha=1}^{p} \vec{\gamma}_\alpha \otimes D^{-1} \circ \vec{G}_\alpha.$$

“Adjoint” of Theorem 1

Theorem 2 *Let R be a hereditary, weakly non-local and weakly normal operator of the form*

$$R = \sum_{i=0}^{r} a_i D^i + \sum_{\alpha=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \frac{\delta \rho_\alpha}{\delta \vec{u}},$$

where $a_i \in gl_n(\mathcal{A})$, $\vec{G}_\alpha \in \mathcal{V}$, $\rho_\alpha \in \mathcal{A}$, and $r \geq 0$.

Further assume that $\rho \in \mathcal{A}$ and R are such that

i) $\delta(\delta\rho/\delta\vec{u} \cdot \vec{G}_\alpha)/\delta\vec{u} = 0$, $\alpha = 1, \ldots, p$

ii) $\exists \tilde{\rho} \in \mathcal{A} : R^\dagger(\delta\rho/\delta\vec{u}) = \delta\tilde{\rho}/\delta\vec{u}$.

Then $\vec{\zeta}_j = (R^\dagger)^j(\delta\rho/\delta\vec{u}) \in \mathcal{V}^*$ are local for all $j = 0, 1, 2, \ldots$.

Amenable operators

We shall say that a weakly nonlocal operator R of the form

$$R = \sum_{i=0}^{r} a_i D^i + \sum_{\alpha=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \frac{\delta \rho_\alpha}{\delta \vec{u}},$$

where $a_i \in gl_n(\mathcal{A})$, $\vec{G}_\alpha \in \mathcal{V}$, $\rho_\alpha \in \mathcal{A}$, and $r \geq 0$, is **amenable** if it is hereditary (on the whole $\mathcal{V}$), weakly nonlocal, weakly normal,

and for all $\alpha = 1, \ldots, p$

i) $\vec{Q} = \vec{G}_\alpha$ satisfy the conditions of Theorem 1

ii) $\rho = \rho_\alpha$ satisfy the conditions of Theorem 2.

Products of weakly nonlocal operators

Let
$$\begin{aligned} R &= \sum_{i=0}^{r} a_i D^i + \sum_{j=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \vec{\gamma}_\alpha, \\ \tilde{R} &= \sum_{i=0}^{\tilde{r}} b_i D^i + \sum_{\alpha=1}^{\tilde{p}} \vec{K}_\alpha \otimes D^{-1} \circ \vec{\zeta}_\alpha. \end{aligned} \tag{6}$$

where $\vec{G}_\alpha, \vec{\gamma}_\alpha, \vec{K}_\alpha, \vec{\zeta}_\alpha \in \mathcal{A}^n$, $r, \tilde{r} \geq 0$.

Enriques, Orlov & Rubtsov (1993):

$$\begin{aligned} (R \circ \tilde{R})_- = & \sum_{\alpha=1}^{\tilde{p}} R(\vec{K}_\alpha) \otimes D^{-1} \circ \vec{\zeta}_\alpha \\ + & \sum_{\alpha=1}^{p} \vec{G}_\alpha \otimes D^{-1} \circ \tilde{R}^\dagger(\vec{\gamma}_\alpha). \end{aligned} \tag{7}$$

Products of weakly nonlocal operators II

Repeatedly using (7) yields for $n, m = 1, 2, \ldots$

$$R^n_- = \sum_{j=0}^{n-1} \sum_{\alpha=1}^{p} C^j_{n-1} R^{n-1-j}(\vec{G}_\alpha) \otimes D^{-1} \circ R^{\dagger\, j}(\vec{\gamma}_\alpha),$$

$$R^{\dagger\, n}_- = -\sum_{j=0}^{n-1} \sum_{\alpha=1}^{p} C^j_{n-1} R^{\dagger\, j}(\vec{\gamma}_\alpha) \otimes D^{-1} \circ R^{n-1-j}(\vec{G}_\alpha),$$

$$(R^n \circ \tilde{R}^m)_- = \sum_{j=0}^{n-1} \sum_{\alpha=1}^{p} C^j_{n-1} R^{n-1-j}(\vec{G}_\alpha) \otimes D^{-1} \circ \tilde{R}^{\dagger\, m} R^{\dagger\, j}(\vec{\gamma}_\alpha)$$

$$+ \sum_{j=0}^{m-1} \sum_{\alpha=1}^{\tilde{p}} C^j_{m-1} R^n \tilde{R}^{m-1-j}(\vec{K}_\alpha) \otimes D^{-1} \circ R^{\dagger\, j}(\vec{\zeta}_\alpha).$$

Here $C^j_{n-1} = \dfrac{(n-1)!}{(n-1-j)!j!}$.

Maltsev–Novikov conjecture

A.Ya. Maltsev & S.P. Novikov,
Physica D **156**(2001),53–80:

If R, P, and J are respectively recursion, Hamiltonian and symplectic operator, of an integrable system, and they are weakly nonlocal, then so are R^n, $R^n \circ P$ and $R^{\dagger\, n} \circ J$ for all $n = 1, 2, \ldots$.

Proving the MN conjecture

Proposition 1 *Suppose that R is amenable, and let $P : \mathcal{V}^* \to \mathcal{V}$, $J : \mathcal{V} \to \mathcal{V}^*$, be* differential *(rather than weakly nonlocal) operators.*

Then R^k, $(R^\dagger)^k$, $R^k \circ P$, and $(R^\dagger)^k \circ J$, are weakly nonlocal for all $k = 0, 1, 2, \ldots$.

When P and J are weakly nonlocal, we also can find sufficient conditions under which $R^k \circ P$, and $(R^\dagger)^k \circ J$ are weakly nonlocal for all $k = 0, 1, 2, \ldots$

(see A. Sergyeyev, J. Phys. A **38** (2005), no. 15, 3397–3407, arXiv:nlin.SI/0410049)

Main results

We found new *sufficient conditions for locality* of hierarchies generated by weakly nonlocal hereditary recursion operators

Main advantage: we need the recursion operator alone, *no additional structures* required.

In particular, *we avoid* the need to know

i) a ZCR or a Lax pair

ii) a scaling symmetry

iii) bi-Hamiltonian structure.

Important application:

proof of the Maltsev–Novikov conjecture

References

1. A. Sergyeyev, Why nonlocal recursion operators produce local symmetries: new results and applications, J. Phys. A **38** (2005), 3397–3407, arXiv:nlin.SI/0410049

2. A. Sergyeyev, Integrable systems in (1+1) dimensions: local symmetries vs. nonlocal recursion operators, in preparation