Beyond-mean-field theories and zero-range effective interactions.

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Objectives

- Dealing with nuclear many-body problem with in beyond-mean-field approaches.
- Evaluating beyond-mean-field corrections of the equation of state of symmetric nuclear matter with zero-range Skyrme-type interaction.
- Treating the ultraviolet divergence that exists at beyond-mean-field level due to the zero-range Skyrme interaction.
- Refitting the parameters of the effective interaction at a beyond-mean-field level.

Skyrme effective interaction

- In 1956, Skyrme proposed an approximate representation of the effective nuclear force with a zero-range two-body and three-body term:

$$V_{12}(i,j) + \sum_{i<j<k} V_{123}(i,j,k)$$

- The three-body term is given by:

$$V_{123}(r_1, r_2, r_3) \quad = \quad t_3 \delta(r_1 - r_2)\delta(r_2 - r_3)$$

- Later, for spin saturated even-even nuclei, the three-body part has been replaced by a density-dependent term:

$$\frac{1}{6} t_3 \left(1 + P^\sigma\right) \delta(r_1 - r_2) \rho^\alpha \left(\frac{\vec{r_1} + \vec{r_2}}{2}\right)$$

Skyrme effective interaction

- In general, effective interactions contain parameters that must be adjusted to reproduce a given set of observables (binding energies and radii).
- Our effective-phenomenological interaction becomes:

$$
\begin{aligned}
V_{12}(r_1, r_2) &= t_0(1 + x_0 P^\sigma)\delta(r_1 - r_2) \\
&+ \frac{t_1}{2}(1 + x_1 P^\sigma)\left[\delta(r_1 - r_2)k^2 + k'^2\delta(r_1 - r_2)\right] \\
&+ t_2 \cdot (1 + x_2 P^\sigma)k' \cdot \delta(r_1 - r_2)k \\
&+ \frac{1}{6}t_3\,(1 + x_3 P^\sigma)\,\delta(r_1 - r_2)\rho^\alpha\left(\frac{\vec{r_1} + \vec{r_2}}{2}\right) \\
&+ iW_0\sigma \cdot k' \times \delta(r_1 - r_2)k
\end{aligned}
$$

- $P^{\sigma} = \frac{1}{2}(1 + \sigma_1 \cdot \sigma_2)$ is the spin exchange operator.
- $k = \frac{1}{2i}(\nabla_1 - \nabla_2)$ is the operator of the relative momentum acting on the right.
- $k^{'} = -\frac{1}{2i}(\nabla_1 - \nabla_2)$ acting on the left.
- $\sigma = \sigma_1 + \sigma_2$ is the total spin operator.

Skyrme effective interaction

- For simplicity, we will only take into considerations t_0 and t_3 parameters.
- we use a two-body interaction written as:

$$v_{12}(r_1, r_2) \quad = \quad t_0\delta(r_1 - r_2) + \frac{1}{6}t_3\delta(r_1 - r_2)\rho^{\alpha}\left(\frac{r_1 + r_2}{2}\right)$$

- Taking now the relative vector $r = r_1 - r_2$, we get:

$$V_{12}(r) \quad = \quad \underbrace{\left(t_0 + \frac{1}{6}t_3\rho^{\alpha}\right)}_{g}\delta(r) = g\delta(r)$$

Symmetric nuclear matter

The infinite-symmetric nuclear matter which is an ideal system composed of nucleons with the following properties:

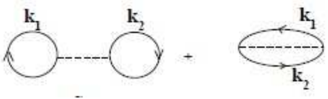
- Equal number of protons and neutrons ($N = Z$).
- No Coulomb interaction between protons.
- No correlations between nucleons.
- Neglecting the surface forces.
- Constant density ρ_0 at equilibrium.

Mean-field framework

- Let us suppose that the ground state is represented by a Slater determinant ϕ of single-particle states ϕ_i:

$$\phi(x_1; x_2, \cdots x_A) = \frac{1}{\sqrt{A!}} \det |\phi_i(x_j)|$$

- The Hartree-Fock energy can be visualized by the following Feynman diagrams:



$$E = \frac{\Omega^2}{(2\pi)^6} \sum_{\sigma,\tau} \int_{k_1,k_2<k_F} d^3k_1 d^3k_2 v(k_1, k_2, k_1, k_2)$$

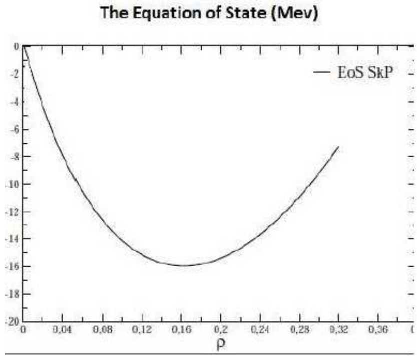
- In symmetric nuclear matter, the mean-field equation of state becomes:

$$\frac{E_0}{A}(\rho) = \frac{3\hbar^2}{10m}\left(\frac{3\pi^2\rho}{2}\right)^{\frac{2}{3}} + \frac{3}{8}t_0\rho + \frac{1}{16}t_3\rho^{\alpha+1}$$

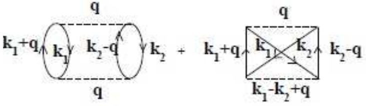
Mean-field framework

- The "SkP" equation of state is the ground state of the zero-range Skyrme effective interaction without adding any corrections.
- The parameters t_0, t_3 and α of the Skyrme interaction:

$$t_0 = -2931.70 \; Mev \; fm^3, t_3 = 18708.97 \; Mev \; fm^{3+3\alpha}, \; \alpha = 1/6.$$



Beyond-mean-field Level



- The 2^{nd} order correction (with exchange):

$$\Delta E = d\frac{\Omega^3}{(2\pi)^9}\int_{C_I} d^3k_1 d^3k_2 d^3q \frac{g^2}{\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_1+q}-\epsilon_{k_2-q}}$$

where $C_I = |k_1|, |k_2| < k_F, |k_1+q|, k_2-q| > k_F$.

- After long-straight forward calculations:

$$\frac{\Delta E}{A} = \chi(\rho)\times I(\rho,\infty), \qquad \chi(\rho) = -\frac{3k_f^7}{4\pi^6}\frac{m}{\hbar^2}\frac{g^2}{\rho}$$

$$I(\rho,\infty) = \int_0^1 du\ u\ f_1(u) + \int_1^{+\infty} du\ u\ f_2(u)$$

- The integral I diverges linearly. i.e we have ultraviolet divergence because:

$$\lim_{u \to \infty} u\ f_2(u) = constant$$

- We will regularize this momentum integral by introducing a cutoff Λ.

$$I(\rho, \Lambda) = \int_0^1 du\ u\ f_1(u) + \int_1^{\frac{\Lambda}{2k_f}} du\ u\ f_2(u)$$

- Indeed, this cutoff Λ must take finite values in such a manner that the integral can be calculated.
- For instance, for low-energy nuclear physics problems, Λ takes a maximum value: $\Lambda_{max} \sim 2fm^{-1}$ because Λ must be smaller than the momentum associated with the nucleon size (point-like nucleons).

An exact form of the I without any approximations:

$$
\begin{aligned}
I(\rho,\Lambda) &= \frac{1}{105}(43-46\ln 2)-\frac{18}{35}+\frac{\Lambda}{35k_F}+\frac{11\Lambda^3}{210k_F^3}+\frac{\Lambda^5}{840k_F^5}\\
&+ \frac{16\ln 2}{35}+\left(\frac{\Lambda^5}{60k_F^5}-\frac{\Lambda^7}{1680k_F^7}\right)\ln\left(\frac{\Lambda}{k_F}\right)\\
&+ \left(\frac{1}{35}-\frac{\Lambda^2}{30k_F^2}+\frac{\Lambda^4}{48k_F^4}-\frac{\Lambda^5}{120k_F^5}+\frac{\Lambda^7}{3360k_F^7}\right)\ln\left(-2+\frac{\Lambda}{k_F}\right)\\
&- \left(\frac{1}{35}-\frac{\Lambda^2}{30k_F^2}+\frac{\Lambda^4}{48k_F^4}+\frac{\Lambda^5}{120k_F^5}-\frac{\Lambda^7}{3360k_F^7}\right)\ln\left(2+\frac{\Lambda}{k_F}\right).
\end{aligned}
$$

Beyond-mean-field Level

- The second-order energy-correction integral diverges because of the zero-range of the Skyrme force: (large values of Λ)

$$I(\rho, \Lambda) \approx \frac{1}{105}(2\log 2 - 11) + \frac{\Lambda}{9k_f} - \frac{2k_f}{45\Lambda} - \frac{8k_f^3}{525\Lambda^3} + \mathcal{O}\left(\frac{k_f^5}{\Lambda^5}\right)$$

- It should be noted that there is no divergence at very low densities due to the density dependent terms in the expression of χ.

$$\frac{\Delta E}{A} = \chi(\rho) \times I(\rho), \qquad \chi(\rho) = -\frac{3k_f^7}{4\pi^6}\frac{m}{\hbar^2}\frac{g^2}{\rho}$$

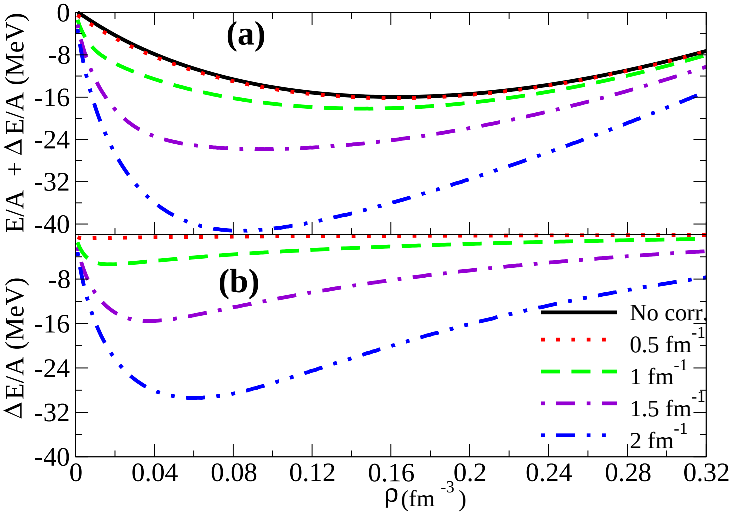
The corrected equation of state

- The corrected equation of state is:

$$\frac{E}{A}(\rho,\Lambda) = \frac{E_0}{A}(\rho) + \frac{\Delta E}{A}(\rho,\Lambda)$$

$$= \underbrace{\frac{3\hbar^2}{10m}\left(\frac{3\pi^2\rho}{2}\right)^{\frac{2}{3}}}_{Kinetic\ energy} + \underbrace{\frac{3}{8}t_0\rho + \frac{1}{16}t_3\rho^{\alpha+1}}_{1^{st}\ order} + \underbrace{\chi(\rho)\times I(\rho,\Lambda)}_{2^{nd}\ order}$$

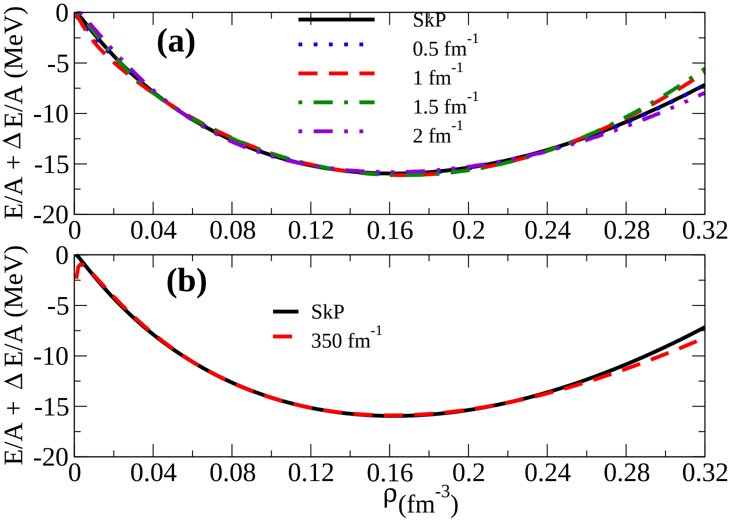
The corrected equation of state



Renormalization and fitting

- To treat this U.V divergence, we are going to refit our parameters t_0, t_3 and α in such a way to recover the "SKP" mean-field equation of state.
- We will use the least square root fit program that needs an initial guess of the parameters ("SkP" parameters) and a set of points to be fitted$(\rho_0/4, \rho_0/2, 3\rho_0/4, \rho_0, 3\rho_0/2)$.
- The quality of the fits can be judged by the corresponding values of $\chi^2 = \sum_{i=1}^{5} \frac{(E_i - O_i)^2}{O_i}$

Renormalization and fitting



- For each value of Λ we can perform a least square fit to determine a new parameter set "SkP_{Λ}" , such that the "EOS" including the second-order correction matches rather well the one obtained with the original force "SkP" at the mean- field level.
- For purely mathematical illustration, the refit done with the extreme value of $\Lambda = 350 fm^{-1}$.

Renormalization and fitting

	t_0 (MeV fm^3)	t_3 (MeV fm$^{3+3\alpha}$)	α	χ^2/N
SkP	-2931.70	18708.97	1/6	
$\Lambda = 0.5$ fm^{-1}	-2352.900	15379.861	0.217	0.00004
$\Lambda = 1$ fm^{-1}	-1155.580	9435.246	0.572	0.00142
$\Lambda = 1.5$ fm^{-1}	-754.131	8278.251	1.011	0.00106
$\Lambda = 2$ fm^{-1}	-632.653	5324.848	0.886	0.00192
$\Lambda = 350$ fm^{-1}	-64.904	360.039	0.425	0.00042

- You can see that the values of $\chi^2 \sim 0.002$ which is a very small quantity. Another definition of χ^2 can lead to values that are comparable to 1.

Conclusions

- The beyond-mean-field equation of state diverges linearly (ultraviolet divergence) with high momenta in our $t_0 - t_3$ model because of the zero-range of the Skyrme force.
- We introduce a momentum cutoff Λ and do the regularization procedure for the integral loop.
- We refit the interaction's parameters using a Least Square Root fit so that the "SKP" mean-field equation of state is recovered.
- **Thanks for you attention!!**