Free Fermi GAS:

All momentum occupied up to the Fermi momenta

deg { spin-isospin degeneracy

$$\rho = \frac{\deg}{(2\pi)^3} \int d^3k \; \Theta(k_F - k)$$

$$\sum_k \Longrightarrow \frac{\Omega}{(2\pi)^3} \int d^3k$$

$\Rightarrow |\phi\rangle_{GS} \Rightarrow$ Slater determinant of plane-waves

$$\frac{1}{\sqrt{\Omega}} e^{i\bar{k}\bar{r}} \chi_{m_s} \chi^T_{m_t}$$

$$H = \sum_k \frac{\hbar^2 k^2}{2m} a^+_k a_k \qquad \boxed{\text{No spin}}$$

$$H |\phi_{GS}\rangle = E_0 |\phi_{GS}\rangle$$

$$H a^+_k |\phi_{GS}\rangle \begin{cases} = \left(E_0 + \frac{\hbar^2 k^2}{2m}\right) a^+_k |\phi_{GS}\rangle & k > k_F \\ = 0 & k < k_F \end{cases}$$

$$H a_k |\phi_{GS}\rangle \begin{cases} = \left(E_0 - \frac{\hbar^2 k^2}{2m}\right) a_k |\phi_{GS}\rangle & k < k_F \\ = 0 & k > k_F \end{cases}$$

$$g^{(0)}(k,E) = \sum_m \frac{\langle \Psi_0^A | a_{\vec{k}} | \Psi_m^{A+1} \rangle \langle \Psi_m^{A+1} | a^+_{\vec{k}} | \Psi_0^A \rangle}{E - (E_m^{A+1} - E_0^A) + i\eta} +$$

$$+ \sum_n \frac{\langle \Psi_0^A | a^+_{\vec{k}} | \Psi_n^{A-1} \rangle \langle \Psi_n^{A-1} | a_{\vec{k}} | \Psi_0^A \rangle}{E - (E_0^A - E_n^{A-1}) - i\eta}$$

only one state in the intermediate states!

$$= \left\{ \frac{\Theta(k-k_F)}{E - \left(\left(E_0^A + \frac{\hbar^2 k^2}{2m} \right) - E_0^A \right) + i\eta} + \frac{\Theta(k_F - k)}{E - \left(E_0^A - \left(E_0^A - \frac{\hbar^2 k^2}{2m} \right) \right) - i\eta} \right\}$$

$$= \boxed{ \frac{\Theta(k-k_F)}{E - \frac{\hbar^2 k^2}{2m} + i\eta} + \frac{\Theta(k_F-k)}{E - \frac{\hbar^2 k^2}{2m} - i\eta} = g^{(0)}(k,E) }$$

Spectral functions $\quad \frac{1}{A \pm i\eta} = P\left(\frac{1}{A}\right) \mp i\pi\delta(A)$

$$\underset{E<\varepsilon_F}{S_h(k,E)} = \frac{1}{\pi} \operatorname{Im} g^{(0)}(k,E) = \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) \Theta(k_F - k)$$

$$\underset{E>\varepsilon_F}{S_p(k,E)} = -\frac{1}{\pi} \operatorname{Im} g^{(0)}(k,E) = \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) \Theta(k - k_F)$$

$$n(k) = \int_{-\infty}^{\varepsilon_F} S_h(k,E) = \Theta(k_F - k) \int_{-\infty}^{\varepsilon_F} \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) dE = \Theta(k_F - k)$$

$$d(k) = \Theta(k - k_F) \quad \Rightarrow \quad \boxed{n(k) + d(k) = 1}$$

Kinetic energy per particle

$$\frac{1}{N}\langle \hat{T}\rangle_{FS} = \frac{1}{N}\underbrace{\frac{\Omega}{(2\pi)^3}\int d^3k}_{\sum_{\vec{k}}}\ \underbrace{\langle \vec{k}|t|\vec{k}\rangle}_{\frac{\hbar^2k^2}{2m}}\ \underbrace{\int_{-\infty}^{\varepsilon_F} S_h(k,E)\,dE}_{\theta(k_F-k)}$$

$$= \frac{1}{\rho}\frac{1}{(2\pi)^3}\int d^3k\,\frac{\hbar^2k^2}{2m}\,\theta(k_F-k) = \frac{1}{\rho}\frac{1}{(2\pi)^3}\left[\frac{\hbar^2k_F^2}{2m}\right]4\pi\frac{1}{5}k_F^3$$

$$= \frac{3}{5}\frac{\hbar^2k_F^2}{2m}$$

$$\frac{1}{N}\langle|\sum_k a_k^\dagger a_k|\rangle = 1 = \frac{1}{N}\frac{\Omega}{(2\pi)^3}\int d^3k\,\theta(k_F-k)$$

$$= \frac{1}{N}\frac{\Omega}{(2\pi)^3}\frac{4}{3}\pi k_F^3 = \frac{1}{\rho}\frac{k_F^3}{6\pi^2} = 1 \Rightarrow \rho = \frac{k_F^3}{6\pi^2}$$

including deg $\Rightarrow \boxed{\rho = \frac{\deg\, k_F^3}{6\pi^2}}$

Koltun sum-rule

$$\frac{1}{N}\langle\hat{H}\rangle = \frac{1}{\rho}\frac{1}{(2\pi)^3}\frac{1}{2}\int d^3k\int_{-\infty}^{\varepsilon_F}dE\left(\frac{\hbar^2k^2}{2m}+E\right)\underbrace{S_h(k,E)}_{\delta\left(E-\frac{\hbar^2k^2}{2m}\right)\theta(k_F-k)}$$

$$= \frac{1}{\rho}\frac{1}{(2\pi)^3}\cancel{\frac{1}{2}}\int d^3k\,\cancel{2}\,\frac{\hbar^2k^2}{2m}\,\theta(k_F-k) = \frac{3}{5}\frac{\hbar^2k_F^2}{2m}$$

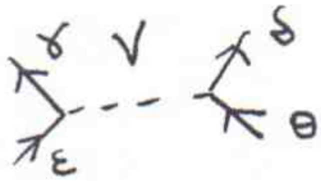
Perturbation expansion of the single-particle propagator

* much more difficult than the single-particle propagator in one-particle quantum mechanics

* Based in two things:

$\boxed{\text{Time evolution in the interaction picture}}$
$\boxed{\text{Wick's theorem to evaluate expectation values}}$ $\Rightarrow$ $\boxed{\text{Diagrammatic rules.}}$

* Different diagrammatic elements

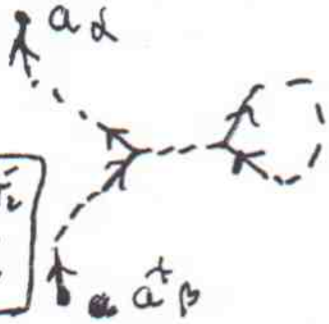


$\Rightarrow$ $\langle \gamma\, \delta \,|\, V \,|\, \varepsilon\, \theta \rangle$

left out, right out, left in, right in

$V(r)$

$$\boxed{\int d^3r_1 d^3r_2\, e^{-i\bar{k}_\gamma \bar{r}_1} e^{-i\bar{k}_\delta \bar{r}_2}\, V(r)\, e^{i\bar{k}_\varepsilon \bar{r}_1} e^{i\bar{k}_\theta \bar{r}_2}}$$



$\uparrow a_\alpha$ $\qquad$ $\uparrow a^+_\beta$

* At a given order "n" we draw $\underline{n}$ horizontal lines, and two external points $\uparrow a^+_\beta$ and $\uparrow a_\alpha$. Now one should joint the lines without finding arrows against $\Rightarrow$ perform the contractions of Wick's theorem

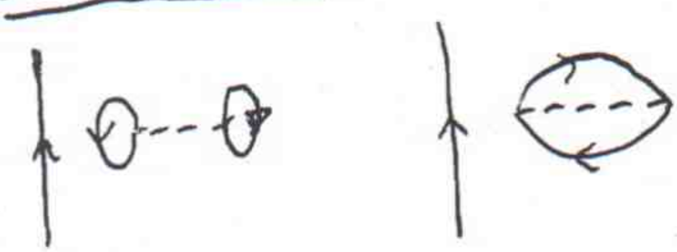
involvedly!

- At order "n" $\Rightarrow$ n interactions V.
  $\Rightarrow$ We have to join $2n+2$ points
  $\Rightarrow$ $2n+1$ lines $\Rightarrow$ $(2n+1)$ g_0 factors!
- At order "n" we have $4n+2$ creation an annihilation operators, $(2n+1)$ creation
  $(2n+1)$ annihilation

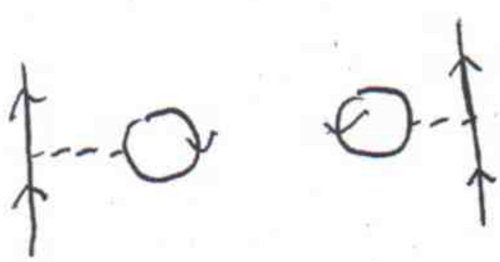
$\Rightarrow$ In principle I can make $(2n+1)!$ terms all contracted!

- $n=1 \Rightarrow (2n+1)! = 6$
  $n=2 \Rightarrow (2n+1)! = 120$

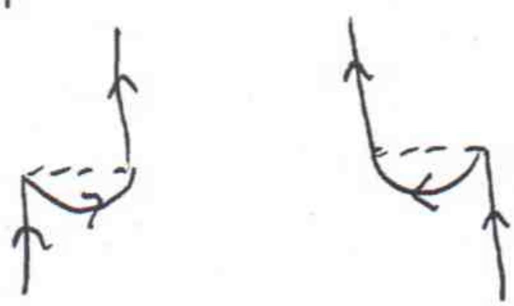
<u>Order $n=1$</u>



"unlinked"
* One needs to separate the pen from the paper.
* Cancelled with the denominator!



"topologically equivalent"
Calculate only one.

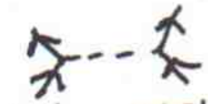


"topologically equivalent"

At order n appear 2^n equivalent diagrams, we calculate <u>only one</u> and cancel the factor $\frac{1}{2}$ which appear at order "n" due to the potential

Rules for ig - Order m

① Draw all topologically different diagrams with m interaction lines and $2m+1$ $g^{(0)}$ propagators.

To this end: a) Draw "m" horizontal lines  with vertices and "in" and "out" arrows, plus the two external points

b) Join all vertices starting from the lower vertex with an in-going and out-going line at each vertex.

• To visualize if two diagrams are equivalent ⇒ try to deform one to get the other one.

• There are Feynman diagrams ⇒ The arrows in the lines indicate the flux of energy and momentum, which are conserved at each vertex! They contain both particles and holes.

② Assign a label $\vec{k}, \sigma, E$ at each line.

③ At each line ($g^{(0)}$) corresponds a factor:

$$i\, g^{(0)}_{\sigma\sigma'}(\bar{k}, E) = i\, \delta_{\sigma\sigma'} \left[\frac{\Theta(|\vec{k}|-k_F)}{E - \frac{\hbar^2 k^2}{2m} + i\eta} + \frac{\Theta(k_F - |\vec{k}|)}{E - \frac{\hbar^2 k^2}{2m} - i\eta} \right]$$

③ Assign a matrix element for each interaction

$(-i) \langle \bar{k}_1 \bar{k}_2 | V | \bar{k}_3 \bar{k}_4 \rangle$ momentum and energy conservation at each vertex!

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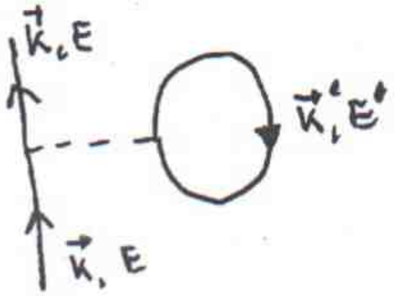
(3) Assign a matrix element for each interaction

$(-i) \langle \vec{k}_1 \vec{k}_2 | V | \vec{k}_3 \vec{k}_4 \rangle$ momentum and energy conservation at each vertex!

Examples: linked diagrams



interaction
↓
i^3 $\quad (-i) \quad (-1) \quad g^{(0)}(\vec{k},E)$

↓ for each $g^{(0)}$ $\qquad$ ↓ fermionic loop

$$\sum_{\vec{k}'} \frac{1}{2\pi} \oint dE' \, g^{(0)}(k',E') \langle \bar{k}\bar{k}'|V|\bar{k}\bar{k}'\rangle \, g^{(0)}(k,E)$$

$$\langle \bar{k}\bar{k}'|V|\bar{k}\bar{k}'\rangle = \frac{1}{\Omega}\int d^3r \, V(r)$$

waves functions normalized to volume!
Direct term.
Independent of k.

$$= g^{(0)}(k,E)\left[\underbrace{\frac{\Omega}{(2\pi)^3}\int d^3k'}_{\sum_{k'}} \frac{1}{2\pi}\oint dE' \, g^{(0)}(k',E')\frac{1}{\Omega}\int d^3r \, V(r)\right] g^{(0)}(k,E)$$

$$\frac{1}{2\pi}\oint dE'\left[\frac{\Theta(k'-k_F)}{E-\varepsilon_{k'}+i\eta}+\frac{\Theta(k_F-k')}{E-\varepsilon_{k'}-i\eta}\right] = \frac{1}{2\pi}2\pi i\sum \mathrm{Res} = i\,\Theta(k_F-k')$$

$$\operatorname*{Res}_{E\to\varepsilon_k+i\eta} g^{(0)}(k',E) = \lim_{E\to\varepsilon_k+i\eta}(E-\varepsilon_k-i\eta)\, g^{(0)}(k',E) = \Theta(k_F-k')$$

$$= i\, g^{(0)}(k,E)\left[\underbrace{\frac{1}{(2\pi)^3}\int d^3k' \, \Theta(k_F-k')}_{\rho}\int d^3r \, V(r)\right] g^{(0)}(k,E)$$

$$= i\, g^{(0)}(k,E)\cdot\rho\int d^3r \, V(r) \, g^{(0)}(k,E)$$

* there is not fermionic loop
* first order diagram
* represents the exchange term of

$$i^3(-i)\, g^{(0)}(k,E)\, \frac{\Omega}{(2\pi)^3}\int d^3k'\, \frac{1}{2\pi}\oint dE'\, g^{(0)}(k',E)\, \langle \vec{k}'\, \vec{k}\,|\,\vartheta\,|\,\vec{k}\, \vec{k}'\rangle\, g^{(0)}(k,E)$$

$$\langle \bar{k}'\, \bar{k}\,|\,\vartheta(r)\,|\,\bar{k}\, \bar{k}'\rangle = \frac{1}{\Omega^2}\int d^3r_1\int d^3r_2\; e^{-i\bar{k}'\bar{r}_1}\, e^{-i\bar{k}\bar{r}_2}\, \vartheta(|\bar{r}_1-\bar{r}_2|)\, e^{i\bar{k}\bar{r}_1}\, e^{i\bar{k}'\bar{r}_2}$$

$$= \frac{1}{\Omega^2}\int d^3R\int d^3r\; e^{-i\vec{k}'(\bar{R}+\frac{\bar{r}}{2})}\, e^{-i\bar{k}(\bar{R}-\frac{\bar{r}}{2})}\, \vartheta(r)\, e^{i\bar{k}(\bar{R}+\frac{\bar{r}}{2})}\, e^{i\bar{k}'(\bar{R}-\frac{\bar{r}}{2})}$$

$$= \frac{1}{\Omega}\int d^3r\; \vartheta(r)\, e^{-i\bar{q}\bar{r}}$$

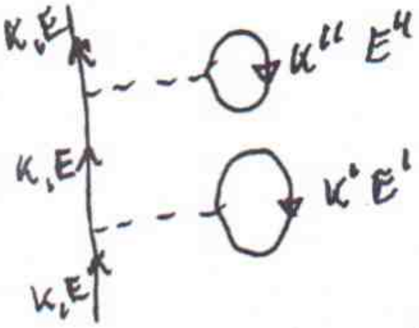
$\bar{q} = \bar{k}' - \bar{k}$ momentum associated to ---

$$(-1)\, g^{(0)}(k,E)\, \frac{\Omega}{(2\pi)^3}\int d^3k'\, \underbrace{i\Theta(k_F-k')}_{\frac{1}{2\pi}\oint dE'\, g^{(0)}(k',E)}\; \frac{1}{\Omega}\int d^3r\, \vartheta(r)\, e^{-iqr}\, g^{(0)}(k,E)$$

$$= -i\, g^{(0)}(k,E)\left[\frac{1}{(2\pi)^3}\int d^3k'\, \Theta(k_F-k')\int d^3r\, \vartheta(r)\, e^{-i(\bar{k}'-\bar{k})\bar{r}}\right] g^{(0)}(k,E)$$

$$\Big(\; + \; + \;\Big) = i\, g^{(0)}(k,E) + i\, g^{(0)}(k,E)\left[\rho\int d^3r\, V(r) - \frac{1}{(2\pi)^3}\int d^3k'\, \Theta(k_F-k')\int d^3r\, \vartheta(r)\, e^{-i(\bar{k}'-\bar{k})\bar{r}}\right] \cdot g^{(0)}(k,E)$$

Second order



$$i^5 \; (-i)^2 \; (-1)^2 \quad g^0(k,E)\,\frac{\Omega}{(2\pi)^3}\int d^3k'\,\frac{1}{2\pi}\oint dE'\, g^{(0)}(k',E')$$

$i^5 \to g^{(0)}$, $(-i)^2 \to$ interaction, $(-1)^2 \to$ loops

$$\langle \bar{k}\bar{k}'|V|\bar{k}\bar{k}'\rangle\, g^{(0)}(k,E)\cdot\frac{\Omega}{(2\pi)^3}\int d^3k''\,\frac{1}{2\pi}\oint dE''\, g^{(0)}(k'',E'')$$

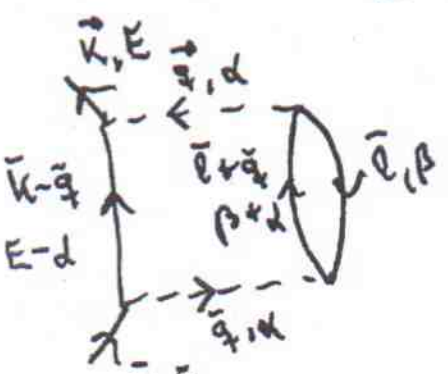
$$\langle \bar{k}\bar{k}''|V|\bar{k}\bar{k}''\rangle\, g^{(0)}(k,E) =$$

$$= -i\, g^{(0)}(k,E)\left[\frac{\cancel{\Omega}}{(2\pi)^3}\int d^3k'\; i\,\theta(k_F-k')\,\frac{1}{\cancel{\Omega}}\int d^3r\, V(r)\, g^{(0)}(k,E)\right.$$
$$\left.\frac{\cancel{\Omega}}{(2\pi)^3}\int d^3k''\; i\,\theta(k_F-k'')\,\frac{1}{\cancel{\Omega}}\int d^3r\, V(r)\right] g^{(0)}(k,E)$$

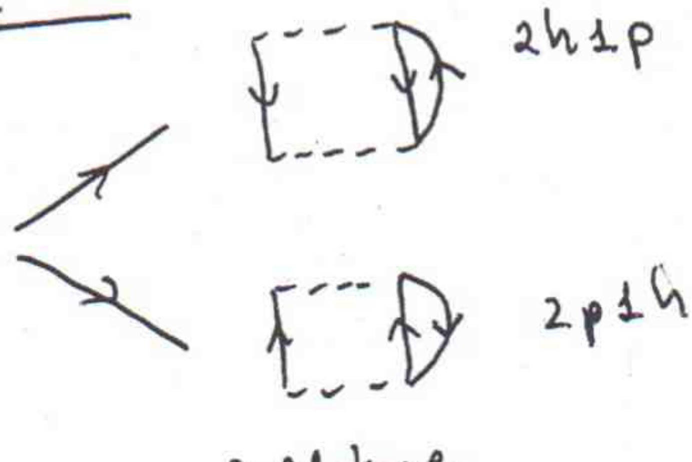
$$\bar{V} = \int d^3r\, V(r)$$

$$= -\,i^3\, g^{(0)}(k,E)\left[\rho\,\bar{V}\; g^{(0)}(k,E)\;\rho\,\bar{V}\right] g^{(0)}(k,\omega)$$

Beyond HF



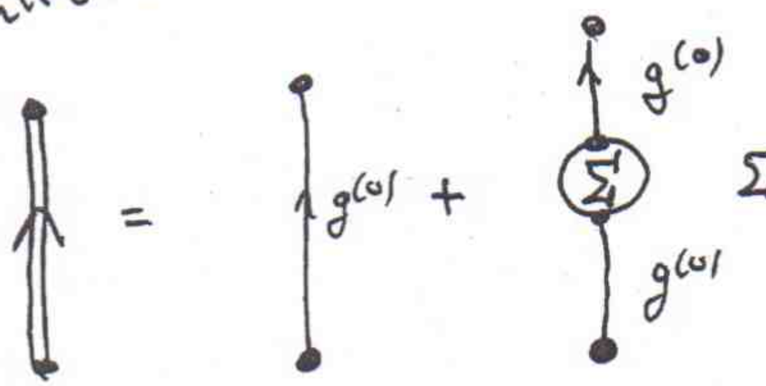
Feynman diagram



Goldstone diagrams.

Dyson equation and self-energy

All diagrams, except (0) order, have a non-interacting propagator at the top and at the bottom of the diagram ⇒ The self-energy represents the sum of all intermediate contributions!



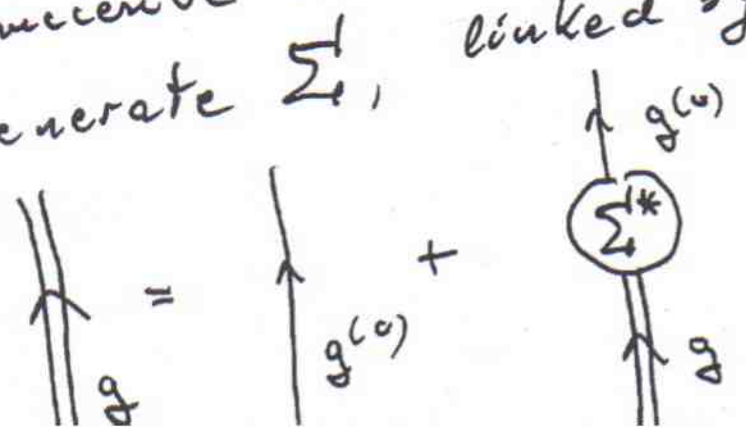
$\Sigma(k,E)$ complex quantity!

Now we classify the self-energy into two categories: irreducible! and reducible.

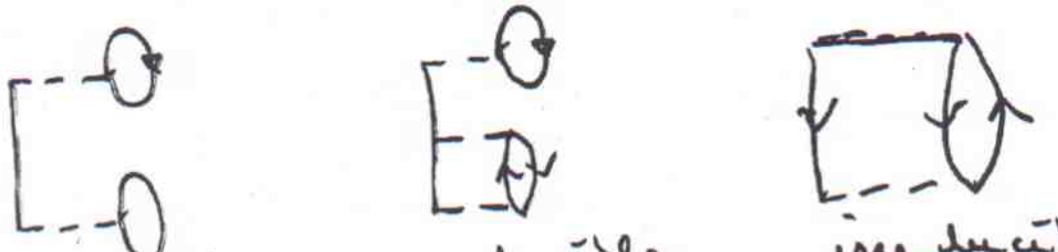
<u>Irreducible:</u> Σ^{*} The diagrams <u>do not</u> contain two (or more) parts that are only connected by an unperturbed sp-propagator $g^{(0)}$.

all other contributions are call reducible

The successive iterations of the irreducible Σ^{*} generate Σ, linked by $g^{(0)}$

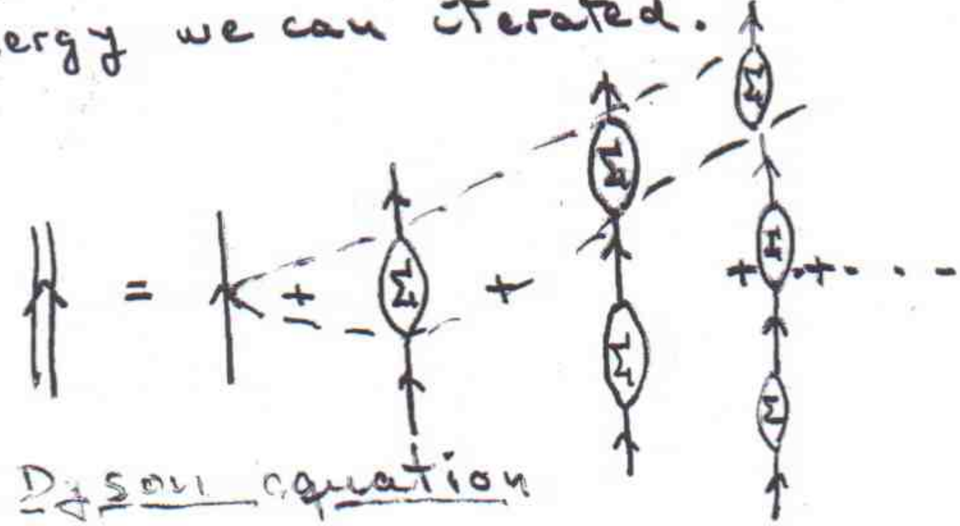


Dyson equation for g.



reducible reducible irreducible.

If we have a piece of the irreducible self-energy we can iterated.



Dyson equation

$$g(k,E) = g^{(0)}(k,E) + g^{(0)}(k,E)\,\Sigma_1(k,E)\,g(k,E)$$

$$g(k,E) = \frac{g^{(0)}}{1 - g^{(0)}\Sigma_1} = \frac{1}{g^{(0)-1} - \Sigma_1(k,E)}$$

$$g(k,E) = \frac{1}{E - \frac{\hbar^2 k^2}{2m} - \Sigma_1(k,E)}$$

$$\boxed{S_h(k,E) = \frac{1}{\pi}\,\frac{\Sigma_{1I}(k,E)}{\left[E - \frac{\hbar^2k^2}{2m} - \Sigma_{1R}(k,E)\right]^2 + \left[\Sigma_{1I}(k,E)\right]^2}}$$

Self-energy $\langle\equiv\rangle$ Green-function