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Stochastic theory of nonlinear auto-oscillator: Spin-torque nano-oscillator

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In collaboration with:

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- **Joo-Von Kim** (Universite Paris-Sud, Orsay , France)

Outline

- Introduction
  - Linear and nonlinear auto-oscillators
  - Spin-torque oscillator (STO)
- Stochastic model of a nonlinear auto-oscillator
- Generation linewidth of a nonlinear auto-oscillator
  - Theoretical results
  - Comparison with experiment (STO)
- Summary

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Models of (weakly perturbed) auto-oscillators

Auto-oscillator – autonomous dynamical system with stable limit cycle

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- Phase model

Unperturbed system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \qquad \mathbf{x} = \mathbf{X}_0(\omega_0 t + \phi_0) \qquad \mathbf{X}_0(\phi + 2\pi) = \mathbf{X}_0(\phi)$$

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Perturbed system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) + \mathbf{f}(t, \mathbf{x}) \qquad \mathbf{x} = \mathbf{X}_0(\phi(t)) + \mathbf{y}(t)$$

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$$\boxed{\frac{d\phi}{dt} - \omega_0 = f(t, \phi)}$$

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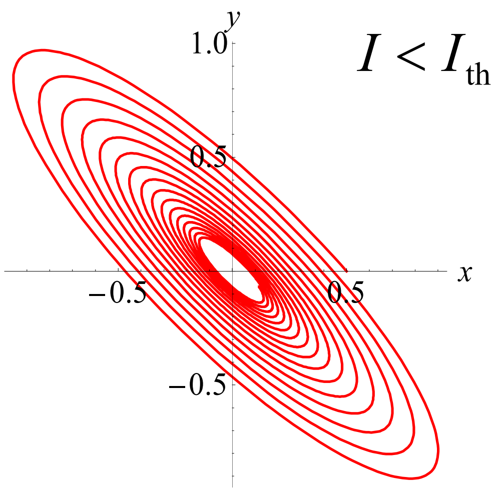
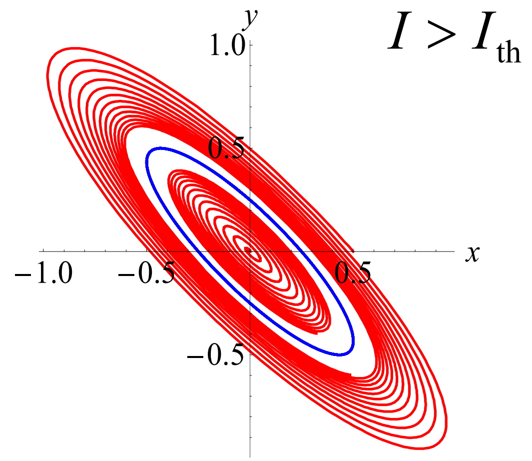
$$\frac{d\phi}{dt} - \omega_0 = f(t,\phi)$$

- Weakly nonlinear model





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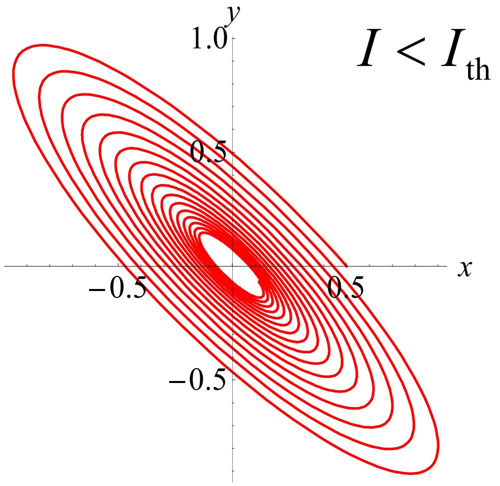

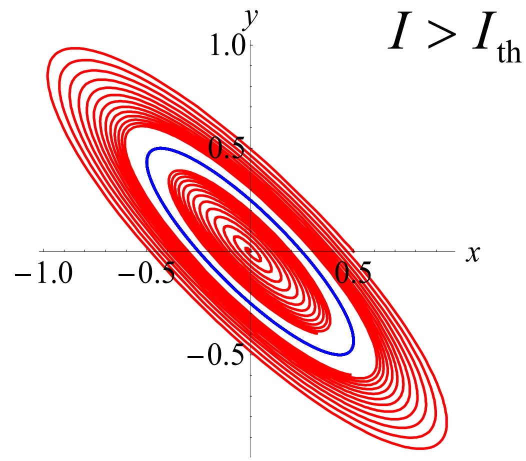

Complex amplitude: $c = x + i\,\alpha\,y$

$$\frac{dc}{dt} + i(\omega_0 + N|c|^2)c + [\alpha(I - I_{\text{th}}) - \beta|c|^2]c = f(t,c)$$

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Linear and nonlinear auto-oscillators

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Nonlinear auto-oscillator: frequency depends on the amplitude

Linear and nonlinear auto-oscillators

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$$\frac{d\phi}{dt} - \omega_0 = f(t,\phi)$$

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$$\frac{dc}{dt} + i(\omega_0 + N|c|^2)c + [\alpha(I - I_{\text{th}}) - \beta|c|^2]c = f(t,c)$$

All conventional auto-oscillators are linear.

$$|N| << \beta$$

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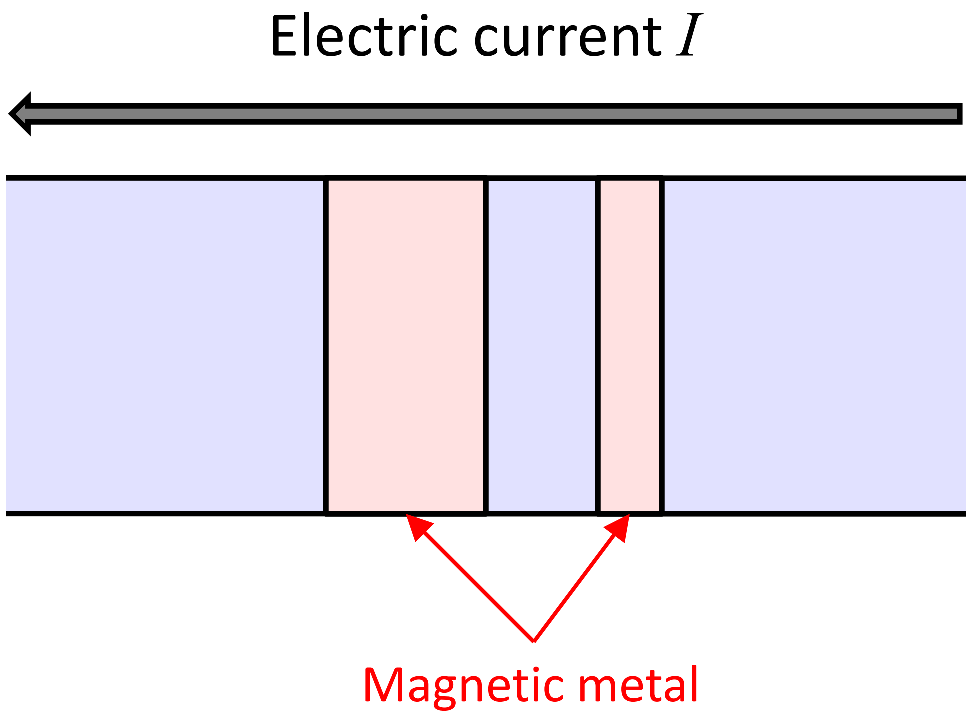
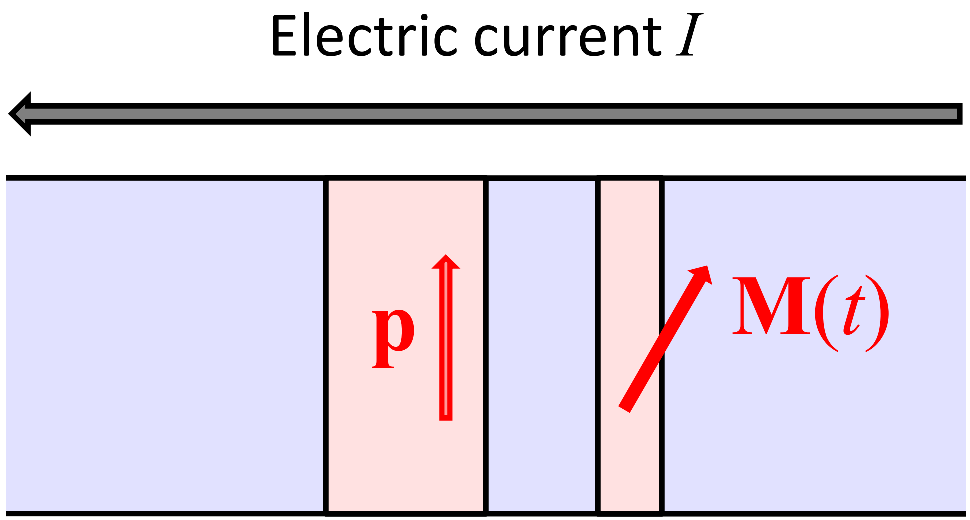
Spin-torque oscillator



Ferromagnetic metal – strong self-interaction – **strong nonlinearity**

Electric current – compensation of dissipation – **auto-oscillatory dynamics**

Spin-torque oscillator



Spin-transfer torque:

$$\left(\frac{d\mathbf{M}}{dt}\right)_{\mathrm{S}} = \mathbf{T}_{\mathrm{S}} = \frac{\sigma I}{M_0}\left[\mathbf{M}\times\left[\mathbf{M}\times\mathbf{p}\right]\right]$$

$$\sigma = \frac{\varepsilon\, g\, \mu_{\mathrm{B}}}{2\, e}\frac{1}{M_0 V}$$

J. Slonczewski, J. Magn. Magn. Mat. **159**, L1 (1996)
L. Berger, Phys. Rev. B. **54**, 9353 (1996)

Equation for magnetization

Landau-Lifshits-Gilbert-Slonczewski equation:

$$\frac{d\mathbf{M}}{dt} = \gamma[\mathbf{H}_{\mathrm{eff}} \times \mathbf{M}] + \mathbf{T}_{\mathrm{G}} + \mathbf{T}_{\mathrm{S}}$$

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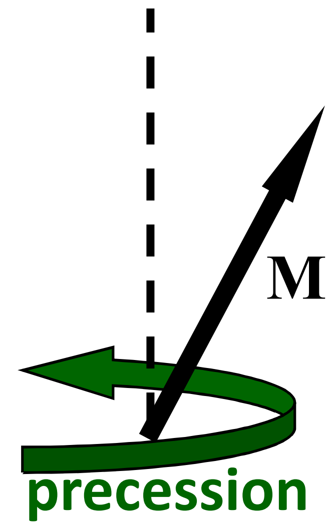

$\gamma[\mathbf{H}_{\text{eff}} \times \mathbf{M}]$ – conservative torque (precession)

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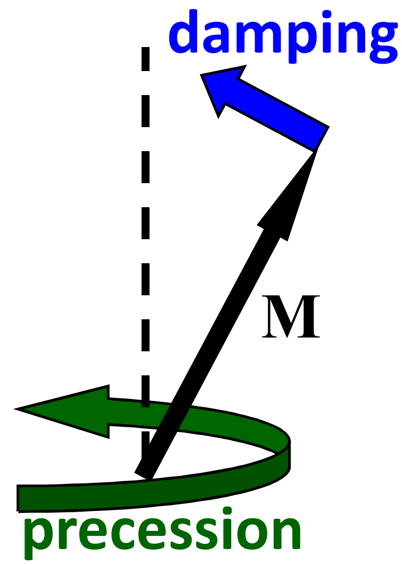

$\gamma[\mathbf{H}_{\mathrm{eff}} \times \mathbf{M}]$ – conservative torque (precession)

$$\mathbf{T}_{\mathrm{G}} = -\frac{\alpha_{\mathrm{G}}\gamma}{M_0}[\mathbf{M} \times [\mathbf{M} \times \mathbf{H}_{\mathrm{eff}}]]$$ – dissipative torque (**positive** damping)

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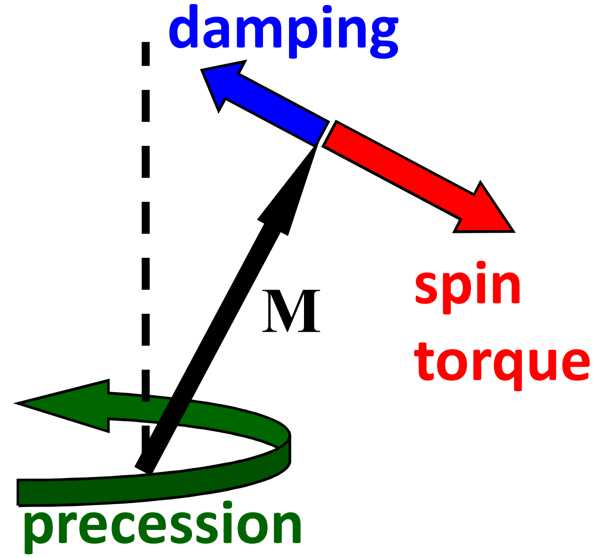

$\gamma[\mathbf{H}_{\text{eff}} \times \mathbf{M}]$ – conservative torque (precession)

$$\mathbf{T}_{\text{G}} = -\frac{\alpha_{\text{G}}\gamma}{M_0}[\mathbf{M} \times [\mathbf{M} \times \mathbf{H}_{\text{eff}}]]$$ – dissipative torque (**positive** damping)

$$\mathbf{T}_{\text{S}} = +\frac{\sigma I}{M_0}[\mathbf{M} \times [\mathbf{M} \times \mathbf{p}]]$$ – spin-transfer torque (**negative** damping)

Equation for magnetization

Landau-Lifshits-Gilbert-Slonczewski equation:

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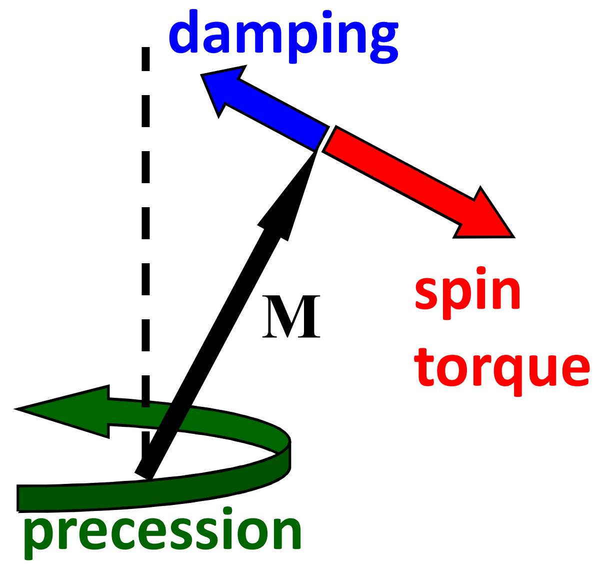
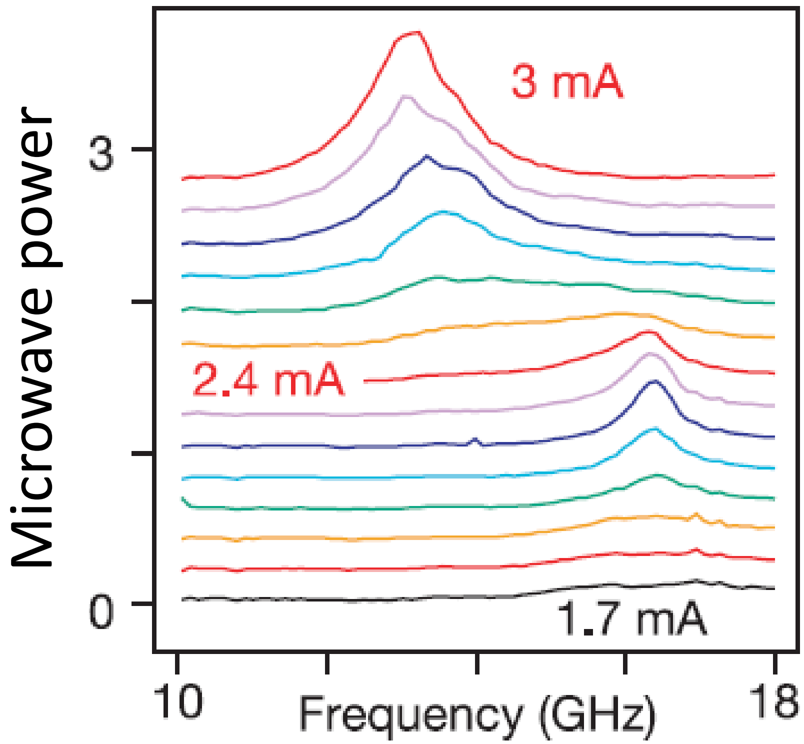


When negative current-induced damping compensates natural magnetic damping, self-sustained precession of magnetization starts.

Experimental studies of spin-torque oscillators



S.I. Kiselev *et al.*, Nature **425**, 380 (2003)

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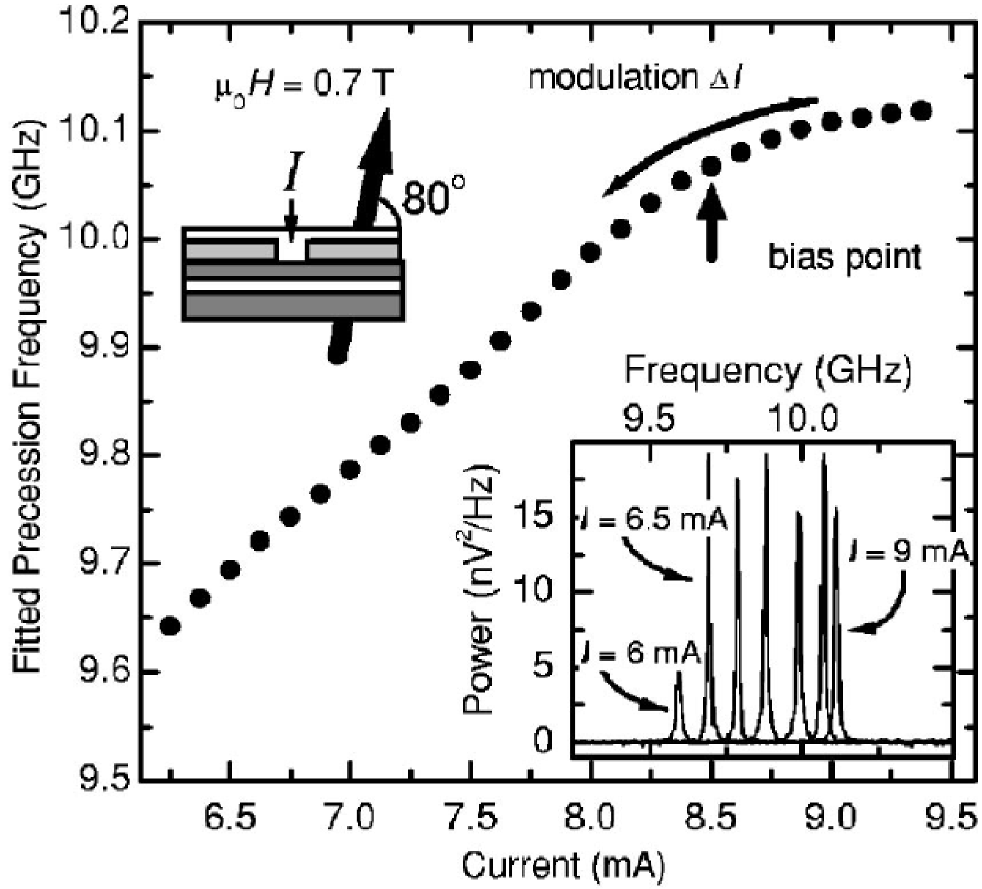
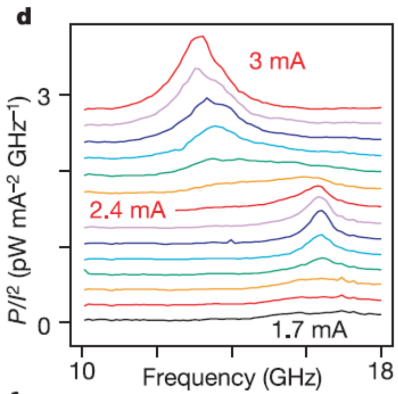


M.R. Pufall *et al.*, Appl. Phys. Lett. **86**, 082506 (2005)

Specific features of spin-torque oscillators



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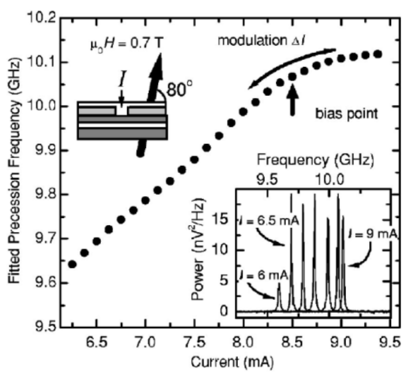

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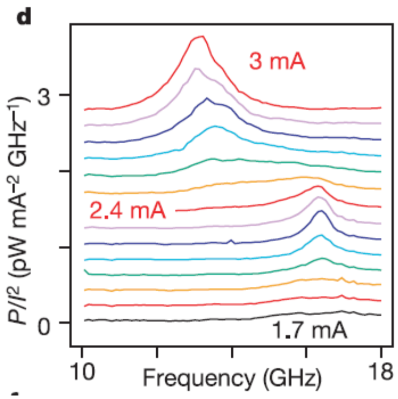

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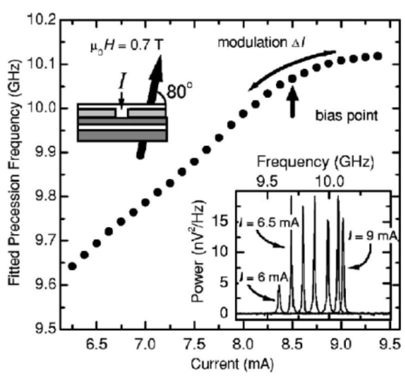

M.R. Pufall *et al.*, Appl. Phys. Lett. **86**, 082506 (2005)

- Strong influence of thermal noise (spin-torque **NANO**-oscillator)

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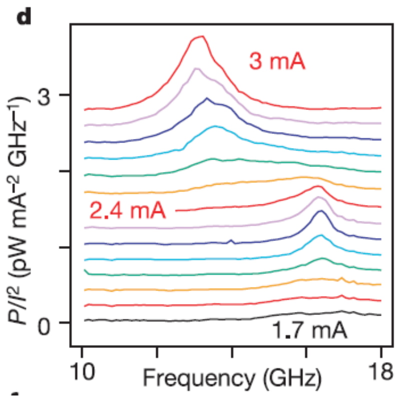

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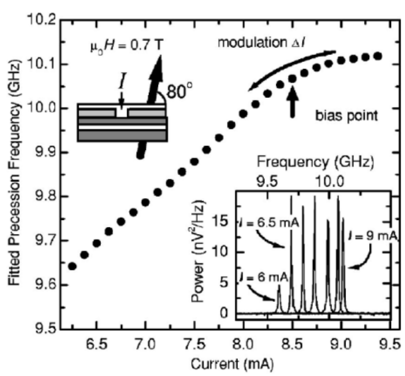

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- Strong influence of thermal noise (spin-torque **NANO**-oscillator)

- Strong frequency nonlinearity

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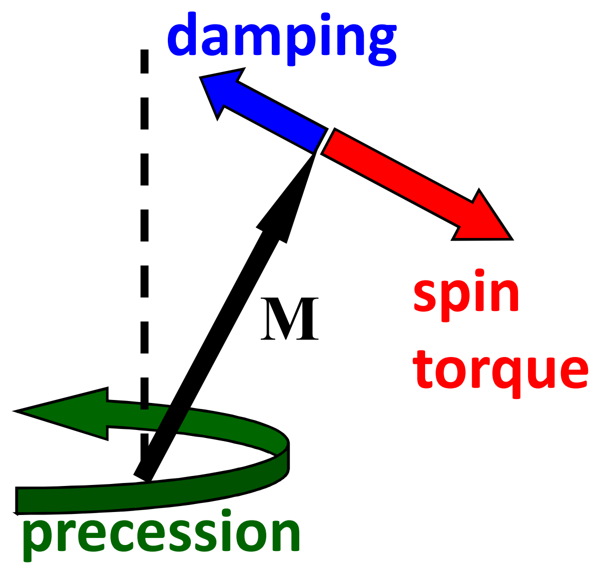
Nonlinear oscillator model

$$\frac{dc}{dt} + i\omega(p)c + \Gamma_{+}(p)c - \Gamma_{-}(p)c = 0$$

$p = |c|^2$ – oscillation **power**

$\phi = \arg(c)$ – oscillation **phase**



Nonlinear oscillator model

precession

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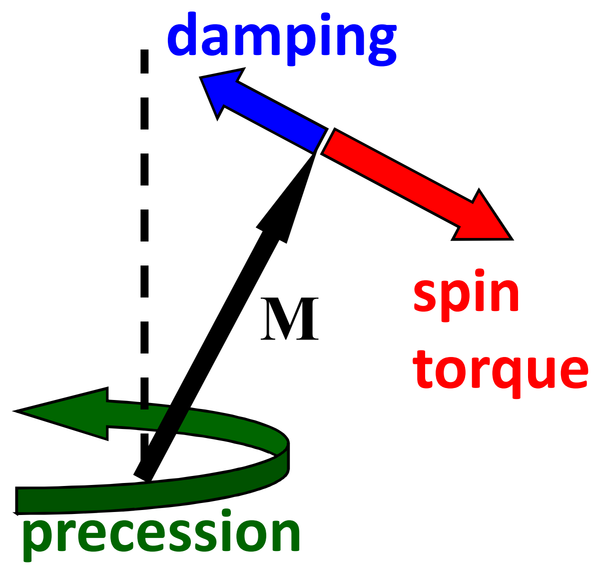

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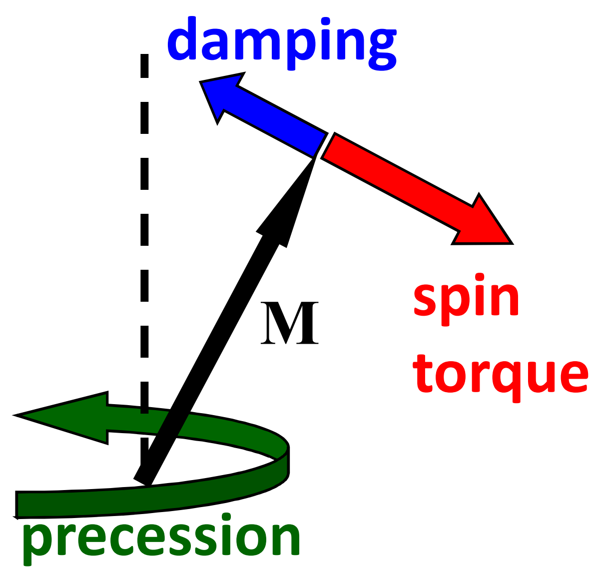

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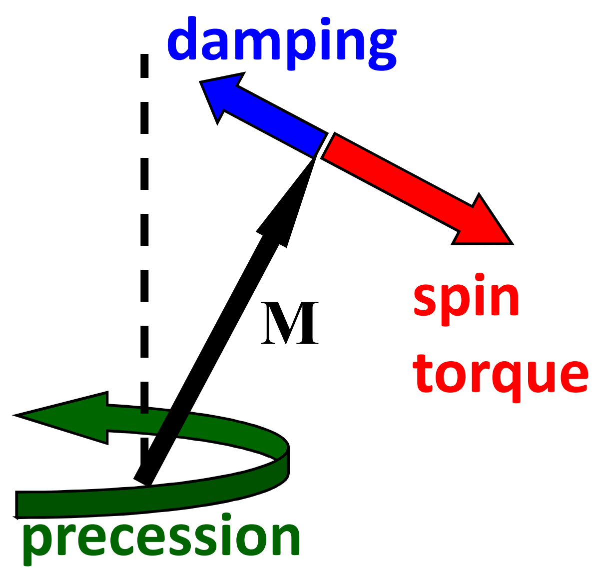

How to find parameters of the model?

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$$\frac{dc}{dt} + i\omega(p)c + \Gamma_+(p)c - \Gamma_-(p)c = 0$$

- Start from the Landau-Lifshits-Gilbert-Slonczewski equation
- Rewrite it in canonical coordinates
- Perform weakly-nonlinear expansion
- Diagonalize linear part of the Hamiltonian
- Perform renormalization of non-resonant three-wave processes
- Take into account only resonant four-wave processes
- Average dissipative terms over “fast” conservative motion
- and obtain analytical results for

$$\omega(p) \qquad \Gamma_+(p) \qquad \Gamma_-(p)$$

Deterministic theory

$$\frac{dc}{dt} + i\omega(p)c + \Gamma_+(p)c - \Gamma_-(p)c = 0$$

$p = |c|^2$ – oscillation power

Stationary solution: $c(t) = \sqrt{p_0}\exp(-i\omega_g t + i\phi_0)$

Stationary power is determined from the condition of vanishing **total** damping:

$$\Gamma_+(p_0) = \Gamma_-(p_0)$$

Stationary frequency is determined by the oscillation power:

$$\omega_g = \omega(p_0)$$

Deterministic theory: Spin-torque oscillator

Stationary power is determined from the condition of vanishing **total** damping:

$$\Gamma_+(p_0) = \Gamma_-(p_0)$$

$$p_0 = \frac{\zeta - 1}{\zeta + Q}$$

$\zeta = I/I_{\rm th}$ – supercriticality parameter

$I_{\rm th} = \Gamma_0/\sigma$ – threshold current

Stationary frequency is determined by the oscillation power:

$$\omega_{\rm g} = \omega(p_0)$$

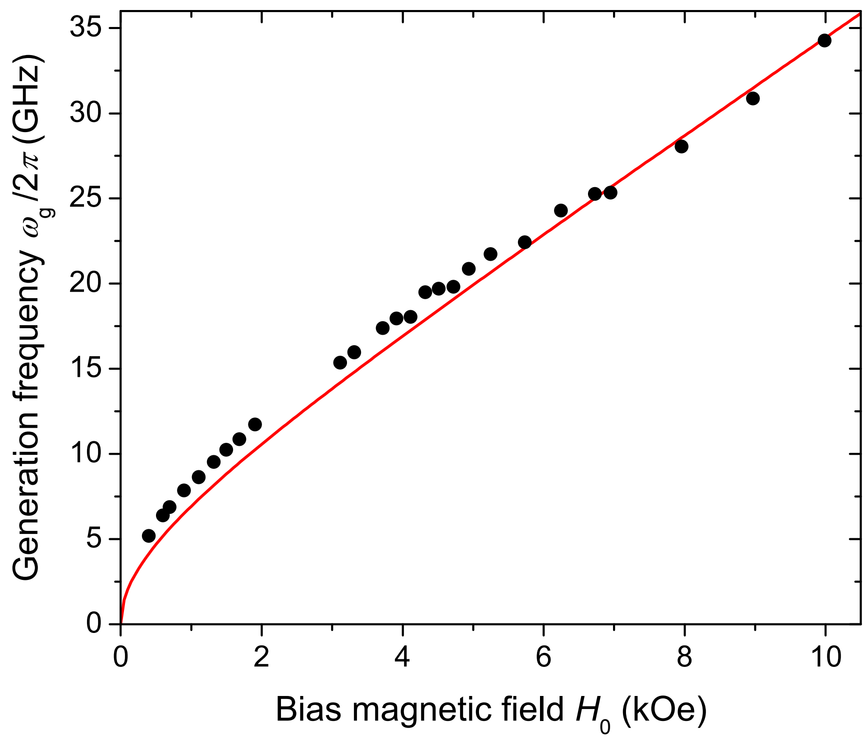
$$\omega_{\rm g} = \omega_0 + N p_0 = \omega_0 + N\frac{\zeta - 1}{\zeta + Q}$$

ω_0 – FMR frequency

N – nonlinear frequency shift

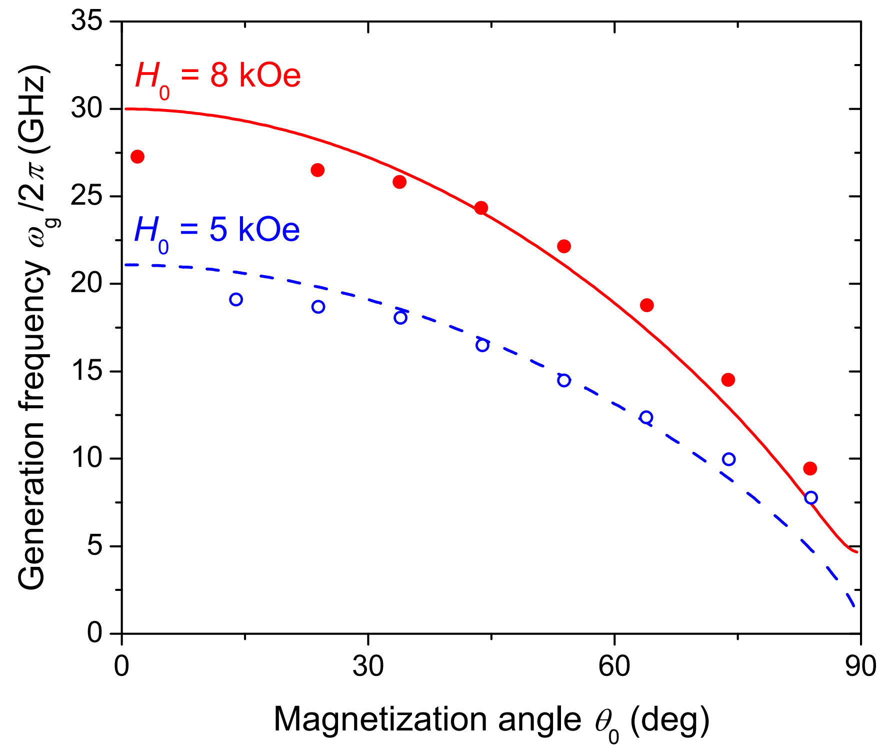
Generated frequency

Frequency as function of applied field



Experiment: W.H. Rippard *et al.*, Phys. Rev. Lett. **92**, 027201 (2004)

Frequency as function of magnetization angle



Experiment: W.H. Rippard *et al.*, Phys. Rev. B **70**, 100406 (2004)

Stochastic nonlinear oscillator model

Stochastic Langevin equation:

$$\frac{dc}{dt} + i\omega(p)c + \Gamma_+(p)c - \Gamma_-(p)c = \sqrt{D_\mathrm{n}} f_\mathrm{n}(t)$$

Random thermal noise:

$$\left\langle f_\mathrm{n}(t) f_\mathrm{n}^*(t') \right\rangle = 2\delta(t - t')$$

$$D_\mathrm{n}(p) = \Gamma_+(p)\eta(p) = \Gamma_+(p)\frac{k_\mathrm{B}T}{\lambda\omega(p)}$$

η – thermal equilibrium power of oscillations: $\left\langle |c|^2 \right\rangle_{\Gamma_-=0} = \left\langle p \right\rangle_{\Gamma_-=0} = \eta$

λ – scale factor that relates energy and power: $E(p) = \lambda \int \omega(p)dp$

Fokker-Planck equation for a nonlinear oscillator

Probability distribution function (PDF) $P(t, p, \phi)$ describes probability that oscillator has the power $p = |c|^2$ and the phase $\phi = \arg(c)$ at the moment of time t.

$$\frac{dP}{dt} - \frac{\partial}{\partial p}\{2p[\Gamma_+(p) - \Gamma_-(p)]P\} - \omega(p)\frac{\partial P}{\partial \phi} = \frac{\partial}{\partial p}\left[2pD_{\mathrm{n}}(p)\frac{\partial P}{\partial p}\right] + \frac{D_{\mathrm{n}}(p)}{2p}\frac{\partial^2 P}{\partial \phi^2}$$

Deterministic terms describe noise-free dynamics of the oscillator

Stochastic terms describe thermal diffusion in the phase space

$$D_{\mathrm{n}}(p) = \Gamma_+(p)\eta(p) = \Gamma_+(p)\frac{k_{\mathrm{B}}T}{\lambda\omega(p)}$$

Stationary PDF

Stationary probability distribution function P_0:

- does not depend on the time t (by definition);
- does not depend on the phase ϕ (by phase-invariance of FP equation).

Ordinary differential equation for $P_0(p)$:

$$-\frac{d}{dp}\left\{2p\left[\Gamma_+(p)-\Gamma_-(p)\right]P_0\right\}=\frac{d}{dp}\left[2pD_\mathrm{n}(p)\frac{dP_0}{dp}\right]$$

Stationary PDF for an arbitrary auto-oscillator:

$$P_0(p)=Z_0\exp\left[-\frac{\lambda}{k_\mathrm{B}T}\int_0^p\omega(p')\left(1-\frac{\Gamma_-(p')}{\Gamma_+(p')}\right)dp'\right]$$

Constant Z_0 is determined by the normalization condition $\int_0^\infty P_0(p)dp=1$

Analytical results for STO

"Material functions" for the case of STO:

$$\Gamma_+(p) = \Gamma_G(1 + Qp) \qquad \Gamma_-(p) = \sigma I(1 - p) \qquad \zeta = I/I_{th} = \sigma I/\Gamma_G$$

Analytical expression for the stationary PDF:

$$P_0(p) = \frac{Q}{(1+Qp)^\beta} \frac{\exp\left[-(\zeta + Q)(1+Qp)/Q^2\eta\right]}{E_\beta\left((\zeta+Q)/Q^2\eta\right)}$$

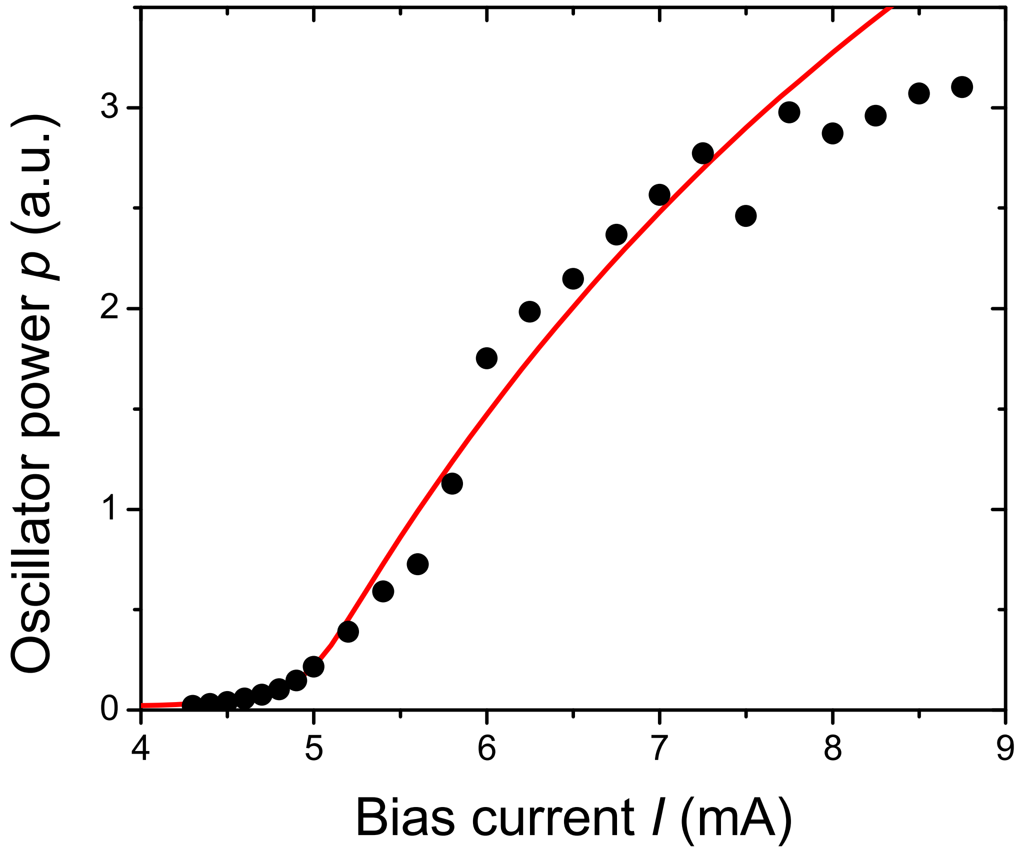
Analytical expression for the mean oscillation power

$$\bar{p} = \frac{Q\eta}{\zeta + Q}\left[1 + \frac{\exp\left(-(\zeta+Q)/Q^2\eta\right)}{E_\beta\left((\zeta+Q)/Q^2\eta\right)}\right] + \frac{\zeta - 1}{\zeta + Q}$$

Here $\beta = (1+Q)\zeta/Q^2\eta$

$E_\beta(x) = \int_1^\infty e^{-xt} t^{-\beta} dt$ – exponential integral function

Power in the near-threshold region



Experiment: Q. Mistral *et al.*, Appl. Phys. Lett. **88**, 192507 (2006).

Theory: V. Tiberkevich *et al.*, Appl. Phys. Lett. **91**, 192506 (2007).

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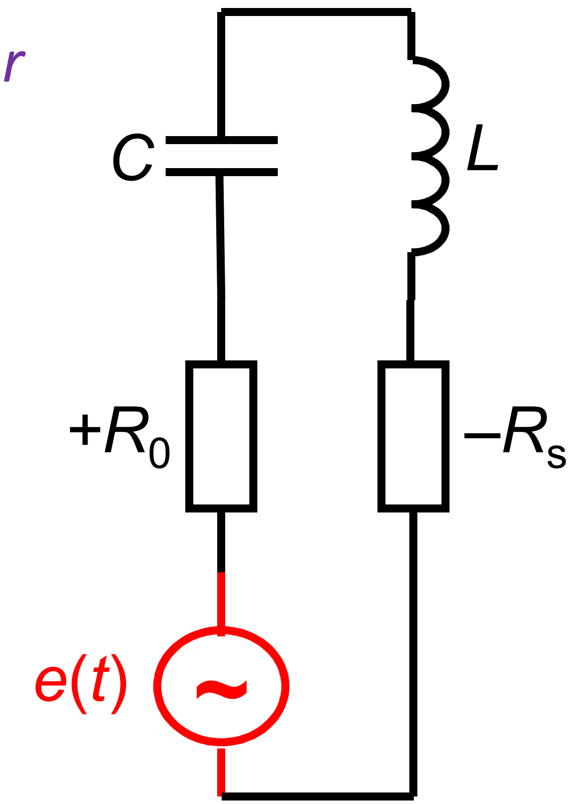
Linewidth of a conventional oscillator

Classical result (see, e.g., A. Blaquiere, *Nonlinear System Analysis* (Acad. Press, N.Y., 1966)):

Full linewidth of an electrical oscillator

$$2\Delta\omega = \frac{k_{\mathrm{B}}T\ R_0\omega_0^2}{U_0^2}$$



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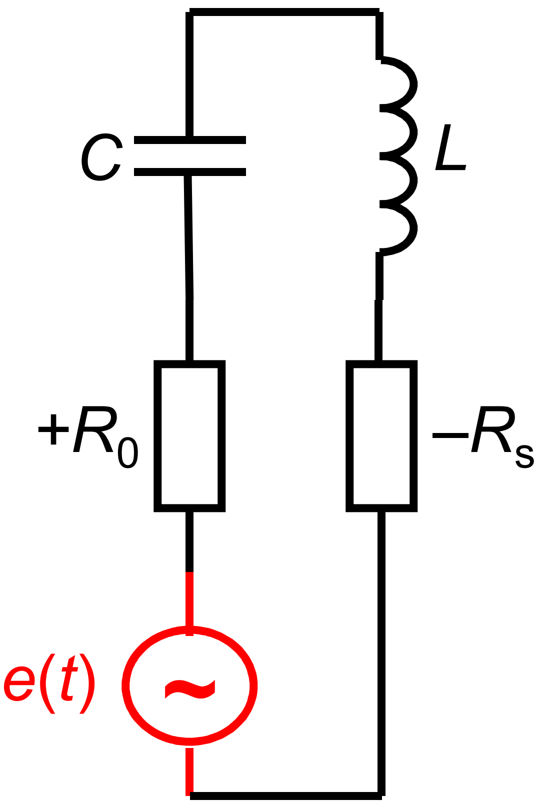


Physical interpretation of the classical result

$$2\Delta\omega = \Gamma_0 \frac{k_B T}{E_0}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Frequency

$$\Gamma_0 = \frac{R_0}{2L}$$

Equilibrium half-linewidth

$$E_0 = \frac{1}{2} C U_0^2$$

Oscillation energy

Linewidth of a laser

Quantum limit for the linewidth of a single-mode laser (Schawlow-Townes limit)
[A.L. Schawlow and C.H. Townes, Phys. Rev. **112**, 1940 (1958)]

$$2\Delta\omega = \frac{\hbar\omega_0\Gamma_0^2}{2P_{\mathrm{out}}}$$

Linewidth of a laser

Quantum limit for the linewidth of a single-mode laser (Schawlow-Townes limit)
[A.L. Schawlow and C.H. Townes, Phys. Rev. **112**, 1940 (1958)]

$$2\Delta\omega = \frac{\hbar\omega_0 \Gamma_0^2}{2P_{\mathrm{out}}}$$

Physical interpretation of the classical result

$$2\Delta\omega = \Gamma_0 \frac{k_{\mathrm{B}}T}{E_0}$$

$$k_{\mathrm{B}}T = \frac{\hbar\omega_0}{2}$$

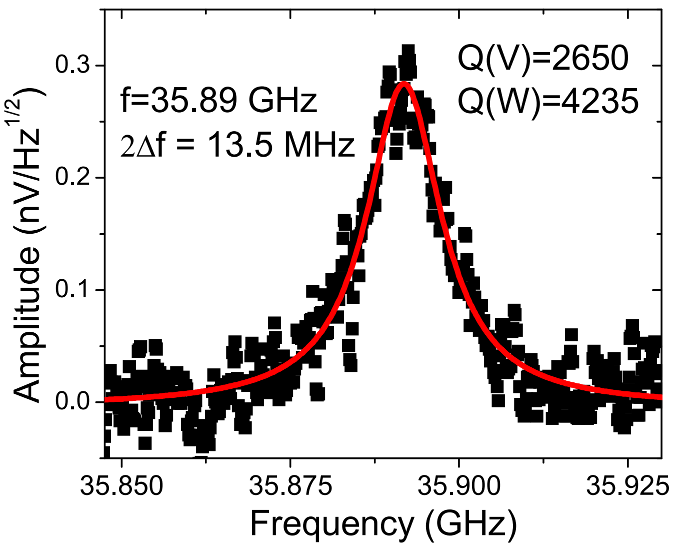
Effective thermal energy

$$E_0 = P_{\mathrm{out}} / \Gamma_0$$

Oscillation energy

Linewidth of a spin-torque oscillator

Experiment by W.H. Rippard *et al.*, *DARPA Review* (2004).



Theoretical result [J.-V. Kim, PRB **73**, 174412 (2006)]:

$$2\Delta\omega \approx \alpha_G \frac{\partial \omega_0}{\partial H} \frac{k_B T}{M_0 V_{eff} p_0}$$

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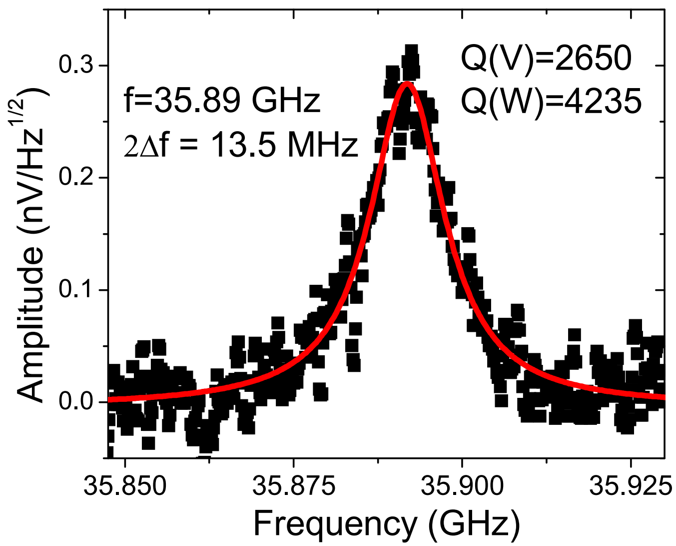

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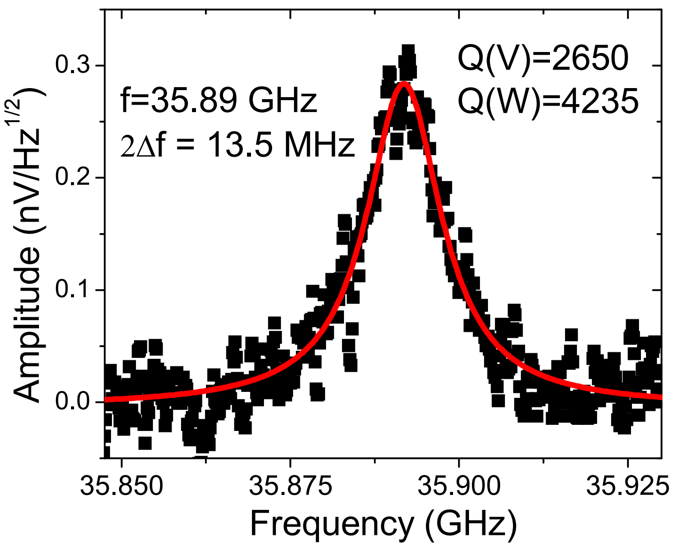

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Physical interpretation:

$$2\Delta\omega = \Gamma_0 \frac{k_B T}{E_0}$$

$$2\Delta f_{exp} = 2\Delta\omega / 2\pi = 13.5 \text{ MHz}$$

$$2\Delta f_{theor} = 2\Delta\omega / 2\pi = 0.35 \text{ MHz}$$

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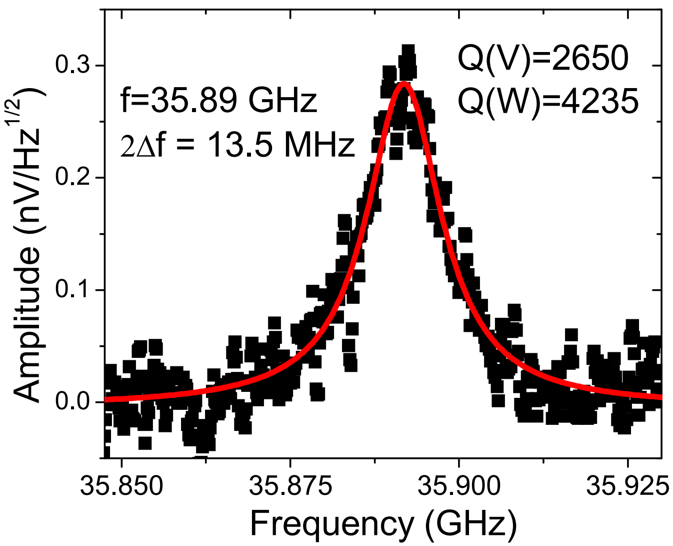

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This theory does assumes that the frequency is constant

$$\omega(p) = \omega_0 = \text{const}$$

How to calculate linewidth?

Power spectrum $S(\Omega)$ is the Fourier transform of the autocorrelation function $K(\tau)$:

$$S(\Omega) = \int K(\tau) e^{i\Omega\tau} d\tau \qquad K(\tau) = \langle c(t+\tau) c^*(t) \rangle$$

two-time function

Autocorrelation function $K(\tau)$ **can not** be found from the **stationary** PDF. Ways to proceed:

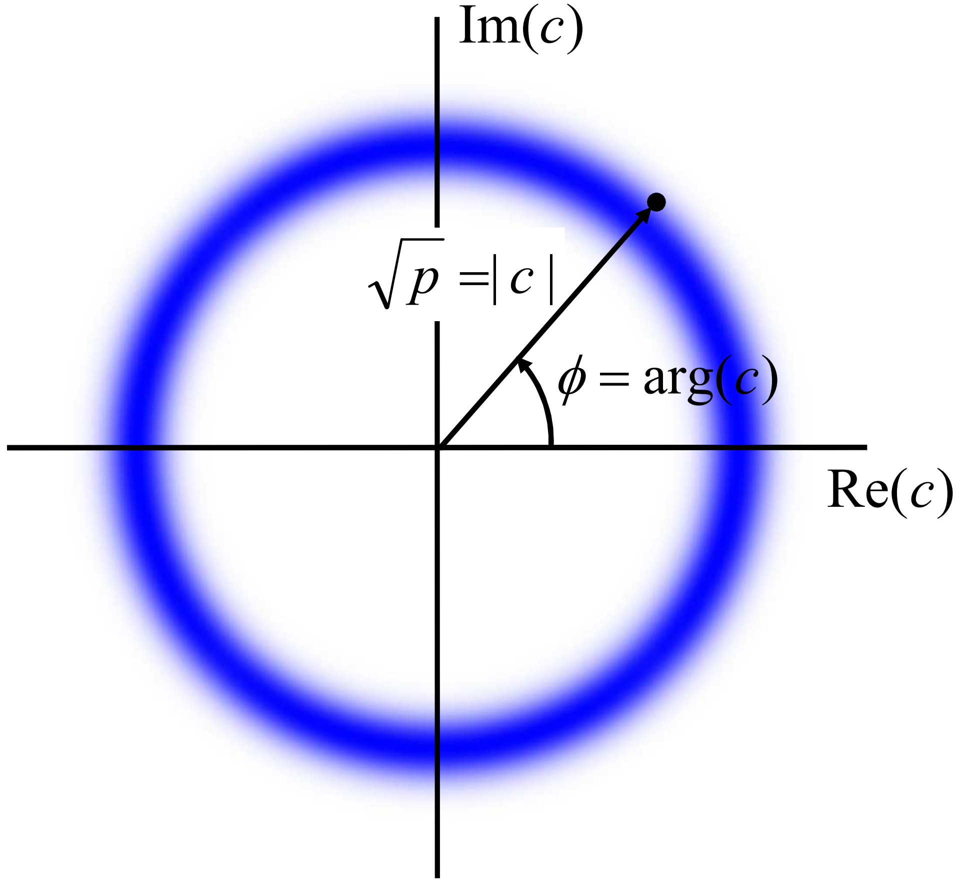
- Solve non-stationary Fokker-Planck equation [J.-V. Kim *et al.*, Phys. Rev. Lett. **100**, 167201 (2008)];

- Analyze stochastic Langevin equation.

$$\frac{dc}{dt} + i\omega(p)c + \Gamma_+(p)c - \Gamma_-(p)c = \sqrt{D_n} f_n(t)$$

Power and phase fluctuations

Two-dimensional plot of stationary PDF



Power fluctuations are weak

$$\frac{\Delta p}{p_0} \approx \sqrt{\frac{k_B T}{E(p_0)}} << 1$$

Oscillations – phase-modulated process:

$$c(t) = \sqrt{p(t)}e^{i\phi(t)} \approx \sqrt{p_0}e^{i\phi(t)}$$

Autocorrelation function:

$$K(\tau) \approx p_0 \langle \exp[i\phi(\tau) - i\phi(0)] \rangle$$

Linewidth of any oscillator is determined by the **phase fluctuations**

Linearized power-phase equations

Stochastic Langevin equation: $$\frac{dc}{dt} + i\omega(p)c + \Gamma_{+}(p)c - \Gamma_{-}(p)c = \sqrt{D_{\mathrm{n}}} f_{\mathrm{n}}(t)$$

Linearized power-phase equations

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Power-phase ansatz: $$c(t) = \sqrt{p_0 + \delta p(t)}\; e^{i\phi(t)} \qquad |\delta p| << p_0$$

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Stochastic power-phase equations:

$$\frac{d\delta p}{dt} + 2\Gamma_{\text{eff}}\, \delta p = 2\sqrt{D_n p_0}\, \text{Re}\left[f_n(t) e^{-i\phi}\right]$$

$$\frac{d\phi}{dt} + \omega(p_0) = \sqrt{\frac{D_n}{p_0}}\, \text{Im}\left[f_n(t) e^{-i\phi}\right] + N\, \delta p$$

Linear system of equations.
Can be solved in general case.

Effective damping: $\Gamma_{\text{eff}} = (G_+ - G_-) p_0$ $\quad G_\pm = d\Gamma_\pm(p)/dp$

Nonlinear frequency shift coefficient: $N = d\omega(p)/dp$

Nonlinear frequency shift creates additional source of the phase noise

Power fluctuations

Correlation function for power fluctuations:

$$\left\langle \delta p(t+\tau)\delta p(t) \right\rangle = p_0^2 \left(\frac{\Gamma_+}{\Gamma_{\mathrm{eff}}} \right) \frac{k_{\mathrm{B}} T}{E(p_0)} e^{-2\Gamma_{\mathrm{eff}} |\tau|}$$

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Power fluctuations are small

Non-zero correlation time of power fluctuations

Additional noise source $-N\,\delta p$ for the phase fluctuations is Gaussian, but not "white"

Phase fluctuations

Averaged value of the phase difference:

$$\langle \phi(t+\tau) - \phi(t) \rangle = -\omega(p_0)\tau$$

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Variance of the phase difference:

$$\Delta\phi^2(\tau) = \Gamma_+ \frac{k_B T}{E(p_0)} \left[|\tau| + \nu^2 \left(|\tau| - \frac{1 - e^{-2\Gamma_{\text{eff}}|\tau|}}{2\Gamma_{\text{eff}}} \right) \right]$$

$$\nu = \frac{N p_0}{\Gamma_{\text{eff}}} = \frac{N}{G_+ - G_-}$$

– dimensionless nonlinear frequency shift

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Determines oscillation linewidth

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– dimensionless nonlinear frequency shift

Auto-correlation function:

$$K(\tau) = p_0 e^{-i\omega(p_0)\tau}\exp\left[-\frac{1}{2}\Delta\phi^2(\tau)\right]$$

Low-temperature asymptote

$$\Delta\phi^2(\tau) \approx \Gamma_+ \frac{k_\mathrm{B}T}{E(p_0)}(1+\nu^2)\,|\,\tau\,|$$

"Random walk" of the phase

Low-temperature asymptote

$$\Delta\phi^2(\tau) \approx \Gamma_+ \frac{k_B T}{E(p_0)} (1+\nu^2)\,|\,\tau\,|$$

"Random walk" of the phase

Lorentzian lineshape with the full linewidth

$$2\Delta\omega = (1+\nu^2)\Gamma_+ \frac{k_B T}{E(p_0)} = (1+\nu^2)\; 2\Delta\omega_{\mathrm{lin}}$$

Frequency nonlinearity broadens linewidth by the factor

$$1+\nu^2 = 1 + \left(\frac{N}{G_+ - G_-}\right)^2$$

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Region of validity $\quad k_B T << \left(\frac{\Gamma_{\mathrm{eff}}}{\Gamma_+}\right)\frac{E(p_0)}{1+\nu^2} \sim 300\ \mathrm{K}$

High-temperature asymptote

$$\Delta\phi^2(\tau) \approx \Gamma_+ \frac{k_\mathrm{B}T}{E(p_0)} \nu^2 \Gamma_\mathrm{eff}\, \tau^2$$

"Inhomogeneous broadening" of the frequencies

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"Inhomogeneous broadening" of the frequencies

Gaussian lineshape with the full linewidth

$$2\Delta\omega_* = 2\,|\,\nu\,|\sqrt{\Gamma_+ \Gamma_{\mathrm{eff}}}\sqrt{\frac{k_B T}{E(p_0)}}$$

Different dependence of the linewidth on the temperature:

$$2\Delta\omega_* \sim \sqrt{T}$$

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  - Linear and nonlinear auto-oscillators
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  - **Comparison with experiment (STO)**
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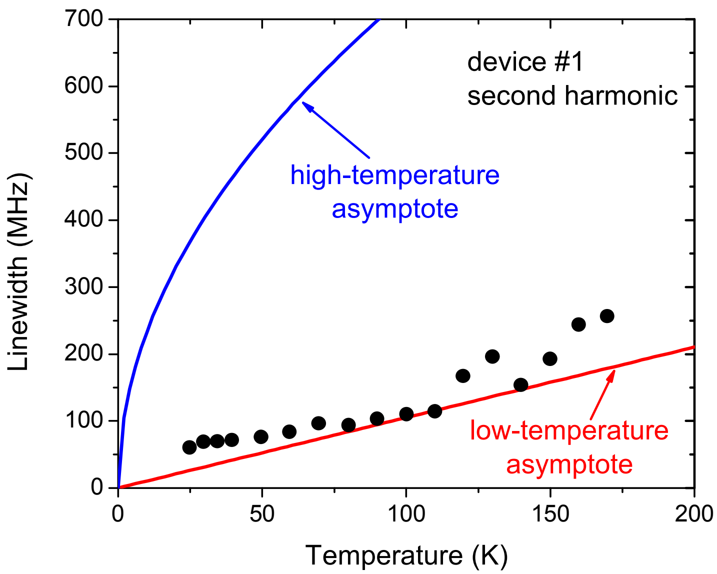
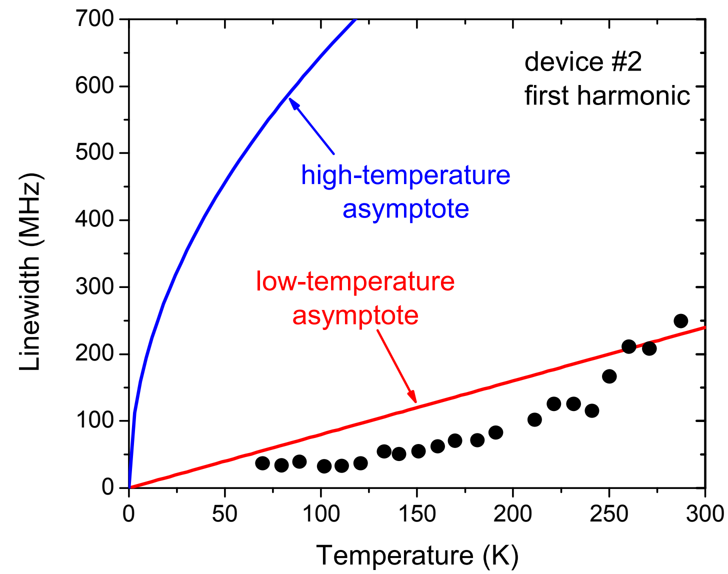
Temperature dependence of STO linewidth

Experiment: J. Sankey *et al.*, Phys. Rev. B **72**, 224427 (2005)





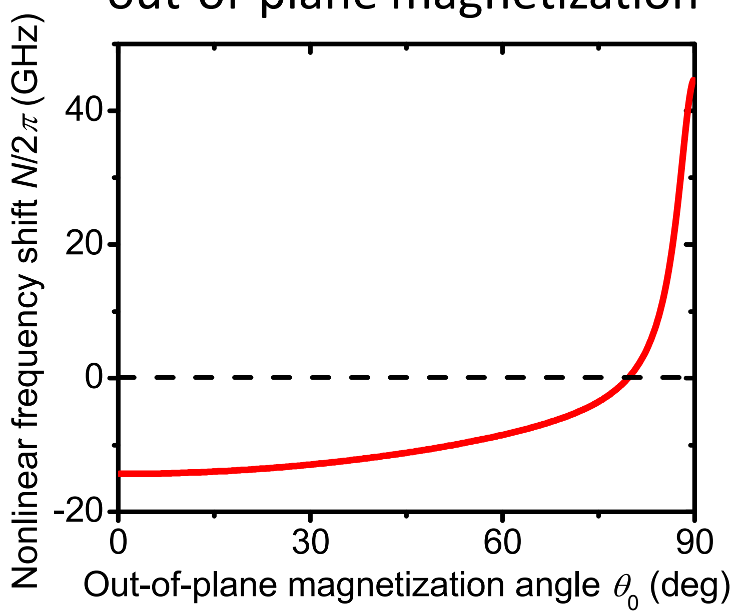
Low-temperature asymptote gives correct order-of-magnitude estimation of the linewidth.

Angular dependence of the linewidth

$$2\Delta\omega = (1+\nu^2)\Gamma_+ \frac{k_B T}{E(p_0)} = \left(1+\left(\frac{N}{G_+ - G_-}\right)^2\right)\Gamma_+ \frac{k_B T}{E(p_0)}$$

Nonlinear frequency shift coefficient ***N*** strongly depends on the orientation of the bias magnetic field

Isotropic film,
out-of-plane magnetization



Anisotropic film,
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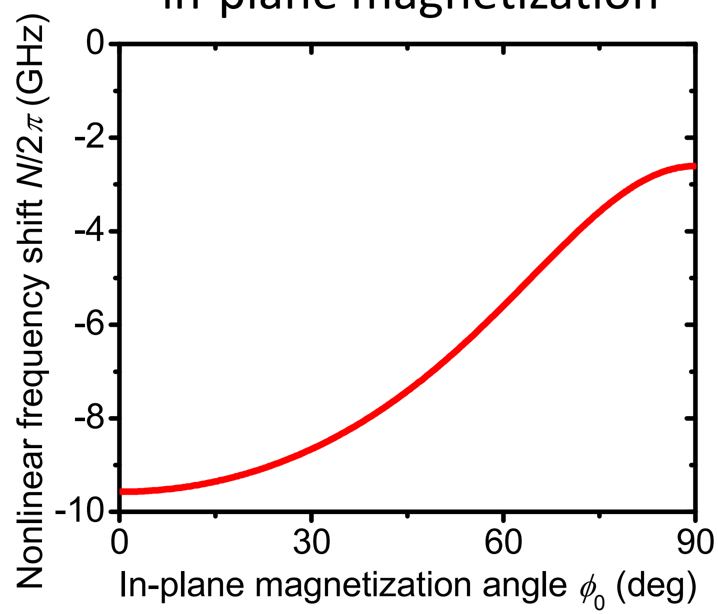

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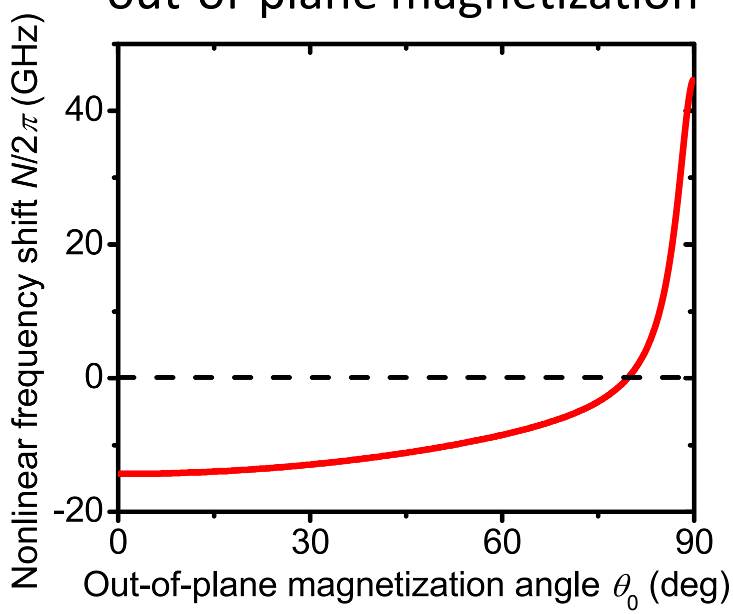

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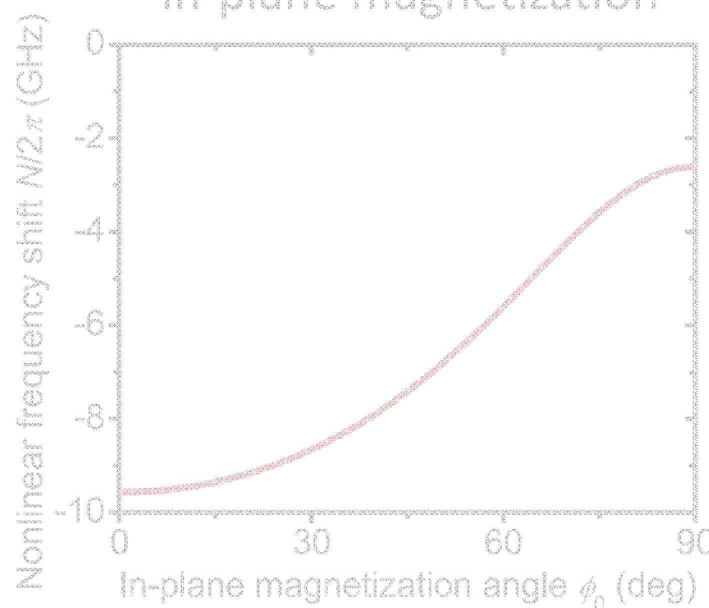

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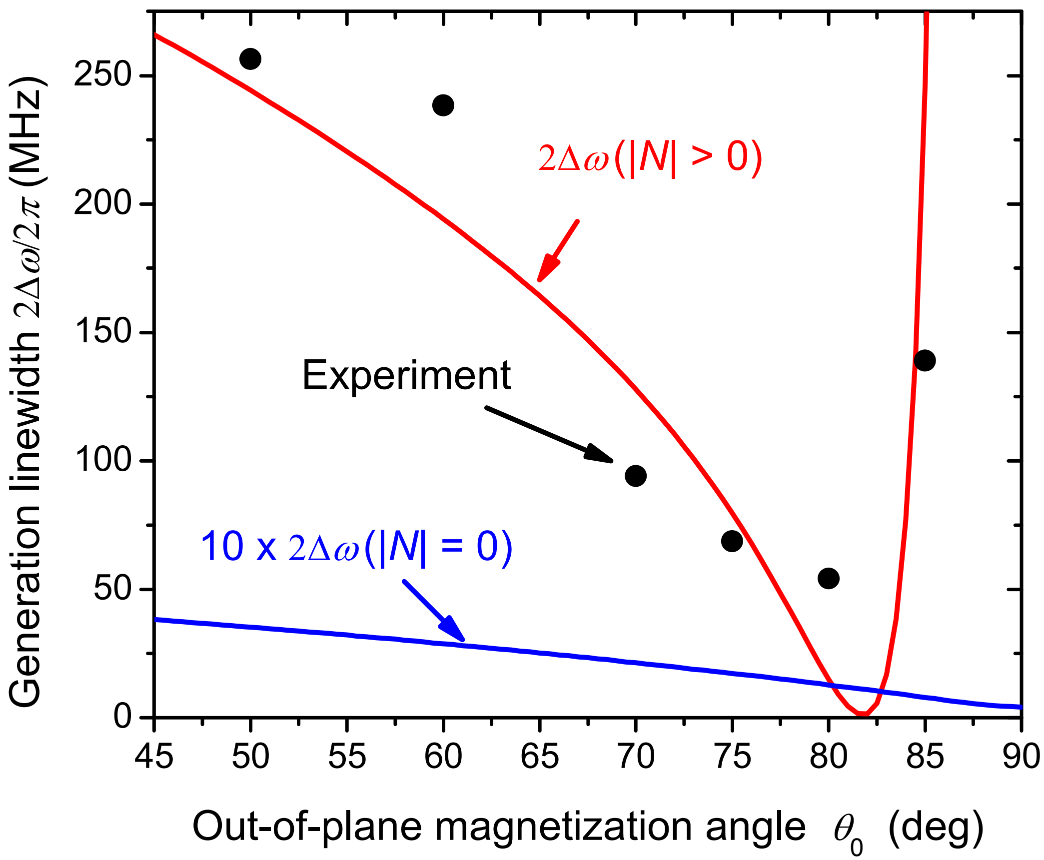
Out-of-plane magnetization



Experiment: W.H. Rippard *et al.*, Phys. Rev. B **74**, 224409 (2006)

Theory: J.-V. Kim, V. Tiberkevich and A. Slavin, Phys. Rev. Lett. **100**, 017207 (2008)

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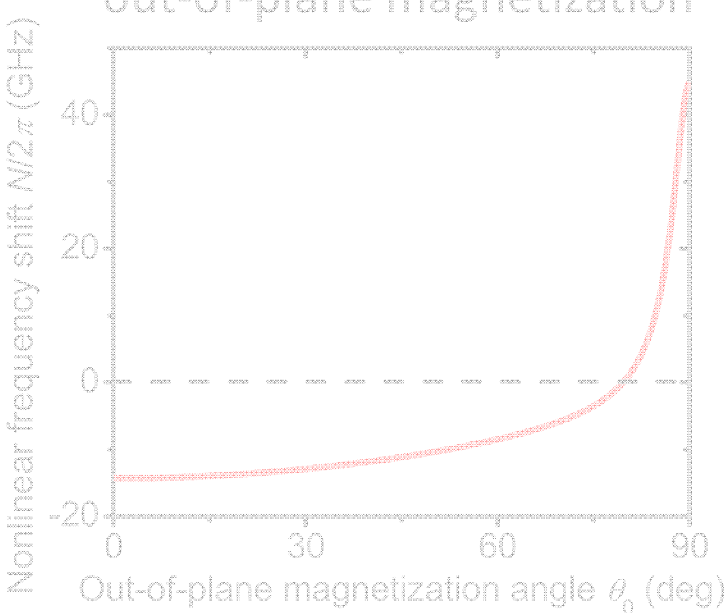

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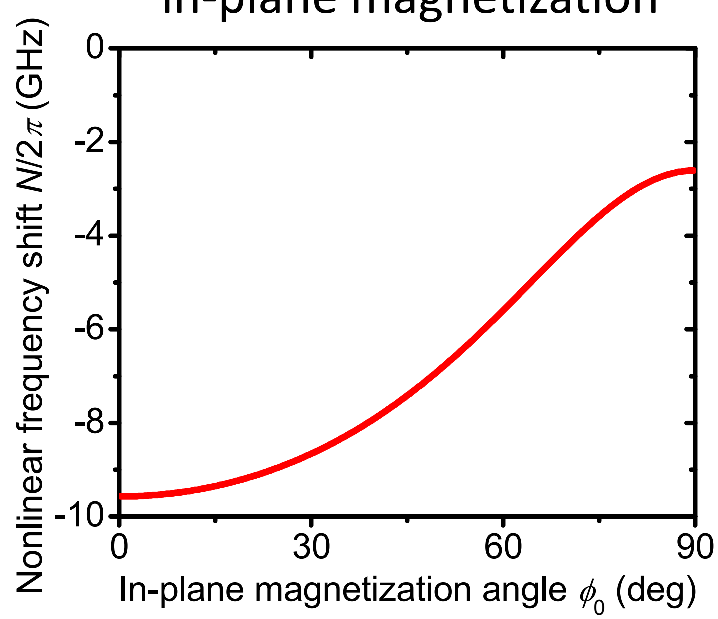

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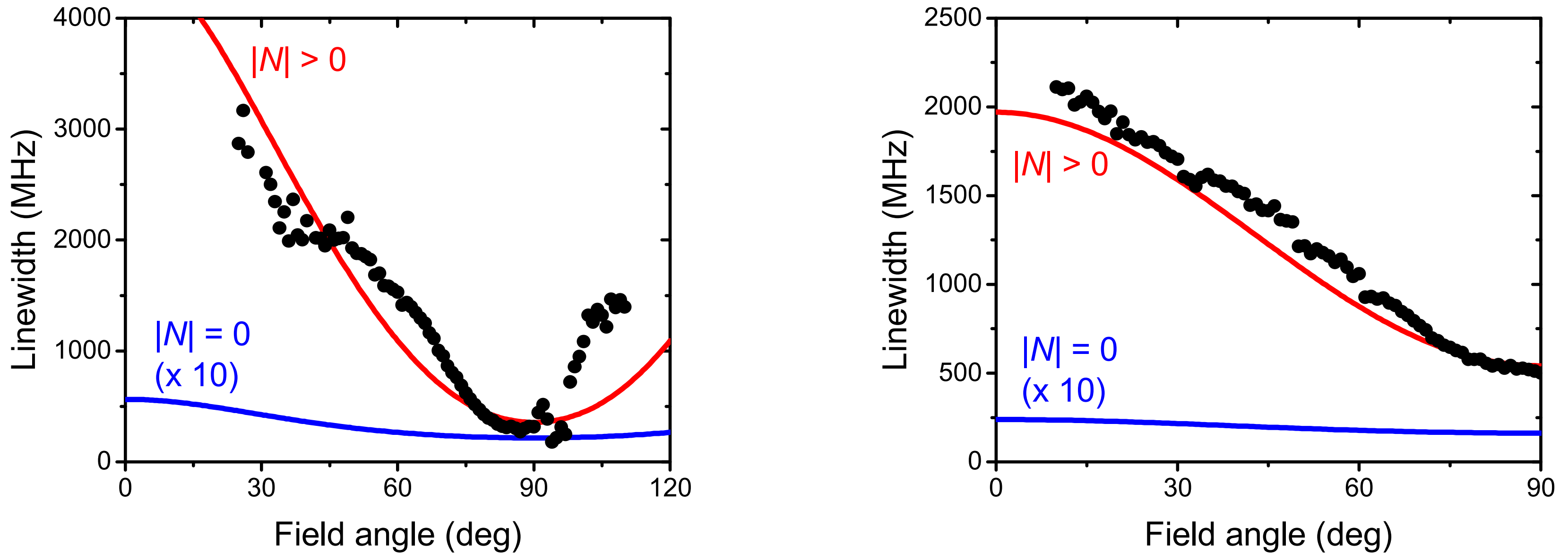

Experiment: K. V. Thadani *et al.*, arXiv: 0803.2871 (2008)

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Summary

- "Nonlinear" auto-oscillators (oscillators with power-dependent frequency) are simple dynamical systems that can be described by universal model, but which have not been studied previously

- There are a number of qualitative differences in the dynamics of "linear" and "nonlinear" oscillators:
    - Different temperature regimes of generation linewidth (low- and high-temperature asymptotes)
    - Different mechanism of phase-locking of "nonlinear" oscillators [A. Slavin and V. Tiberkevich, Phys. Rev. B **72**, 092407 (2005); *ibid.*, **74**, 104401 (2006)]
    - Possibility of chaotic regime of mutual phase-locking [preliminary results]

- Spin-torque oscillator (STO) is the first experimental realization of a "nonlinear" oscillator. Analytical nonlinear oscillator model correctly describes both deterministic and stochastic properties of STOs.