Madelung Fluid Description of gNLS Eq. Special Solutions and their Stability

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Contents

$$i\frac{\partial \Psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2} + 2i\Psi^2\frac{\partial \Psi^*}{\partial x} + \alpha U(|\Psi|^2)\Psi = 0$$

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- Madelung Hydrodynamic Description
- Madelung's Approach - Generalized NLS Equations
- Generalized NLS Equation

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- Bright Soliton Solution
- Periodic Solutions
- Solution Stability

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Madelung Hydrodynamic Description

Hydrodynamic description of quantum mechanics

E. Madelung - 1926

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + mU(x)\Psi$$

$$\Psi = \sqrt{\rho}e^{i\theta}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)v = \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right) - \frac{\partial U}{\partial x}$$

$\rho = \|\Psi\|^2$	fluid density
$v = \frac{\hbar}{m}\frac{\partial\theta}{\partial x}$	fluid velocity
$\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right)$	quantum Bohm's potential

$$j = \frac{\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right) = \rho v$$

Madelung's Approach - Generalized NLS Equations

Series of papers R. Fedele, H. Schamel, P.K. Shukla, V.I. Karpman (2002) used MA to discuss generalized NLS eq.

$$i\alpha \frac{\partial \Psi}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - U(|\Psi|^2)\Psi = 0$$

For $U(|\Psi|^2) = |\Psi|^2$ is the well known completely integrable cubic NLS eq.

$$U(|\Psi|^2) = q_1 |\Psi|^{2\beta} + q_2 |\Psi|^{4\beta}$$

($\beta = 1$, cubic + quintic nonlinearity)

Recently we used MA to discuss generalized derivative NLS eqs. (D. Grecu, A.T. Grecu, A. Visinescu, R. Fedele, S. DeNicola - 2008)

$$i\alpha\frac{\partial\Psi}{\partial t}+\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2}+i\gamma\frac{\partial}{\partial x}\left(U(|\Psi|^2)\Psi\right)=0 \quad gdNLS-1$$

$$i\alpha\frac{\partial\Psi}{\partial t}+\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2}+i\gamma U(|\Psi|^2)\frac{\partial\Psi}{\partial x}=0 \qquad gdNLS-2$$

For $U(|\Psi|^2)=|\Psi|^2$ they transform into completely integrable equations.

1. R. Fedele, H. Schamel, *Eur. Phys. J* B **27**, 313 (2002)
2. R. Fedele, *Physica Scripta* **65**, 502 (2002)
3. R. Fedele, H. Schamel, P.K. Shukla, *Physica Scripta* T **98**, 18 (2002)
4. R. Fedele, H. Schamel, V.I. Karpman, P.K. Shukla, *J. Phys. A: Math. Gen* **36**, 1169 (2003)
5. D. Grecu, A.T. Grecu, A. Visinescu, R. Fedele, S.DeNicola, *J. Nonlinear Math. Phys* - to be published (Proc. NEEDS-2007)
6. A. Visinescu, S. DeNicola, R. Fedele, D. Grecu, *Proc.Int. Conf. FARPhys 2007, Iassy* - to be published

gNLS Equation

We are studying now mixed eqs. like

$$i\frac{\partial\Psi}{\partial t} + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + 2i\Psi^2\frac{\partial\Psi^*}{\partial x} + \alpha U(|\Psi|^2)\Psi = 0$$

For $U(|\Psi|^2) = |\Psi|^4 = |\Psi|^{2\beta}$ it is completely integrable (Ablowitz et al - 1980, Gerdjikov and Ivanov - 1983, Tsuschida - 2002)

$$\Psi = \sqrt{\rho}e^{i\theta}$$

$$\rho(x,t) = |\Psi|^2, \qquad v(x,t) = \frac{\partial\theta(x,t)}{\partial x}$$

Separating the real and imaginary part we get

- equation of continuity

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\rho v + \rho^2\right)$$

and

- equation of motion for fluid velocity

$$\left(-\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) v = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial x^2}\right) + 2\frac{\partial (v\rho)}{\partial x} + \alpha\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x}$$

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Using a series of transformations (Fedele et al. 2002), the equation of motion can be written

$$\rho\frac{\partial v}{\partial t} - v\frac{\partial \rho}{\partial t} + 2\left[c_0(t) + \int \frac{\partial v}{\partial t}dx\right]\frac{\partial \rho}{\partial x} + \\ +\frac{1}{4}\frac{\partial^3 \rho}{\partial x^3} + \alpha\left(\rho\frac{dU}{d\rho} + 2U\right)\frac{\partial \rho}{\partial x} + 2\rho^2\frac{\partial v}{\partial x} + 4v\frac{\partial(\rho^2)}{\partial x} = 0$$

a KdV-type equation

NO Solutions with Constant Current Velocity

Indeed if $v = v_0$ from the equation of continuity we get

$$\rho_t - (v_0 + 2\rho)\rho_x = 0$$

If

$$f(x) = \rho(x, t = 0)$$

is the initial solution, then

$$\rho(x, t) = f[x + (v_0 + 2\rho)t]$$

which is incompatible with the dispersive equation of motion

Motion with Stationary Profile Current Velocity

$$\rho(x,t) = \rho(\xi) \qquad v(x,t) = v(\xi)$$

$$\xi = x + u_0 t$$

Eq. of continuity becomes

$$\frac{d}{d\xi}\left(-u_0\rho + v\rho + \rho^2\right) = 0$$

$$v = u_0 - \rho + \frac{A_0}{\rho}$$

If $\rho \to 0$ for $|\xi| \to \infty$ then $A_0 \equiv 0$
(bright soliton case)

$$v = u_0 - \rho$$

the KdV-type equation becomes an ordinary differential equation

$$\frac{1}{4}\frac{d^3\rho}{d\xi^3} + \alpha\left(\rho\frac{dU}{d\rho} + 2U\right)\frac{d\rho}{d\xi} + 2\left(c_0 + \frac{u_0^2}{2}\right)\frac{d\rho}{d\xi} +$$

$$+3u_0\frac{d(\rho^2)}{d\xi} - \frac{10}{3}\frac{d(\rho^3)}{d\xi} = 0$$

Solution for $\beta = 2, \quad \alpha = 1$

$$\rho = \frac{b^2}{\sqrt{b^2 - u_0^2}\ sinh\, 2b(\xi - \xi_0) + u_0}$$

and the phase is given by $\theta = \int (u_0 - \rho) d\xi$

$$\theta = \theta_0 + u_0 \xi - 3\ tanh^{-1}\ \frac{u_0\ tanh\ \frac{\xi}{2} - \sqrt{b^2 - u_0^2}}{b}$$

where

$$b^2 = -(2c_0 + u_0^2) > 0$$

It is easily seen that we can find solutions for $\beta = 1, 2, 3$ and different values for α. The case $\beta = 2$ and $\alpha = 4$ corresponds to the integrable equation.

Periodic Solution for $\alpha = 4$ and $\beta = 2$

From the KdV-type equation

$$\rho \frac{\partial v}{\partial t} - v \frac{\partial \rho}{\partial t} + 2 \left[c_0(t) + \int \frac{\partial v}{\partial t} dx \right] \frac{\partial \rho}{\partial x} +$$
$$+ \frac{1}{4} \frac{\partial^3 \rho}{\partial x^3} + \alpha \left(\rho \frac{dU}{d\rho} + 2U \right) \frac{\partial \rho}{\partial x} + 2\rho^2 \frac{\partial v}{\partial x} + 4v \frac{\partial (\rho^2)}{\partial x} = 0$$

by substituting

$$\xi = x + u_0 t, \qquad v = u_0 - \rho + \frac{A}{\rho}$$

after two integration we get

$$\frac{1}{4} \left(\frac{d\rho}{dx} \right)^2 = -[\rho^4 + 2u_0 \rho^3 + (c_0 + \frac{u_0^2}{2} + 5A_0)\rho^2 + B\rho + C] = -P_4(\rho)$$

We consider the case with 4 real roots of P_4, $(\rho_1,\ \rho_2,\ \rho_3, \rho_4)$, at least 2 positive, $0 \leq \rho_2 < \rho < \rho_1$. The periodic solution of gNLS can be expressed in terms of Jacobi elliptic functions, as

$$\rho = \rho_3 + \frac{\rho_2 - \rho_3}{1 - \lambda^2 sn^2 u}$$

where

$$k^2 = \frac{(\rho_1 - \rho_2)(\rho_3 - \rho_4)}{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}, \qquad k^2 < \lambda^2 = \frac{\rho_1 - \rho_2}{\rho_1 - \rho_3} < 1$$

$$u = \frac{2}{g}\xi, \qquad g = \frac{2}{\sqrt{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}}$$

Stability - Vakhitov- Kolokolov Criterion

$$i\frac{\partial\Psi}{\partial t}+\frac{\partial^2\Psi}{\partial x^2}+f(|\Psi|)\Psi=0$$

ground state (stationary state)

$$\Psi=e^{i\omega t}\Phi(x,\omega)$$

$$N(\omega)=\int\Phi^2(x,\omega)dx$$

Vakhitov- Kolokolov criterion: instability if

$$\frac{dN(\omega)}{d\omega}<0$$

M.G. Vakhitov, A.A. Kolokolov, *Izv. Vyssh. Uch. Izv. Radiofizica* **16**, 1020 (1973)

M.I. Weinstein, *Comm. Pure Appl. Math* **39**, 51 (1986)

D.E. Pelinovski, V.V. Afanasyev, Yu.S. Kivshar *Phys.Rev.*E **53**, 1940 (1996) Going beyond linear analysis

$$\Psi(x,t) = \phi(x,t)e^{i\int_0^t \omega(t')dt'}$$

$$\phi(x,t) = \Phi(x,\omega(t)) + \phi_1(x,t) + \phi_2(x,t) + ...$$

instability-induced evolution is

- slow in time
- almost adiabatic

result: an equation (second order differential nonlinear equation) for $\omega(t)$; various instability scenarios

D.E. Pelinovski, V.V. Afanasyev, Yu.S. Kivshar *Phys.Rev.*E **54**, 2015 (1996) - dark soliton stability

Vakhitov-Kolokolov Criterion for gNLS eq.

Apply Vakhitov-Kolokolov criterion for

$$iq_t = \frac{1}{2} q_{xx} + 2iq^2 q_x^* + 2q_0 |q|^2 q$$

Stationary solution

$$q(x,t) = e^{-i\frac{\omega}{2}t} e^{i\theta(x,\omega)} \Phi(x,\omega)$$

Separating imaginary and real part, we obtain from the imaginary one

$$\frac{d\theta}{dx} = -\Phi^2$$

and from the real part

$$\left(\frac{d\Phi}{dx}\right)^2 = \omega\Phi^2 - 2q_0\Phi^4 + \frac{5}{3}\Phi^6$$

$$\Phi = \frac{\sqrt{\omega}}{\left(\frac{\sqrt{\Delta}}{2} sinh\, 2\sqrt{\omega}x + q_0\right)^{\frac{1}{2}}}$$

where

$$\Delta = \frac{5\omega}{3} - q_0^2 > 0$$

$$N(\omega) = \int_{-\infty}^{\infty} \Phi^2 dx = \sqrt{\frac{\omega}{\gamma}} \int_{-\infty}^{\infty} \frac{dx}{sinh\, x + \gamma}$$

$$\gamma = \frac{2q_0}{\sqrt{\Delta}}$$

$\gamma > 1$ means $\frac{3q_0^2}{5} < \omega < 3q_0^2$

$$N(\omega) = \sqrt{\frac{3\omega}{5\omega - 3q_0^2}} \frac{1}{\sqrt{\gamma^2 - 1}} ln\frac{\gamma + \sqrt{\gamma^2 - 1}}{\gamma - \sqrt{\gamma^2 - 1}}$$

$\frac{dN}{d\omega} < 0$ unstable

$\gamma < 1$ means $\omega > 3q_0^2$

$$N(\omega) = \sqrt{\frac{3\omega}{5\omega - 3q_0^2}} \frac{2}{\sqrt{\gamma^2 - 1}} artg\frac{\gamma}{\sqrt{1 - \gamma^2}}$$

$\frac{dN}{d\omega} > 0$ stable

Work in Progress

Difficult to extend to other nonlinearities. We are trying to study the stability (using the VK criterion) for a simpler mixed dNLS+NLS equation

$$iq_t + q_{xx} + ia_1|q|^{2\beta_1}q_x + a_2|q|^{2\beta_2}q = 0$$