Plan of lectures

Introduction

The Yukawa potential in classical and quantum mechanics

One- and two-meson exchange amplitudes

- Corrections to the Yukawa potential
- Chiral effective field theory
- Interaction Hamiltonians and vertices
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- The χEFT potential to Q^2
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Introduction

Nuclei made up by protons and neutrons (nucleons). These interact via the strong interactions (QCD), and also via the electroweak (EW) interactions, which are much weaker.

Nucleons are not the fundamental degrees of freedom (d.o.f.) of QCD.

These lectures: nuclei as bound states of nucleons interacting amongst themselves via two- and three-body forces and with external EW fields via one-, two-, and three-body currents; this is the "basic model". Other nuclear models, for example the shell model, are approximations of the basic model valid in certain mass number ranges and/or energy regimes.

Basic model:

$$H = \sum_{i=1}^{A} \Big(\underbrace{m_i}_{\text{rest mass}} + \underbrace{\frac{\vec{p}_i^{\,2}}{2m_i}}_{\text{NR kinetic energy}} \Big) + \sum_{i<j} \overbrace{v_{ij}}^{\text{dominant pair potential (theory + data)}} + \sum_{i<j<k} \underbrace{V_{ijk}}_{\text{weaker three-body potential (theory + data)}} + \dots$$

Protons and neutrons are spin $s = 1/2$ particles. If the mass difference between neutron and proton ($m_n - m_p \simeq 1.3$ MeV) is ignored, then the proton and neutron can be considered as two possible states of a particle we call the nucleon (N), of rest mass

$$m = (m_p + m_n)/2 ,$$

having "isospin" $t = 1/2$ and "isospin projections" $t_z = 1/2$ for $|p\rangle$ and $t_z = -1/2$ for $|n\rangle$. The operator $\vec{t} = \frac{1}{2}\vec{\tau}$ is mathematically identical to $\vec{s} = \frac{1}{2}\vec{\sigma}$,

$$[t_i, t_j] = i\varepsilon_{ijk} t_k ,$$

but acts in "isospin space" rather than in ordinary space like $\vec{s}$.

Strong interactions have approximate isospin invariance:

$$H = \underbrace{H_0}_{\text{scalar in isospin space}} + H_{IB} \leftarrow \text{isospin symmetry breaking terms (small)}$$

State of N: $\vec{r}, \sigma_z \tau_z \equiv \vec{x}$; v_{ij} depends on $\vec{x}_i, \vec{x}_j$ and V_{ijk} depends on $\vec{x}_i, \vec{x}_j, \vec{x}_k$.

The basic model assumes that the series of potentials converge rapidly; this is expected from theory (χEFT) and borne out by calculations.

Nucleons are baryons; baryons and meson, i.e. the hadrons, are bound states of quarks (q) and gluons (g), the fundamental d.o.f. of QCD; q and g are confined within the hadrons (confinement)

q come in 6 flavors $\underbrace{\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}}_{\text{3 families}}$ $\begin{matrix} \text{charge} = 2/3\, e \\ = -1/3\, e \end{matrix}$

and each q carries a unit of color charge, and is in one of 3 possible color states, conventionally R, B, G.

Hadrons consist of color singlet states of q and g:

baryons: $|qqq\rangle + |qqqq'\bar{q}'\rangle + \cdots$

mesons: $\underbrace{|q\bar{q}\rangle}_{\text{valence structure}} + |q\bar{q}\,\underbrace{q'\bar{q}'}_{\text{additional } q'\bar{q}' \text{ pairs}}\rangle + \cdots$

For example,

$$|p\rangle = |uud\rangle + \underbrace{|uuddd\rangle}_{\simeq |n\rangle|\pi^+\rangle} + \overbrace{|uuds\bar{s}\rangle}^{\simeq |\Lambda\rangle|K^+\rangle} + \cdots$$

and therefore nucleons have "meson clouds".

Lowest energy baryons:

E(MeV)		J^π	t
1440	N(1440)	$\frac{1}{2}^+$	$\frac{1}{2}$
1232	Δ	$\frac{3}{2}^+$	$\frac{3}{2}$
939	N	$\frac{1}{2}^+$	$\frac{1}{2}$

300 MeV

The basic model assumes that the q and $\bar{q}$ in nuclei are confined in color singlet states close to those of free nucleons. They need not be exactly in free N states in nuclei. The v_{ij} is constrained by experimental data and hence contains all the effects of the quarks in the interacting clusters being in excited states of the nucleon.

Energy scales in nuclei : (binding energy)/A $\lesssim$ 9 MeV

$$\frac{1}{A}\Big(\underbrace{\sum_{i<j} \langle v_{ij} \rangle}_{} + \sum_{i<j<k} \langle V_{ijk} \rangle \Big) \simeq -(10\text{–}50)\,\text{MeV}$$

largely cancelled by
(kinetic energy)/A $\simeq$ 300 MeV

|(potential energy)/A|, (kinetic energy)/A $\ll m_\Delta - m$, and this presumably helps the series of potentials converge rapidly.

The Yukawa potential

Dominant terms in v_{ij} due to exchange of pions; they have a complex dependence on spins and isospins of interacting NN pair because pions have $J^\pi = 0^-$ and $t = 1$ (see later)

We want to investigate the connection between meson exchange interactions and their representation in terms of an instantaneous potential. We simplify the problem (for the time being) by ignoring spins and isospins and by considering scalar meson exchange.

The Yukawa potential in classical mechanics

Consider a scalar field $\phi(\vec{r},t)$ interacting with static particles at positions $\vec{r}_1, \ldots, \vec{r}_A$

$$\mathcal{L} = \underbrace{\frac{1}{2}\left[\dot{\phi}^2 - (\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) - \mu^2\phi^2\right]}_{\mathcal{L}_0} \underbrace{- g\phi \sum_{i=1}^{A} \delta(\vec{r}-\vec{r}_i)}_{\mathcal{L}_{int}}$$

Euler-Lagrange equation gives

$$\ddot{\phi} - \vec{\nabla}^2\phi + \mu^2\phi = -g\sum_i \delta(\vec{r}-\vec{r}_i)$$

and the lowest-energy configuration of the field occurs in the static limit

$$\phi(\vec{r},t) \longrightarrow \bar{\phi}(\vec{r})$$

with

$$\vec{\nabla}^2\bar{\phi} - \mu^2\bar{\phi} = g\sum_i \delta(\vec{r}-\vec{r}_i) \quad ,$$

which, apart from the mass term, looks like the Poisson equation for the electrostatic potential of a system of point-like charges at rest. To solve this equation, assume there is one particle at $\vec{r}_j$ and obtain $\bar{\phi}_j$. Then use superposition

$$\bar{\phi} = \sum_{j=1}^{A} \bar{\phi}_j$$

Translational invariance requires $\bar{\phi}_j \equiv \bar{\phi}_j(\vec{r}-\vec{r}_j)$, and introducing the Fourier tranform

$$\bar{\phi}_j(\vec{r}-\vec{r}_j) = \int_{\vec{k}} e^{i\vec{k}\cdot(\vec{r}-\vec{r}_j)}\,\bar{\phi}_j(\vec{k}) \quad , \quad \int_{\vec{k}} \equiv \int \frac{d\vec{k}}{(2\pi)^3}$$

one finds

$$(k^2+\mu^2)\,\phi_j(\vec{k}) = -g$$

and

$$\bar{\phi}_j(\vec{r}-\vec{r}_j) = -\frac{g}{4\pi}\,\frac{e^{-\mu|\vec{r}-\vec{r}_j|}}{|\vec{r}-\vec{r}_j|} .$$

The energy of the field is given by

$$E_\phi = \int d\vec{r}\; \mathcal{H} \leftarrow \text{Hamiltonian density}$$

which in the static limit reduces to

$$E_{\bar{\phi}} = -\int d\vec{r}\left\{-\frac{1}{2}\left[\underbrace{(\vec{\nabla}\bar{\phi})\cdot(\vec{\nabla}\bar{\phi})}_{\vec{\nabla}\cdot(\bar{\phi}\,\vec{\nabla}\bar{\phi})-\bar{\phi}\,\vec{\nabla}^2\bar{\phi}} + \mu^2\bar{\phi}^2\right] - g\bar{\phi}\sum_i \delta(\vec{r}-\vec{r}_i)\right\}$$

$$= \int d\vec{r}\; \frac{1}{2}g\bar{\phi}\sum_i \delta(\vec{r}-\vec{r}_i)$$

$$= \frac{1}{2}g\sum_i \bar{\phi}(\vec{r}_i) = -\frac{g^2}{8\pi}\sum_{i,j}\frac{e^{-\mu r_{ij}}}{r_{ij}}$$

or

$$E_{\bar{\phi}} = \sum_{i<j} v_Y(r_{ij}) + \text{self energies}$$

$$v_Y(r) = -\frac{g^2}{4\pi}\frac{e^{-\mu r}}{r}.$$

The Yukawa potential in quantum mechanics

We now treat the problem in quantum mechanics, and show that in the limit of static particles the ground-state energy of the field (i.e., the exact ground state in the interacting theory) is the same as in the classical treatment

$$E_0 = -\sum_{i<j}\frac{g^2}{4\pi}\frac{e^{-\mu r_{ij}}}{r_{ij}}.$$

This result is important because it implies that the concept of potential is also valid in quantum mechanics, at least for static or slowly moving particles.

The Hamiltonian reads

$$H = H_0 + H_{int}$$

$$H_0 = \sum_{\vec{k}} \omega_k a^\dagger_{\vec{k}} a_{\vec{k}}, \quad \omega_k = (k^2+\mu^2)^{1/2}$$

$$H_{int} = \int d\vec{r}\, g\phi(\vec{r}) \sum_i \delta(\vec{r}-\vec{r}_i)$$

$$= g\sum_i \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k V}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_i} + a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_i} \right)$$

One could use standard perturbation theory to calculate E_0 (all orders but for the 2^{nd} vanish). Instead, define

$$\underbrace{\alpha_{\vec{k}}}_{\text{c-number}} \equiv \frac{g}{\sqrt{2\omega_k V}} \frac{1}{\omega_k} \sum_i e^{-i\vec{k}\cdot\vec{r}_i}$$

and express the full H as

$$H = \sum_{\vec{k}} \omega_k \Big[\underbrace{(a^\dagger_{\vec{k}} + \alpha^*_{\vec{k}})}_{A^\dagger_{\vec{k}}} \underbrace{(a_{\vec{k}} + \alpha_{\vec{k}})}_{A_{\vec{k}}} - |\alpha_{\vec{k}}|^2 \Big]$$

$$= \underbrace{\sum_{\vec{k}} \omega_k A^\dagger_{\vec{k}} A_{\vec{k}}}_{\text{same spectrum of } H_0} \; \underbrace{- \sum_{\vec{k}} \omega_k |\alpha_{\vec{k}}|^2}_{\text{overall energy shift}}$$

The ground-state energy is then given by the energy shift

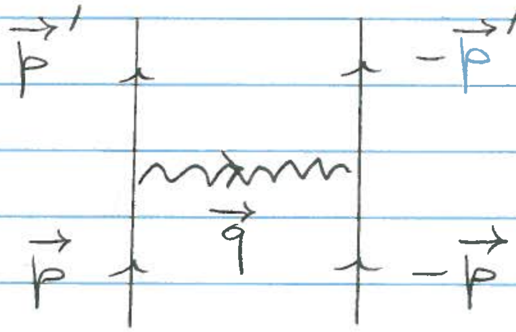
$$E_0 = -\sum_{\vec{k}} \omega_k |\alpha_{\vec{k}}|^2 = -\frac{g^2}{2} \sum_{ij} \int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{r}_i - \vec{r}_j)}}{\omega_k^2}$$

which is exactly the same as the classical result. Therefore in both classical and quantum theories the scalar field energy in the presence of static particles can be replaced by a sum of Yukawa potentials between the particles.

Scattering in Born approximation

Consider scattering between two slowly moving particles interacting via the coupling to the scalar field

$$H = -\frac{1}{2m}\left(\vec{\nabla}_1^2 + \vec{\nabla}_2^2\right) + v_Y(r)$$ ← assumes that ϕ remains in ground state (slow particles)



$$E_i = \frac{p^2}{m}, \quad E_f = \frac{p'^2}{m}$$

$$T^B_{fi} = \int d\vec{r}\, e^{-i\vec{p}\,'\cdot\vec{r}}\, v_Y(r)\, e^{i\vec{p}\cdot\vec{r}}$$

$$= -g^2/(q^2+\mu^2) \equiv \tilde{v}_Y(q), \quad \vec{q} = \vec{p} - \vec{p}\,'$$

One-meson exchange scattering

Calculate the scattering amplitude to order g^2 from meson exchange processes, i.e. in field theory (without using ν_Y). We have

$$H_0 = -\frac{1}{2m}(\vec{\nabla}_1^2 + \vec{\nabla}_2^2) + \sum_{\vec{k}} \omega_k a^\dagger_{\vec{k}} a_{\vec{k}}$$

$$H_{int} = g \sum_{i=1,2} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} (a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_i} + h.c.)$$

and volumes have been set to 1, since they drop out. Scattering theory gives

$$T_{fi} \underset{\eta \to 0^+}{=} \langle f | H_{int} + H_{int} \frac{1}{E_i - H_0 + i\eta} H_{int} + \dots | i \rangle$$

with $E_i = E_f$, and

no meson in initial or final state

$$|i\rangle = |\vec{p}, -\vec{p}; 0\rangle \quad \text{and} \quad |f\rangle = |\vec{p}\,', -\vec{p}\,'; 0\rangle .$$

The first order vanishes, and to order g^2

$$T_{fi} \underset{\eta \to 0^+}{=} \sum_I \frac{\langle \vec{p}\,', -\vec{p}\,'; 0 | H_{int} | I \rangle \langle I | H_{int} | \vec{p}, -\vec{p}; 0 \rangle}{E_i - E_I + i\eta}$$

By considering $H_{int}|\vec{p}, -\vec{p}; 0\rangle$, it is clear that the only intermediate states that can contribute are those with one meson. Since three-momentum is conserved by H_{int}, they are of the type

$$|I_1\rangle = |\vec{p} - \vec{q}, -\vec{p}; \vec{q}\rangle \quad \text{or} \quad |I_2\rangle = |\vec{p}, -\vec{p} - \vec{q}\,'; \vec{q}\,'\rangle$$

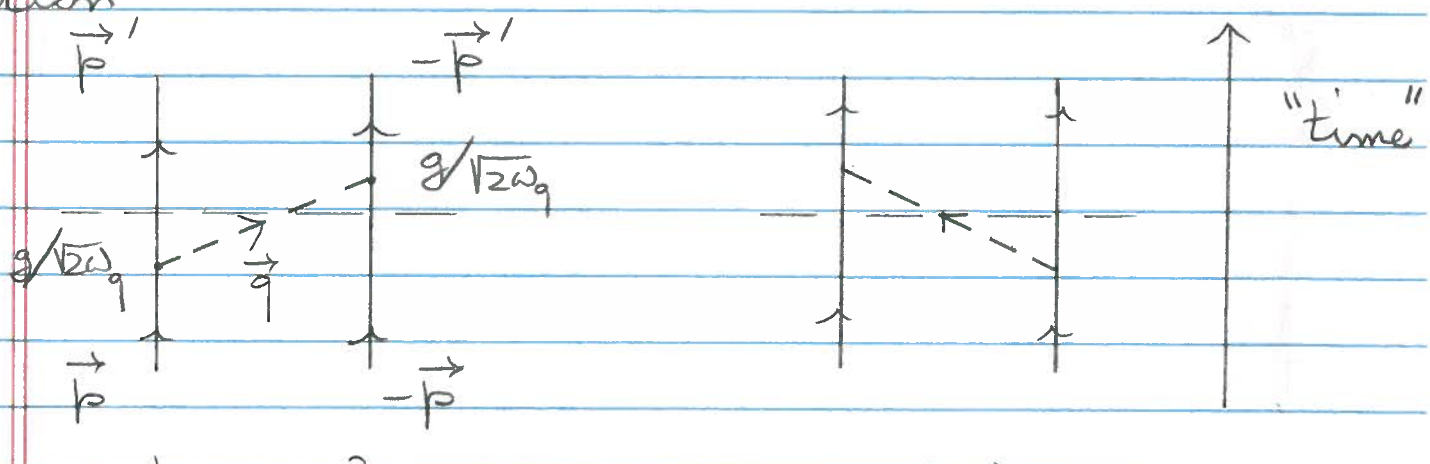
particle 1 emits meson particle 2 emits meson and similarly for the remaining matrix elements. Therefore we find

$$T_{fi} = \underbrace{g/\sqrt{2\omega_q}\;\frac{1}{-\omega_q}\;g/\sqrt{2\omega_q}}_{(a)} + \underbrace{g/\sqrt{2\omega_q}\;\frac{1}{-\omega_q}\;g/\sqrt{2\omega_q}}_{(b)}$$

and the $\eta \to 0^+$ limit can be taken safely here. The two terms (a) and (b) combine to

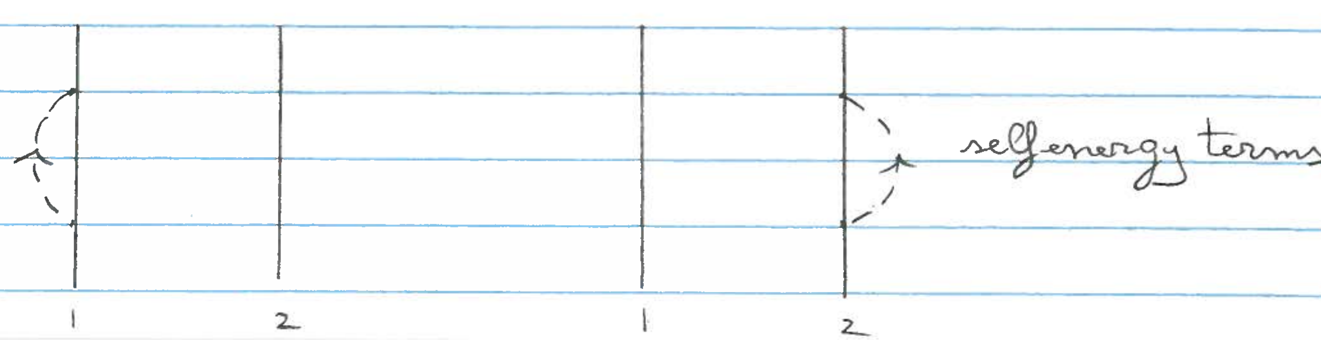
$$T_{fi} = -g^2/\omega_q^2 = \tilde{v}_Y(q) .$$

The terms (a) and (b) can be given a diagrammatic representation



(a) (b)

Note that in T_{fi} are also present the two contributions



Now $\langle \vec{p}\,', -\vec{p}\,'; 0 | H_{int} | I_1 \rangle \neq 0$ only if the meson in $|I_1\rangle$ is absorbed by particle 2 and $\vec{q} = \vec{p} - \vec{p}\,'$.
Similarly, $\langle \vec{p}\,', -\vec{p}\,'; 0 | H_{int} | I_2 \rangle \neq 0$ only if the meson in $|I_2\rangle$ is absorbed by particle 1 and $\vec{q}\,' = \vec{p}\,' - \vec{p} = -\vec{q}$

$$T_{fi} \underset{\eta\to 0^+}{=} \underbrace{\frac{\langle \vec{p}\,', -\vec{p}\,'; 0 | H_{int} | \vec{p}\,', -\vec{p}; \vec{q} \rangle \langle \vec{p}\,', -\vec{p}; \vec{q} \,| H_{int} | \vec{p}, -\vec{p}; 0 \rangle}{E_i - E_{I_1} + i\eta}}_{(a)} \qquad \vec{q} = \vec{p} - \vec{p}\,'$$

$$+ \underbrace{\frac{\langle \vec{p}\,', -\vec{p}\,'; 0 | H_{int} | \vec{p}, -\vec{p}\,'; \vec{q}\,' \rangle \langle \vec{p}, -\vec{p}\,'; \vec{q}\,' | H_{int} | \vec{p}, -\vec{p}; 0 \rangle}{E_i - E_{I_2} + i\eta}}_{(b)} \qquad \vec{q} = \vec{p}\,' - \vec{p} = -\vec{q}$$

$$E_{I_1} = \omega_q + \frac{p'^2}{2m} + \frac{p^2}{2m} \qquad\qquad E_{I_2} = \omega_q + \frac{p^2}{2m} + \frac{p'^2}{2m}$$

$$= \omega_q + E_i \qquad\qquad = \omega_q + E_i$$

since $p = p'$ because of overall energy conservation.

The matrix elements of Hint are readily evaluated:

$$\langle \vec{p}\,', -\vec{p}; \vec{q} \,| H_{int} | \vec{p}, -\vec{p}; 0 \rangle = g \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \langle \cdots; \vec{q} \,| e^{-i\vec{k}\cdot\vec{r}_1} a^\dagger_{\vec{k}} | \cdots; 0 \rangle$$

$$= \frac{g}{\sqrt{2\omega_q}} \langle \vec{p}\,', -\vec{p} \,| e^{-i\vec{q}\cdot\vec{r}_1} | \vec{p}, -\vec{p} \rangle$$

$$= \frac{g}{\sqrt{2\omega_q}}$$

The Yukawa potential in Born approximation gives the same amplitude as the meson-exchange field theory in leading order. Indeed, one can define the Yukawa potential via

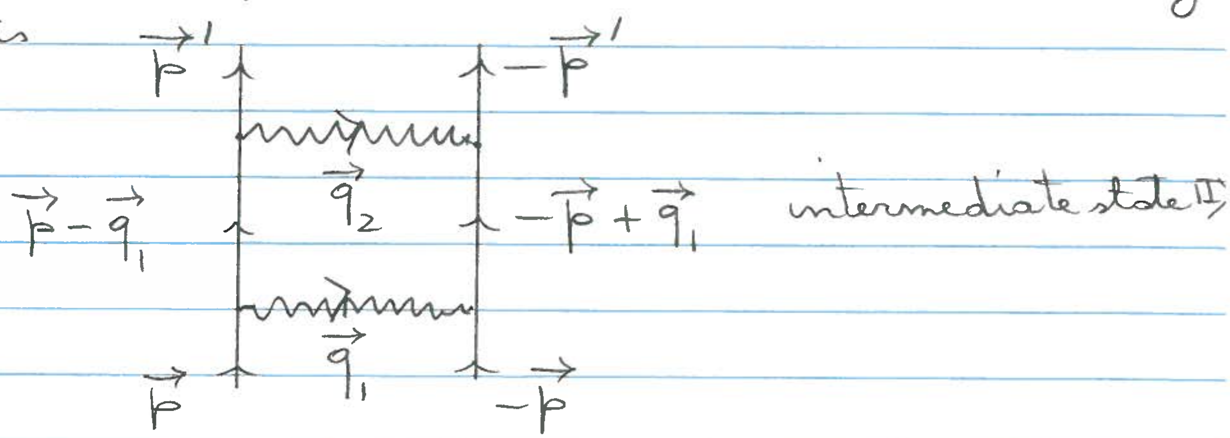
$$\langle f | v_Y | i \rangle \equiv \sum_I \frac{\langle f | H_{int} | I \rangle \langle I | H_{int} | i \rangle}{E_i - E_I}$$

The range of v_Y is $\sim 1/\mu$: $|I\rangle$ violates energy conservation by $\Delta E \sim \omega_q \sim \mu$, and can only exist for a time $\Delta t \sim 1/\mu$, during which the virtual meson propagates for a distance $\lesssim 1/\mu$.

Two-meson exchange amplitude

What happens beyond one-meson exchange? When the coupling constant g is "large", one needs to go beyond the leading order and consider two-meson exchange processes. To what extent is the iterated v_Y a good representation of these processes? We will try answering these questions here.

The 2nd order Yukawa potential contribution to the scattering amplitude is



(Y2)

...e that $\vec{q}_1$ is free to change but, given $\vec{q}_1$, $\vec{q}_2$ is fixed by ...omentum conservation, since

$$\vec{p} - \vec{q}_1 = \vec{p}\,' + \vec{q}_2 \quad \text{or} \quad \vec{q}_2 = \underbrace{\vec{p} - \vec{p}\,'}_{\vec{q}} - \vec{q}_1$$

...e contribution (Y2) reads

$$^{(Y2)}_{\;\;i} = \lim_{\eta \to 0^+} \sum_I \frac{\langle f | v_Y | I \rangle \langle I | v_Y | i \rangle}{E_i - E_I + i\eta} \equiv \sum_{\vec{q}_1} A_Y^{(2)}(\vec{q}_1, \vec{q}_2)$$

...ith

$$\langle \vec{p} - \vec{q}_1 | v_Y | \vec{p} \rangle = \int d\vec{r}\; e^{-i(\cancel{\vec{p}} - \vec{q}_1)\cdot\vec{r}}\; v_Y(r)\; e^{\cancel{i\vec{p}\cdot\vec{r}}} = -g^2/\omega_1^2$$

$$\langle \vec{p}\,' | v_Y | \vec{p} - \vec{q}_1 \rangle = \int d\vec{r}\; e^{-i\vec{p}\,'\cdot\vec{r}}\; v_Y(r)\; e^{i(\vec{p} - \vec{q}_1)\cdot\vec{r}} = -g^2/\omega_2^2$$

$$E_i - E_I = \frac{p^2}{m} - \frac{(\vec{p} - \vec{q}_1)^2}{m} = (-q_1^2 + 2\vec{p}\cdot\vec{q}_1)/m$$

...he amplitude $A_Y^{(2)}$ simply reads:

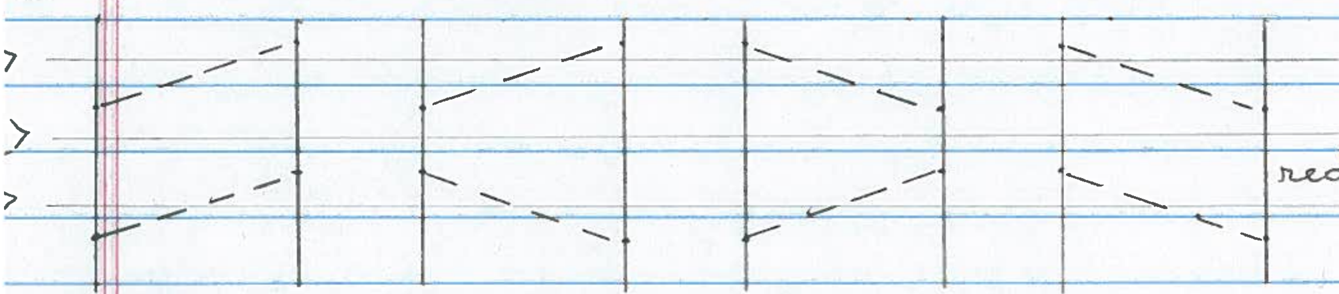
$$A_Y^{(2)} = -\frac{g^4}{\omega_1^2 \omega_2^2}\,\frac{1}{2\delta}\,, \quad 2\delta \equiv (q_1^2 - 2\vec{p}\cdot\vec{q}_1)/m - i\eta$$

...e exact two-meson exchange amplitude comes from

$$= \langle f | H_{int} \frac{1}{E_i - H_0 + i\eta} H_{int} \frac{1}{E_i - H_0 + i\eta} H_{int} \frac{1}{E_i - H_0 + i\eta} H_{int} | i \rangle$$

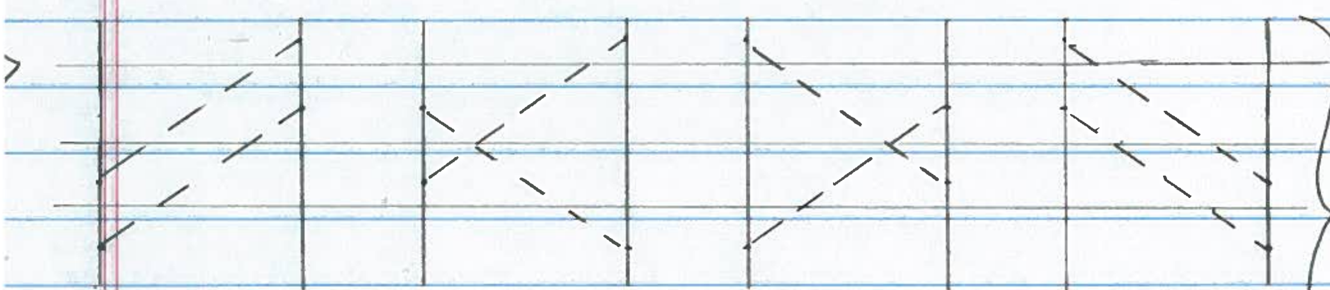
$$= \sum_{I_1 I_2 I_3} \underbrace{\frac{\langle f | H_{int} | I_3 \rangle \langle I_3 | H_{int} | I_2 \rangle \langle I_2 | H_{int} | I_1 \rangle \langle I_1 | H_{int} | i \rangle}{(E_i - E_{I_3} + i\eta)(E_i - E_{I_2} + i\eta)(E_i - E_{I_1} + i\eta)}}_{A^{(2)}}$$

is easier to analyze the contribution $A^{(2)}$ diagrammatically. There are 12 diagrams that contribute

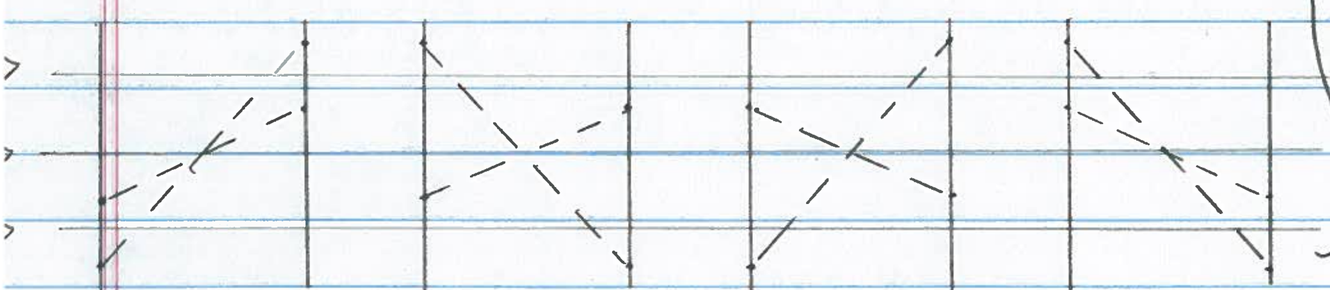


(a1) (a2) (a3) (a4)

reducible



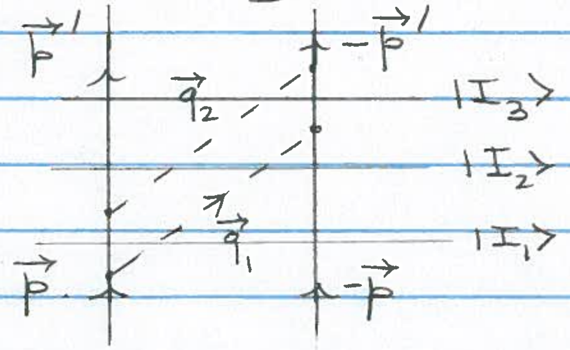
(b1) (b2) (b3) (b4)



(c1) (c2) (c3) (c4)

irreducible

ir contributions are easily worked out. Consider (b1) as example



$$) = \frac{g}{\sqrt{2\omega_2}}\frac{1}{E_i - E_{I_3} + i\eta}\frac{g}{\sqrt{2\omega_1}}\frac{1}{E_i - E_{I_2} + i\eta}\frac{g}{\sqrt{2\omega_2}}\frac{1}{E_i - E_{I_1} + i\eta}\frac{g}{\sqrt{2\omega_1}}$$

$$= \frac{g^4}{4\omega_1^2\omega_2^2}\frac{1}{(\ \cdot\ \cdot\ \cdot\ )}$$

$$= \frac{(\vec{p}-\vec{q}_1)^2}{2m} + \frac{p^2}{2m} + \omega_1 \qquad E_{I_1} - E_i - i\eta = \delta + \omega_1$$

$$= \frac{p'^2}{2m} + \frac{p^2}{2m} + \omega_1 + \omega_2 \qquad E_{I_2} - E_i - i\eta = \omega_1 + \omega_2 - i\eta$$

$$I_3 = \frac{p'^2}{2m} + \frac{(-\vec{p}+\vec{q}_1)^2}{2m} + \omega_2 \qquad E_{I_3} - E_i - i\eta = \delta + \omega_2$$

$$) = -\frac{g^4}{4\omega_1^2\omega_2^2}\frac{1}{(\delta+\omega_1)(\omega_1+\omega_2)(\delta+\omega_2)}$$

$$= \frac{1}{2}A_Y^{(2)}\frac{\delta}{(1+\delta/\omega_1)(\omega_1+\omega_2)(1+\delta/\omega_2)}$$

Define
$$A^{(2)}_{M}(\vec{q}_1,\vec{q}_2) \equiv \sum_{i=1}^{4} \left[(ai) + (bi) + (ci) \right]$$

and in the slow-particles limit (m "large") and $\delta \ll \omega_1, \omega_2$, we find

$$A^{(2)}_{M} = A^{(2)}_{Y}\left[1 - \frac{(\omega_1-\omega_2)^2\,\delta}{\omega_1\omega_2(\omega_1+\omega_2)} + \mathcal{O}(\delta^2) \right]$$

Note that for slow particles $|\vec{p}-\vec{p}\,'| = q = |\vec{q}_1+\vec{q}_2|$ is small, implying $\vec{q}_1 \simeq -\vec{q}_2$ and $\omega_1 \simeq \omega_2$, which shows that in this limit corrections to $A^{(2)}_{Y}$ due to two-meson exchange contributions are of order δ^2.

Corrections to the Yukawa potential

From the scattering amplitudes $T^{(1/2)}_{fi}$ and $T^{(2)}_{fi}$ one obtains a correction term

$$\tilde{v}^{(2)}(\vec{p},\vec{q}) = \int_{\vec{q}_1} \left[A^{(2)}_{M}(\vec{q}_1, \vec{q}-\vec{q}_1) - A^{(2)}_{Y}(\vec{q}_1, \vec{q}-\vec{q}_1) \right]$$

and can define a (non-local) potential

$$\tilde{v}(\vec{p},\vec{q}) = \tilde{v}_Y(\vec{q}) + \tilde{v}^{(2)}(\vec{p},\vec{q})$$

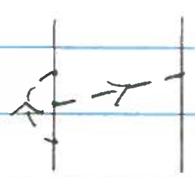
such that it reproduces the exact scattering amplitude up to and including two-meson exchanges. One can doviously extend this procedure by considering the scattering amplitude originating from three-meson exchanges, and so on. The series of potentials

$$\tilde{v} = \tilde{v}_\gamma + \sum_{\ell=2}^{n} \tilde{v}^{(\ell)}$$

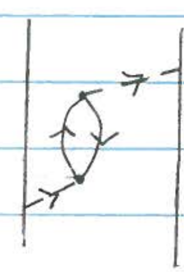
presumably converges if g is small and/or m is large. The range of $\tilde{v}^{(\ell)}$ is $1/(\ell\mu)$, and thus the $\tilde{v}^{(\ell)}$ form a series of potentials with decreasing ranges.

The above discussion is valid for point-like (truly elementary) particles (electrons and photons, for example). However, nucleons and mesons have internal structure and hence many additional processes contribute to $\tilde{v}^{(2)}$. Models of $\tilde{v}$ are obtained by reproducing NN scattering data, which naturally contain all possible many-meson exchange amplitudes. In contrast, the corrections to the Coulomb potential between electrons due to two-photon exchange processes can be calculated with the techniques presented above.

Additional corrections to $\tilde{v}_\gamma$ of order g^4 are obtained from vertex



and vacuum polarization



diagrams. These corrections change the shape of $V_Y(r)$ at small r, and are important in both QED and QCD. Their role in shaping nuclear forces is not obvious. Nucleons and mesons have a finite size. The short-distance behavior of meson-exchange interactions is modified by their size and internal structure. It is likely that only the long-range parts of nuclear forces can be conveniently described as due to exchange of mesons. In practice, the short-range interaction between nucleons is obtained from data along with two-pion exchange interactions.

Chiral effective field theory

Over the past two decades, chiral effective field theory (χEFT), originally proposed by Weinberg in a series of papers in early nineties, has blossomed into a very active field of research. The chiral symmetry exhibited by QCD severely restricts the form of the interactions of pions amongst themselves and with other particles. In particular, the pion couples to baryons, such as nucleons and Δ-isobars, by powers of its momentum Q, and the Lagrangian describing these interactions can be expanded in powers of Q/Λ_χ, where $\Lambda_\chi \sim 1$ GeV specifies the chiral-symmetry breaking scale. As a result, classes of Lagrangians emerge, each characterized by a given power of Q/Λ_χ and each involving a certain number of unknown coefficients, so called low-energy constants (LEC's), which are then determined by fits to experimental data.

We will limit our considerations to an effective Lagrangian of the type

$$\mathcal{L}_{eff} = \mathcal{L}_{\pi N} + \mathcal{L}_{\pi\pi}$$

$\mathcal{L}_{\pi N}$: describes interactions between N's and π's

$\mathcal{L}_{\pi\pi}$: describes interactions amongst π's

and chiral symmetry implies that the individual Lagrangians can be organised as

$$\mathcal{L}_{\pi N} = \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi N} + \cdots$$

$$\mathcal{L}_{\pi\pi} = \mathcal{L}^{(2)}_{\pi\pi} + \mathcal{L}^{(4)}_{\pi\pi} + \cdots$$

where the superscript refers to the number of derivatives or pion mass factors. The procedure used to construct $\mathcal{L}^{(n)}_{\pi N}$ and $\mathcal{L}^{(n)}_{\pi\pi}$ has been reviewed in a number of papers (see Meissner and collaborators). Apart from being invariant under chiral transformations, they are also invariant under the other symmetries of QCD, including parity and charge conjugation (and hence time reversal).

In the limit of slow nucleons, the leading order πN Lagrangian reads

$$\mathcal{L}^{(1)}_{\pi N} = N^\dagger\Big[i\partial_0 - \frac{g_A}{2f_\pi}\tau_a(\vec{\sigma}\cdot\vec{\nabla}\pi_a) - \underbrace{\frac{1}{4f_\pi^2}\vec{\tau}\cdot(\vec{\pi}\times\partial_0\vec{\pi})}_{\text{Weinberg-Tomozawa (WT) term}} + \text{terms with} \geq 3\,\pi\text{'s}\Big]N$$

while the $\pi\pi$ Lagrangian is simply

$$\mathcal{L}^{(2)}_{\pi\pi} = \frac{1}{2}(\partial_\mu \vec{\pi})\cdot(\partial^\mu \vec{\pi}) - \frac{m_\pi^2}{2}\vec{\pi}^2 + \text{terms with} \geq 4\pi\text{'s}$$

Here N_τ $(\tau = \pm 1)$ and π_a $(a = 1, 2, 3)$ represent, respectively, the isodoublet nucleon field and istriplet pion field, g_A $(\simeq 1.25)$ and f_π $(\simeq 93\ \text{MeV})$ are the nucleon axial coupling constant and pion decay amplitude, and m_π is the pion mass.

In addition to these there are contact Lagrangians involving four nucleon fields. Such terms are needed to parametrize the unresolved short-distance dynamics of the nuclear force. They are also required to renormalize loop integrals and to make results reasonably independent of regulators.

Because of parity, nucleon contact interactions come only in even powers of derivatives

$$\mathcal{L}_{NN} = \mathcal{L}^{(0)}_{NN} + \mathcal{L}^{(2)}_{NN} + \cdots$$

with

$$\mathcal{L}^{(0)}_{NN} = -\frac{1}{2}C_S (N^\dagger N)(N^\dagger N) - \frac{1}{2}C_T (N^\dagger \vec{\sigma} N)\cdot(N^\dagger \vec{\sigma} N)$$

$$\mathcal{L}^{(2)}_{NN} = -\sum_{i=1}^{12} C_i' O_i$$

where $\mathcal{O}_1 = (N^\dagger \overrightarrow{\nabla} N)^2 + h.c.$

$\mathcal{O}_2 = (N^\dagger \overrightarrow{\nabla} N)\cdot(N^\dagger \overleftarrow{\nabla} N)$, ...

for the complete list see Girlanda et al. (2010).

Interaction Hamiltonians and vertices

The conventional perturbative expansion for the NN amplitude is formulated in terms of Hamiltonians. These follow from ($\mathcal{L} = \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi\pi}$):

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\pi}_a}, \quad \Pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{N}}, \quad \mathcal{H} = \Pi^\dagger \dot{N} + \Pi_a \dot{\pi}_a - \mathcal{L}$$

where
$$\Pi^\dagger = i N^\dagger$$

from WT term

$$\Pi_a = \dot{\pi}_a - \frac{1}{4f_\pi^2} N^\dagger(\vec{\tau} \times \vec{\pi})_a N$$

We find
$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$$

where $\mathcal{H}_0$ is the free pion Hamiltonian (the free nucleon Hamiltonian vanishes in the static limit) and $\mathcal{H}_{int}$ has several terms

$$\mathcal{H}_{int} = N^\dagger \left[\frac{g_A}{2f_\pi} \tau_a \vec{\sigma}\cdot(\vec{\nabla}\pi)_a + \frac{1}{4f_\pi^2} \vec{\tau}\cdot(\vec{\pi} \times \vec{\Pi}) \right] N +$$

$$+\frac{1}{32 f_\pi^4} N^\dagger(\vec{\tau}\times\vec{\pi})N\cdot N^\dagger(\vec{\tau}\times\vec{\pi})N + \frac{1}{2}C_S (N^\dagger N)(N^\dagger N)$$

$$+\frac{1}{2}C_T (N^\dagger\vec{\sigma}N)\cdot(N^\dagger\vec{\sigma}N)$$

where the contact contributions have also been included. The explicit expressions for the various fields (all in Schrö-dinger picture) are

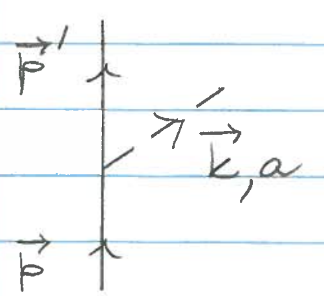
$$\pi_a(\vec{x}) = \sum_{\vec{p}} \frac{1}{\sqrt{2\omega_p}}\left[c_{\vec{p},a}\, e^{i\vec{p}\cdot\vec{x}} + h.c.\right]$$

$$\Pi_a(\vec{x}) = \sum_{\vec{p}} (-i)\sqrt{\frac{\omega_p}{2}}\left[c_{\vec{p},a}\, e^{i\vec{p}\cdot\vec{x}} - h.c.\right]$$

$$N_{\sigma\tau}(\vec{x}) = \sum_{\vec{p}} b_{\vec{p}\sigma\tau}\, e^{i\vec{p}\cdot\vec{x}}\, \chi_{\sigma\tau} \quad \text{spin-isospin state}$$

and the c's and $c^\dagger$'s satisfy commutation relations, while the b's and $b^\dagger$'s satisfy anticommutation relations.

The Hamiltonian $\int d\vec{x}\, \mathcal{H}_{int}$ leads to the following vertices:



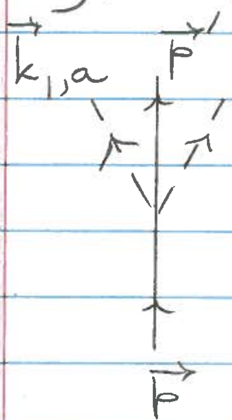
$$= -i\frac{g_A}{2f_\pi}\frac{\vec{\sigma}\cdot\vec{k}}{\sqrt{2\omega_k}}\tau_a \sim Q$$

It follows from evaluating the matrix element

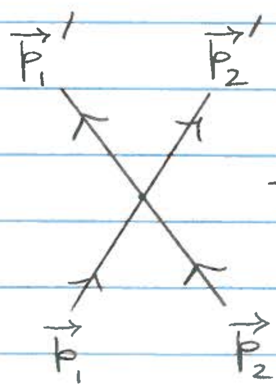
$$\langle \vec{p}\,'\sigma'\tau';\vec{k},a|\int d\vec{x}\, N^\dagger \frac{g_A}{2f_\pi}\tau_a\vec{\sigma}\cdot(\vec{\nabla}\pi_a)N\,|\vec{p}\,\sigma\tau\rangle.$$

Note that a $\delta(\vec{p}-\vec{p}\,'-\vec{k})$ has been dropped for simplicity*. This vertex is "counted" as of order Q since it involves a single power of the momentum. Note that the factor $1/\sqrt{2\omega_k}$ with $\omega_k \sim Q$ from the normalization of the pion field is not included in the counting at this stage (see below).

Similarly, one finds



$$= -\frac{i}{4f_\pi^2}\,\frac{\omega_{k_1}-\omega_{k_2}}{\sqrt{4\omega_{k_1}\omega_{k_2}}}\,\varepsilon_{abc}\tau_c \sim Q$$



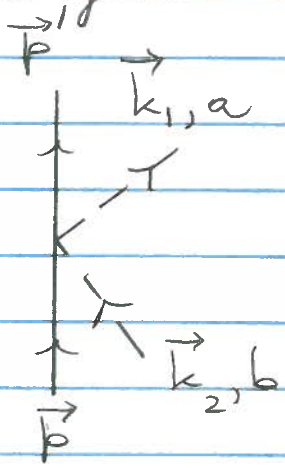
$$= C_S + C_T\,\vec{\sigma}_1\cdot\vec{\sigma}_2 \sim Q^0$$

and we neglect the vertex proportional to $1/f_\pi^4$ since it does not enter at the order of interest (see below). Finally, vertices in which one or more pions are in the initial state are

---

* We have also dropped the spin-isospin states of the initial and final nucleons.

obtained from those above by replacing $\vec{k}_i \longrightarrow -\vec{k}_i$ or $\omega_{k_i} \longrightarrow -\omega_{k_i}$ (the energy replacements are not to be carried out in pion-field normalization factors). For example,



$$= -\frac{i}{4f_\pi^2}\,\frac{\omega_{k_1}+\omega_{k_2}}{\sqrt{4\omega_{k_1}\omega_{k_2}}}\,\varepsilon_{abc}\,\tau_c \sim Q$$

Power counting

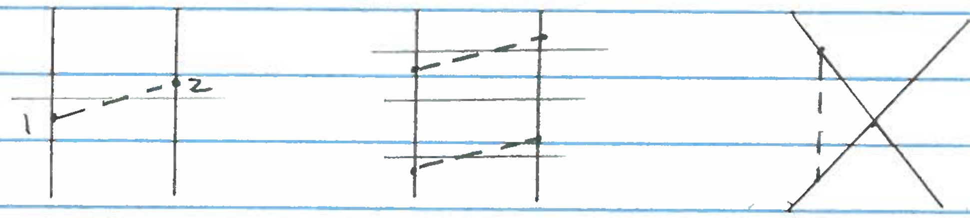
We use the conventional perturbative expansion for the NN scattering amplitude

$$\langle f|T|i\rangle = \langle f|\, H_{int} \sum_{n=1}^{\infty} \left(\frac{1}{E_i - H_0 + i\eta} H_{int} \right)^{n-1} |i\rangle$$

where $|i\rangle$ and $|f\rangle$ represent the initial and final NN states of energy $E_i = E_f$, H_0 is the Hamiltonian describing free pions and nucleons, and H_{int} is the Hamiltonian describing interactions among these particles. Power counting is used to organize the expansion in powers of Q/Λ_χ.

The evaluation of T_{fi} is carried out in practice by inserting complete sets of H_0 eigenstates between successive terms of H_{int}.

As already discussed, the resulting contributions are conveniently represented by diagrams, for example



(a) (b) (c)

Each diagram is characterized by, say, N vertices, $(N-1)$ energy denominators, and possibly L loops; in (b) and (c) above, for example, $L=1$.

A vertex i has
$$Q^{\alpha_i} \times Q^{-\beta_i/2}$$
where α_i is the power counting implied by the relevant term in H_{int} and β_i is the number of pions in and/or out of the vertex. In (a), vertex 1 has $Q \times Q^{-1/2}$

Of the $(N-1)$ energy denominators, a subset N_K of them will possibly) involve only nucleon kinetic energies, which scale as Q^2, and $N-N_K-1$ will involve, in addition pion energies, which are of order Q. In (b), $N_K=1$ and $N-N_K-1=2$.

Each loop contains an integration over the three momentum, and therefore is counted as Q^3.

In summary, the power counting associated with a given contribution is $\left(\prod_{i=1}^{N} Q^{\alpha_i-\beta_i/2}\right)\left[Q^{-(N-N_K-1)}\, Q^{-2N_K}\right] Q^{3L}$.

Clearly, each of the $N - N_K - 1$ energy denominators can be further expanded as

$$\frac{1}{E_i - E_I - \omega_\pi} = -\frac{1}{\omega_\pi}\left[1 + \underbrace{\frac{E_i - E_I}{\omega_\pi}}_{\text{this ratio is order } Q} + \frac{(E_i - E_I)^2}{\omega_\pi^2} + \cdots\right]$$

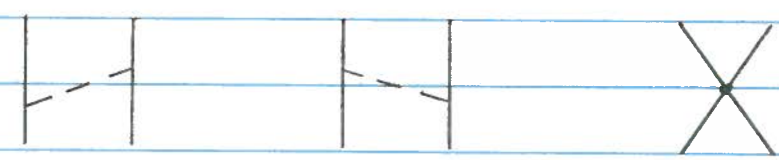
nucleon kinetic energies

Therefore, we organize the expansion for T as

$$T = T^{(0)} + T^{(1)} + T^{(2)} + \cdots, \quad T^{(n)} \sim Q^n$$

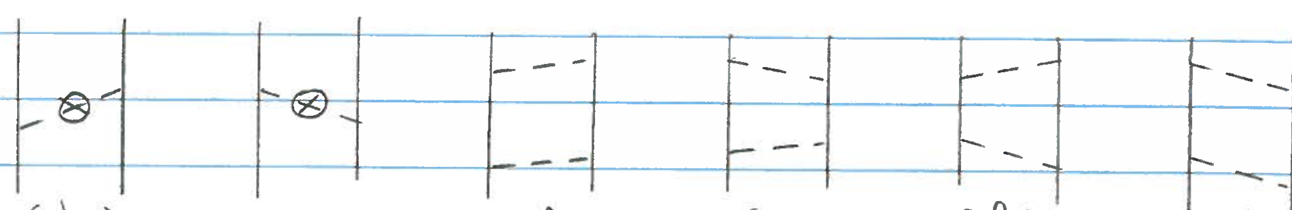
For example,

$T^{(0)} =$

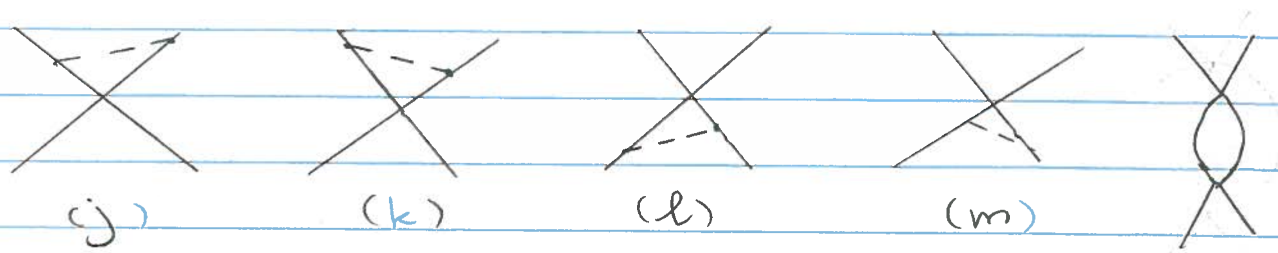


(a) (b) (c)

$T^{(1)} =$



(d) (e) (f) (g) (h) (i)



(j) (k) (l) (m) (n)

where the dash lines w/o or with $\otimes$ indicate that only the leading $-1/\omega_\pi$ term or next-to-leading $-(E_i-E_I)/\omega_\pi^2$ term is retained in the expansion of the associated energy denominator.

From amplitudes to potentials

We want to put on more formal grounds the considerations made on the connection between field theory amplitudes and potentials and illustrated for the case of the scalar mesons interacting with slow moving particles.

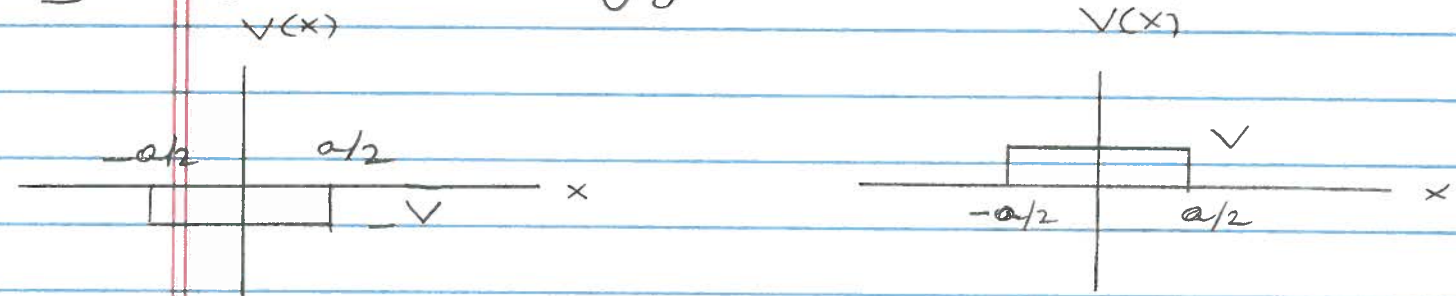
Our objective is to derive a two-nucleon potential v, which when iterated in the Schrödinger (S) or Lippmann-Schwinger (LS) equation,

$$v + v G_0 v + v G_0 v G_0 v + \dots$$

G_0 is the two-nucleon free propagator $1/(E_i - E_I + i\eta)$

leads to the T-matrix calculated in field theory order by order in the power counting. In practice, this requirement can only be satisfied up to a given order n^*, and the resulting potential, when inserted into the LS or S equation will generate contributions of order $n > n^*$ which do not match $T^{(n)}$.

One may wonder why we want to iterate v to all orders instead of doing perturbation theory up to n^*. This perturbative treatnent may be (and has been) used to study low energy scattering of nucleons in higher partial waves ($F, G, H, \ldots$) in which phase shifts are small (so called peripheral waves). However, bound states of an (even weak) attractive potential never come out of doing perturbation theory on free-particle (plane-wave) states. Expanding the states in a power series implies that the physics of the following problems, for example



there is always at least one bound state with energy
$E_0 = -\frac{m(aV)^2}{2\hbar^2}$ regardless of $V(>0)$,

is very similar for small V. But this is certainly not the case for a bound state: a weak attractive potential can have a bound state, but a weak repulsive potential cannot, a much different situation.

We now return to the problem of constructing v. We assume that v has the expansion

$$v = v^{(0)} + v^{(1)} + v^{(2)} + \ldots \quad , \quad v^{(n)} \sim Q^{n}.$$

We note that a term like

$$v^{(m)} G_0 v^{(n)} \sim Q^{m+n+1}$$

since in momentum space ($\vec{p}, \vec{p}'$ and relative momenta)

$$\underbrace{\int_{\vec{k}}}_{Q^3} \underbrace{\langle \vec{p}'| v^{(m)} |\vec{k}\rangle}_{Q^m} \underbrace{\frac{1}{E_i - \frac{k^2}{m} + i\eta}}_{Q^{-2}} \underbrace{\langle \vec{k}| v^{(n)} |\vec{p}\rangle}_{Q^n} \sim Q^{m+n+1}$$

We require

$$v^{(0)} = T^{(0)}$$

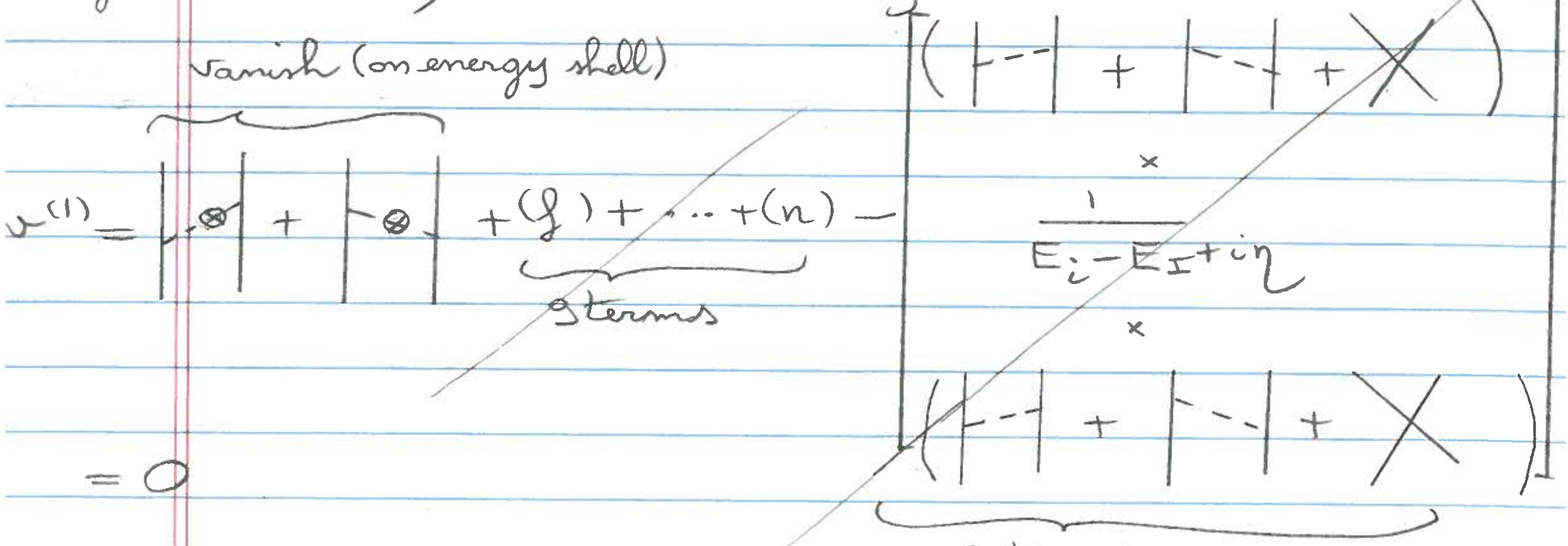
$$v^{(1)} + v^{(0)} G_0 v^{(0)} = T^{(1)}$$

$$v^{(2)} + v^{(0)} G_0 v^{(0)} G_0 v^{(0)} + v^{(1)} G_0 v^{(0)} + v^{(0)} G_0 v^{(1)} = T^{(2)}, \ldots$$

which are easily solved for the $v^{(n)}$, for example

$$v^{(1)} = T^{(1)} - v^{(0)} G_0 v^{(0)}.$$

Diagrammatically, this relation says



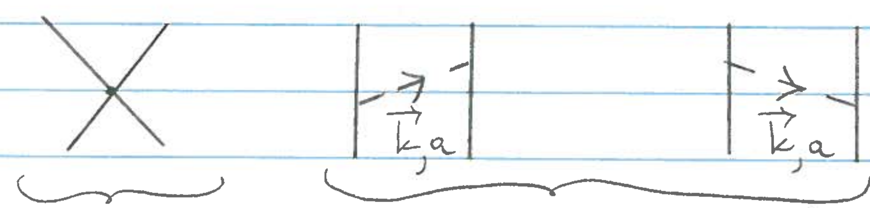
Complete or partial cancellations of this type persist at higher ($n \geq 2$) orders. In fact, for $n \geq 2$ ambiguities arise depending on the off-the-energy-shell extrapolation adopted for the non-static correction for the one-pion-exchange (OPE) potential. It turns out [see Pastore et al. (2011)] that in the center-of-mass system

$$v_\pi^{(2)}(\nu) = (1-2\nu)\, \underbrace{v_\pi^{(0)}(\vec{k})}_{\text{OPE potential}}\, \frac{1}{\omega_k^2}\, \frac{(\vec{k}\cdot\vec{K})^2}{m^2}\,, \qquad \vec{K} = (\vec{p}\,' + \vec{p})/2 \qquad \vec{k} = \vec{p}\,' - \vec{p}$$

with ν an arbitrary parameter. However, these ν-dependent (and hence different) extrapolations are related to each other by a unitary transformation [Friar (1977), Pastore et al. (2011)].

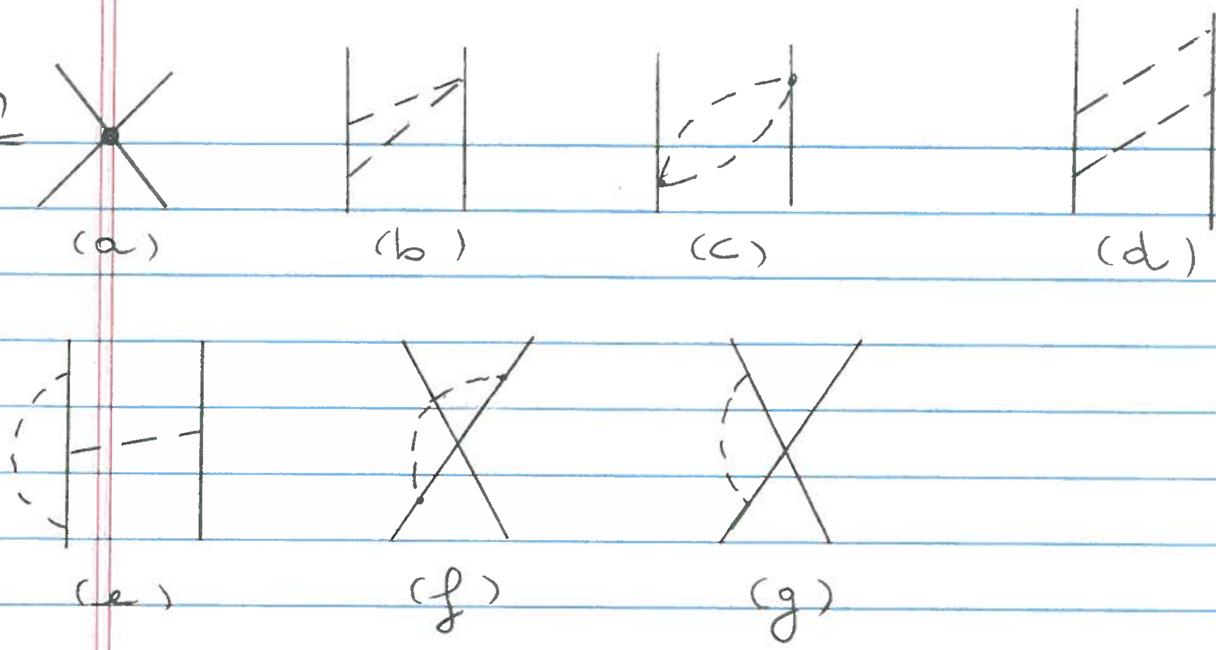
The χEFT potential to Q^2

It is now straightforward to construct the potential up to order Q^2. Diagrammatically, we find

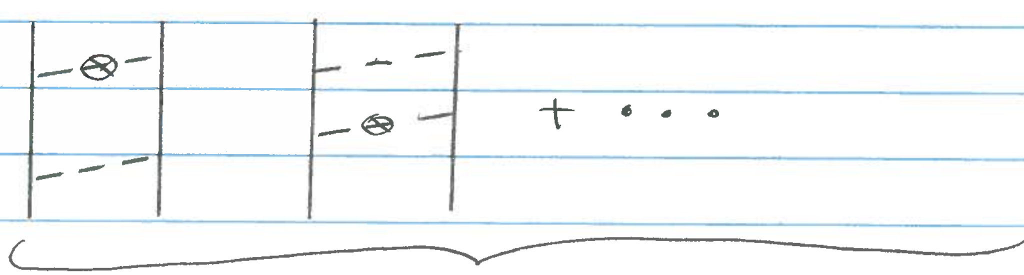
$v^{(0)} =$ 

$$= C_S + C_T\, \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{g_A^2}{4 f_\pi^2}\, \vec{\tau}_1 \cdot \vec{\tau}_2\, \frac{\vec{\sigma}_1 \cdot \vec{k}\;\; \vec{\sigma}_2 \cdot \vec{k}}{\omega_k^2}$$

$v^{(2)} =$



Only one among the possible time orderings is shown for contributions (b)–(g). Diagrams (b) and (c) are irreducible and energy denominators are evaluated in the static limit $1/(E_i - E_I - \omega_\pi) \longrightarrow -1/\omega_\pi$. In contrast, (d)–(g) have reducible and irreducible topologies. For the latter, energy denominators are evaluated in the static limit. However, for the former the next-to-leading order term is retained in the expansion of the energy denominators, $1/(E_i - E_I - \omega_\pi) \longrightarrow -(E_i - E_I)/\omega_\pi^2$. For example,

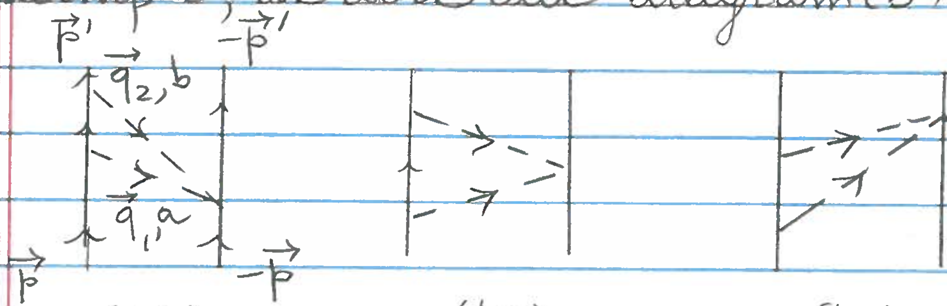


reducible box diagrams

counting:

$$\sim \underbrace{\left(\frac{Q^4}{Q^2}\right)}_{\text{vertices}} \underbrace{\left[Q^0 \frac{1}{Q^2}\frac{1}{Q}\right]}_{\text{energy denominators}} \underbrace{\left(Q^3\right)}_{\text{loop}} \sim Q^2$$

As an example, we work out diagram (b):



(b1) (b2) (b3)

$$= \left(-i\frac{g_A}{2f_\pi}\frac{\sigma_1\cdot q_2}{\sqrt{2\omega_2}}\tau_{1b}\right)\left(-i\frac{g_A}{2f_\pi}\frac{\vec\sigma_1\cdot\vec q_1}{\sqrt{2\omega_1}}\tau_{1a}\right)\left(-\frac{i}{4f_\pi^2}\varepsilon_{abc}\tau_{2c}\frac{1}{\sqrt{4\omega_1\omega_2}}\right)$$

$$\times\left[\frac{(\omega_1-\omega_2)}{\omega_2(\omega_1+\omega_2)} - \frac{(\omega_1+\omega_2)}{\omega_2\omega_1} - \frac{(\omega_1-\omega_2)}{(\omega_1+\omega_2)\omega_1}\right]$$

Combine energy denominators and carry out Pauli matrix algebra to find

$\bar\delta = (2\pi)^3\delta$

$$\underset{\vec p\,'-\vec p}{v_\Delta(\vec k)} = \int_{\vec q_1}\int_{\vec q_2} \bar\delta(\vec q_1+\vec q_2+\vec k)\left(-\frac{g_A^2}{8f_\pi^4}\right)\vec\tau_1\cdot\vec\tau_2\,\frac{1}{\omega_1\omega_2(\omega_1+\omega_2)}\left[\vec q_1\cdot\vec q_2 \cancel{- i\vec\sigma_1\cdot(\vec q_1\times\vec q_2)}\right] + 1\to 2$$

odd under $\vec q_1 \rightleftarrows \vec q_2$ exchange

Change variables to $\vec P = \vec q_1 + \vec q_2$ and $\vec p = \vec p_1 - \vec p_2$ to obtain (after summing over $1\to 2$):

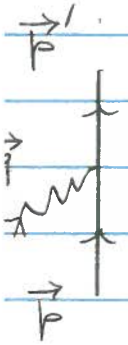
$$v_\Delta(k) = -\frac{g_A^2}{16 f_\pi^4}\vec\tau_1\cdot\vec\tau_2\int_{\vec p}\frac{k^2-p^2}{\omega_+\omega_-(\omega_++\omega_-)}, \qquad \omega_\pm = \sqrt{(\vec k\pm\vec p)^2+4m_\pi^2}$$

The integral is divergent and needs to be regularized. Dimensional regularization is often used to this end [see Pastore et al. (2009)]. The divergent pieces are reabsorbed in the LEC's characterizing the Q^0 and Q^2 contact interactions.

Coupling to EM fields and EM vertices

The coupling to an EM field A^μ follows from minimal substitution: $\partial^\mu \longrightarrow \partial^\mu - ieA^\mu$ ($e>0$). However, the chiral Lagrangians also lead to non-minimal couplings via $F^{\mu\nu}$. The derivation of the relevant interaction Hamiltonians with EM fields from the chiral Lagrangians $\mathcal{L}^{(1)}_{\pi N}$, $\mathcal{L}^{(2)}_{\pi N}$, $\mathcal{L}^{(3)}_{\pi N}$ and $\mathcal{L}^{(2)}_{\pi\pi}$ and $\mathcal{L}^{(4)}_{\pi\pi}$ is discussed in a number of references, for example Kölling et al (2011), Pastore et al. (2009 and 2011). It is fairly straightforward, and will not be given here. Rather we list the leading order vertices and their power counting

ρ $\qquad\qquad\qquad\qquad\qquad$ $\vec{j}$

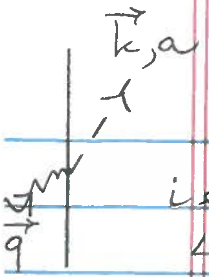


$$e\,e_N \sim eQ^0\,; \qquad \frac{e}{2m}\left[e_N(\vec{p}\,'+\vec{p}) + i\mu_N\,\vec{\sigma}\times\vec{q}\,\right] \sim eQ$$

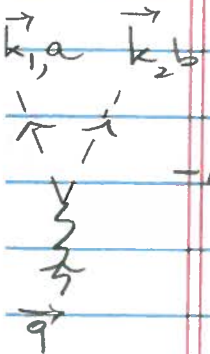
Where

$$e_N = \frac{1+\tau_z}{2}\,, \qquad \mu_N = \frac{\mu_S+\mu_V\tau_z}{2}$$

ρ $\qquad\qquad\qquad\qquad$ $\vec{j}$

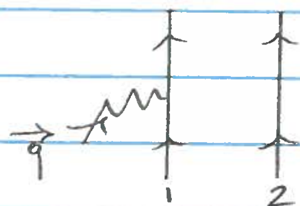


$$\frac{ie\,g_A}{4m f_\pi}\frac{\vec{\sigma}\cdot\vec{q}}{\sqrt{2\omega_k}}(\tau_a+\delta_{az}) \sim eQ \,; \quad -\frac{e\,g_A}{2 f_\pi}\frac{\vec{\sigma}}{\sqrt{2\omega_k}}\,\epsilon_{zab}\tau_b \sim eQ^0$$



$$-ie\,\frac{\omega_{k_1}-\omega_{k_2}}{\sqrt{4\omega_{k_1}\omega_{k_2}}}\,\epsilon_{zab} \sim eQ \,; \quad -ie\,\frac{\vec{k}_1-\vec{k}_2}{\sqrt{4\omega_{k_1}\omega_{k_2}}}\,\epsilon_{zab} \sim eQ$$

Note that the power counting associated with the charge and current is generally different. Note also that a disconnected diagram like



has the counting $Q^{\alpha}Q^{-3}$, where $\alpha = 0$ or 1 depending on whether the EM field couples to the charge or current of the nucleon, and the factor Q^{-3} follows from the momentum δ-function implicit in this type of (disconnected) diagrams. Specifically, we have in momentum space

$$\bar{\delta}(\vec{p}_1+\vec{q}-\vec{p}_1{}')\,\underbrace{V^{\gamma}(1)}_{\text{vertex } eQ^{\alpha}}\,\underbrace{\bar{\delta}(p_2'-p_2)}_{Q^{-3}}$$

There are also contributions originating from gauging the gradients in $\mathcal{L}^{(2)}_{NN}$ ($\vec{\nabla}N \longrightarrow \vec{\nabla}N - i\,e\,e_N\vec{A}$) as well as from non-minimal couplings of the type

$$H^{CT}_{nm} = \frac{e}{2}\int d\vec{x}\left[c'_{15}\, N^\dagger \sigma_k N\, N^\dagger N + c'_{16}\left(N^\dagger \sigma_k \tau_z N\, N^\dagger N - N^\dagger \sigma_k N\, N^\dagger \tau_z N\right)\right]$$

$$\times\, \varepsilon_{ijk} F_{ij}$$

We will represent the current due to these contact terms as

$\sim e\,Q$

The LEC's in the "minimal" current are in principle known from fits to the NN data (they enter in the potential $v^{(2)}$). However, c'_{15} and c'_{16} are fixed by reproducing photonuclear data.

For the purpose of illustration, consider the term proportional to c'_2 in the contact Lagrangian $\mathcal{L}^{(2)}_{NN}$ (see p. 21):

$$\mathcal{L}_{c'_2} = -c'_2\left[N^\dagger(\vec{\nabla}N)\cdot(\vec{\nabla}N)^\dagger \cdot N\right]$$

Minimal substitution leads to

$$\mathcal{L}_{c'_2} \longrightarrow -c'_2\left[N^\dagger(\vec{\nabla} - i e e_N \vec{A})N \cdot N^\dagger(\overleftarrow{\nabla} + i e e_N \vec{A})N\right]$$

$$= \mathcal{L}_{c'_2} \underbrace{- c'_2\, i e\left[N^\dagger(\vec{\nabla}N)\, N^\dagger \underbrace{e_N}_{(1+\tau_z)/2} N - N^\dagger e_N N\,(N^\dagger \overleftarrow{\nabla} N)\right]}_{\mathcal{L}^A_{c'_2}}$$

and terms quadratic in A_i are dropped, since EM interactions are treated in 1st order perturbation theory. Hence

$$\mathcal{L}^A_{C_2'} = -ie\,C_2' \sum_1\sum_2\sum_3\sum_4 e^{-i\vec{x}\cdot(\vec{q}_1-\vec{q}_2+\vec{q}_3-\vec{q}_4)}\, :b_1^\dagger b_2 b_3^\dagger b_4:$$

normal order

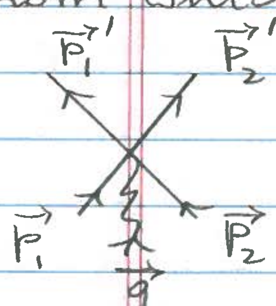
$$\vec{A}\cdot\left[\chi_1^\dagger\, i\vec{q}_2\,\chi_2\,\chi_3^\dagger\, e_N\,\chi_4 + \chi_1^\dagger\, e_N\,\chi_2\,\chi_3^\dagger\, i\vec{q}_3\,\chi_4\right]$$

The associated vertex follows from

$$\langle f|\underbrace{-\int d\vec{x}\,\mathcal{L}^A_{C_2'}}_{H^A_{C_2'}}|i\rangle = \langle 0|\, b_{\vec{p}_2'}\, b_{\vec{p}_1'}\, H^A_{C_2'}\, b^\dagger_{\vec{p}_1}\, b^\dagger_{\vec{p}_2}\,|0\rangle$$

$$= -\vec{J}_\gamma\cdot\hat{\varepsilon}$$

from which



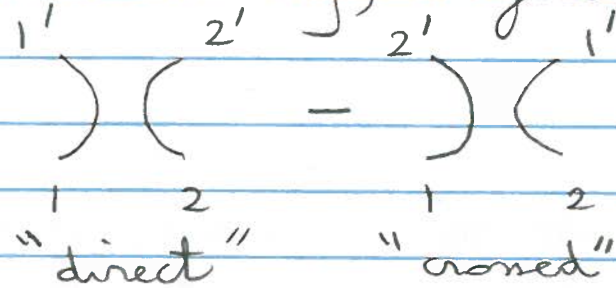
$$= e\,C_2'\,\overbrace{\left[e_{N,1}(\vec{p}_2{}'+\vec{p}_2) + e_{N,2}(\vec{p}_1{}'+\vec{p}_1)\right]}^{\vec{J}_\gamma}.$$

and the spin-isospin states of the initial and final nucleons have been dropped for brevity, for example

$$e_{N,1} \longrightarrow \underbrace{\chi_{1'}^\dagger}_{\text{final}}\, e_N\, \underbrace{\chi_1}_{\text{initial}}$$

spin isospin state

A $\bar{\delta}(\vec{p}_1{}'+\vec{p}_2{}'-\vec{q}-\vec{p}_1-\vec{p}_2)$ has also been dropped. Note also that the evaluation of $\langle f | H^{A}_{c2} | i \rangle$ leads not only to the 'direct' terms given above, but also to the "crossed" ones. Schematically, one gets



The crossed terms are ignored. This is because operators like $\vec{j}_\gamma$ above are sandwiched between antisymmetric states. Hence including also the "crossed" term in $\vec{j}_\gamma$ would double count its contribution.

Fierz identities

Before moving on to a discussion of the EM operators up to one loop, we want to mention briefly one technical point. We illustrate it in the context of the contact current, or rather a piece of it:

$$\langle \vec{p}_1{}' \vec{p}_2{}' | \vec{j} | \vec{p}_1 \vec{p}_2 \rangle = \bar{\delta}(\vec{q} + \vec{p}_1 + \vec{p}_2 - \vec{p}_1{}' - \vec{p}_2{}') \, \frac{C_2}{8} (\tau_{1z} - \tau_{2z})$$
$$\times (\vec{p}_1{}' + \vec{p}_1 - \vec{p}_2{}' - \vec{p}_2)$$

This current is sandwiched between initial and final two-nucleon states that are antisymmetric

$$P^{ex} | i \rangle = - | i \rangle \quad \text{and} \quad P^{ex} | f \rangle = - | f \rangle .$$

where the space-spin-isospin exchange operator is given by

$$P^{ex} = \underbrace{\frac{1+\vec{\tau}_1\cdot\vec{\tau}_2}{2}}_{P^{\tau}} \; \underbrace{\frac{1+\sigma_1\cdot\sigma_2}{2}}_{P^{\sigma}} \; P^{space} \, , \qquad P^{ex\dagger} = P^{ex}$$

It follows that the matrix element of an operator $\mathcal{O}$ satisfies

$$\langle f|\mathcal{O}|i\rangle = -\langle f|P^{ex}\,\mathcal{O}|i\rangle .$$

For the momentum space representation of the current above we have

$$\langle \vec{p}_1{}'\vec{p}_2{}'|\vec{j}|\vec{p}_1\vec{p}_2\rangle = -P^{\tau}P^{\sigma}\langle \underbrace{\vec{p}_2{}'\vec{p}_1{}'}_{\text{note space exchange}}|\vec{j}|\vec{p}_1\vec{p}_2\rangle$$

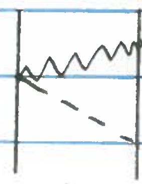
$$= -\frac{C_2}{8}\,\bar{\delta}(\vec{q}+\vec{p}_1+\vec{p}_2-\vec{p}_1{}'-\vec{p}_2{}')(\vec{p}_2{}'+\vec{p}_1-\vec{p}_1{}'-\vec{p}_2)$$

$$\times \frac{1+\vec{\sigma}_1\cdot\vec{\sigma}_2}{2}\;\underbrace{\frac{1+\vec{\tau}_1\cdot\vec{\tau}_2}{2}(\tau_{1z}-\tau_{2z})}_{i(\vec{\tau}_1\times\vec{\tau}_2)_z}$$

Therefore at this stage, the $\vec{j}$ given in p. 38 and that above are equivalent. What is the advantage? The expression on p. 38 leads to a non-local <u>operator in configuration space</u> (a momentum-dependent one), while that above is local in configuration space. These "Fierz identities" can be (and have been) used to remove the non-locality of the contact potential at order Q^2 too.

Construction of nuclear EM operators

We want to write down the perturbative expansion for the transition amplitude T_γ in the presence of EM interactions. We treat the latter in first order (in practice, this means that only a single γ-vertex is allowed in a given diagram, so a contribution like

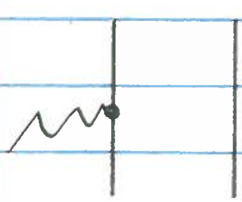


is not to be considered). Just as before, the expansion for T_γ is organised as a power series in powers of eQ^n (Q is the low momentum scale):

$$T_{\gamma L} = T_{\gamma L}^{(-3)} + T_{\gamma L}^{(-2)} + T_{\gamma L}^{(-1)} + \cdots$$

$$T_{\gamma T} = T_{\gamma T}^{(-2)} + T_{\gamma T}^{(-1)} + T_{\gamma T}^{(0)} + \cdots$$

where γL and γT denote, respectively, couplings to the charge (L) and current (T). In particular, the lowest order is



$$= \begin{cases} T_{\gamma L}^{(-3)} \\ T_{\gamma T}^{(-2)} \end{cases}$$

We then require that the EM potentials $v_{\gamma L} = \rho A^0$ and $v_{\gamma T} = -\vec{j}\cdot\vec{A}$ have a power series expansion in eQ^n that matches $T_{\gamma L}$ and $T_{\gamma T}$ order by order in the power counting, i.e.

$$v_{\gamma L} = v_{\gamma L}^{(-3)} + v_{\gamma L}^{(-2)} + v_{\gamma L}^{(-1)} + \cdots \, , \quad v_{\gamma L}^{(n)} = \rho^{(n)} A^0$$

$$v_{\gamma T} = v_{\gamma T}^{(-2)} + v_{\gamma T}^{(-1)} + v_{\gamma T}^{(0)} + \cdots \, , \quad v_{\gamma T}^{(n)} = -\vec{j}^{\,(n)} \cdot \vec{A}$$

with, for $v_{\gamma T}$ for example,

$$v_{\gamma T}^{(-2)} = T_{\gamma T}^{(-2)}$$ $T_{\gamma T}^{(n)}$ are the field theory amplitudes

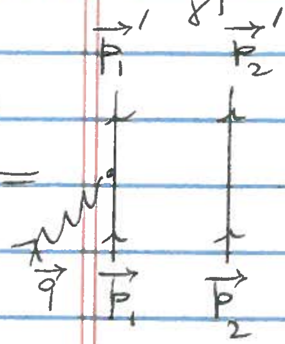
$$v_{\gamma T}^{(-1)} = T_{\gamma T}^{(-1)} - \left[v_{\gamma T}^{(-2)} G_0 v^{(0)} + v^{(0)} G_0 v_{\gamma T}^{(-2)} \right]$$

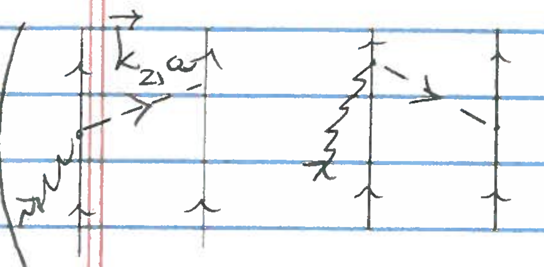
$$v_{\gamma T}^{(0)} = T_{\gamma T}^{(0)} - \left[v_{\gamma T}^{(-2)} G_0 v^{(0)} G_0 v^{(0)} + \text{permutations} \right] - \left[v_{\gamma T}^{(-1)} G_0 v^{(0)} + v^{(0)} G_0 v_{\gamma T}^{(-1)} \right] ,$$

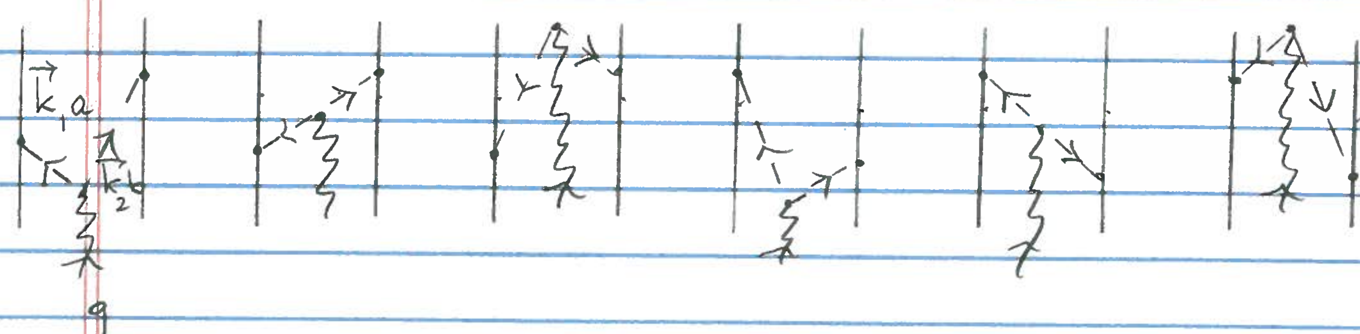
where $v^{(n)}$ (without subscripts) are the strong interaction potentials we have constructed previously (we have used the fact that $v^{(1)}$ vanishes). In the propagator G_0, the initial energy E_i includes also the photon energy ω_γ (which is counted as Q^2), and

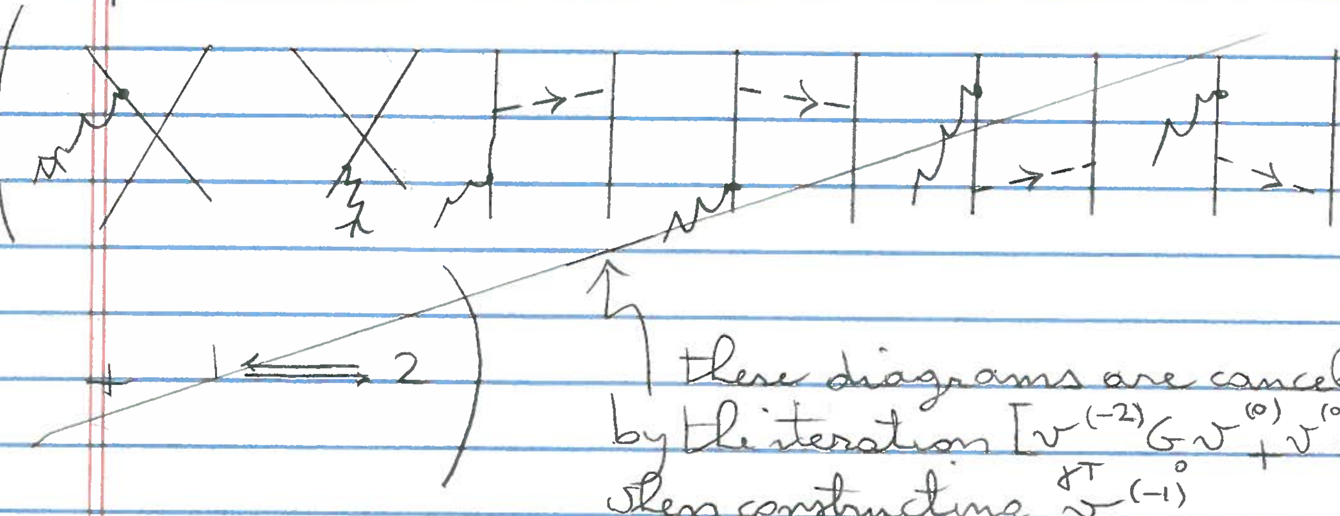
$$\underbrace{E_1 + E_2 + \omega_\gamma}_{E_i} = \underbrace{E_1' + E_2'}_{E_f}$$

The intermediate energy E_I in G_0 may include in addition to the kinetic energies of the intermediate nucleons, also the photon energy, depending on the specific time ordering being considered (see below). As one example, we discuss the construction of $v^{(-2)}_{\gamma T}$ and $v^{(-1)}_{\gamma T}$. We begin by noting

$T^{(-2)}_{\gamma T} =$  $+ \; 1 \rightleftarrows 2$

$T^{(-1)}_{\gamma T} = \Big($  $+ \; 1 \rightleftarrows 2 \Big)$

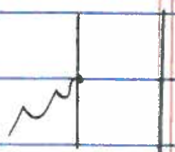
$+$ 

$+ \Big($  $+ \; 1 \rightleftarrows 2 \Big)$

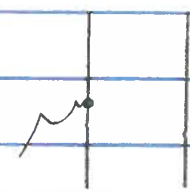
these diagrams are cancelled by the iteration $[v^{(-2)}_{\gamma T} G_0 v^{(0)} + v^{(0)} G_0 v^{(-2)}_{\gamma T}]$ when constructing $v^{(-1)}_{\gamma T}$.

The charge and current operators up to eQ

$\rho^{(n)}$ | $\vec{J}^{(n)}$ | order

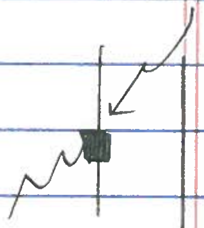


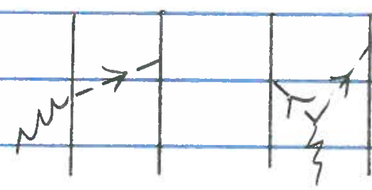
eQ^{-3}



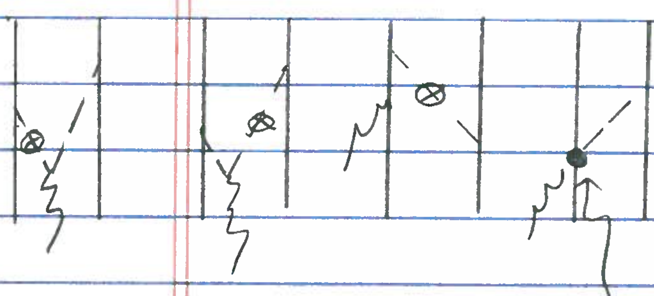
eQ^{-2}

relativistic correction (eQ^2)

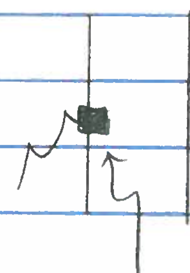




eQ^{-1}



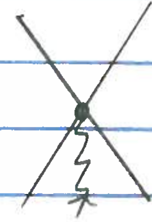
vertex eQ^0 (see p. 35)



relativistic correction (eQ^3)

eQ^0

loops



+ loops eQ.

The loops need to be regularised, and the divergent pieces either cancel out in ρ or can be reabsorbed into a redefinition of the C_i LEC's in the contact current. Note that there are no contact terms in ρ. There is the constraint of charge conservation, however,

$$\rho(\vec{q}=0) = \int d\vec{x}\, \rho(\vec{x}) = \underbrace{\sum_i e\, e_{N_i}}_{\rho^{(-3)}(\vec{q}=0)}$$

which implies that $\rho^{(n)}$ $n \geq -2$ must vanish at $\vec{q}=0$.

Finally, current conservation requires

$$\vec{q} \cdot \vec{j} = [\underbrace{T}_{\text{nucleon kinetic energy}} + v, \rho]$$

and in momentum space

$$T = T^{(-1)} \overset{\text{power counting}}{=} \delta(\vec{p}\,' - \vec{p})\, p^2/m .$$

Hence

$$\vec{q} \cdot [\vec{j}^{(-2)} + \vec{j}^{(-1)} + \vec{j}^{(0)} + \vec{j}^{(1)}] = [T^{(-1)} + v^{(0)} + v^{(2)}, \rho^{(-3)} + \rho^{(-1)}]$$

Matching powers (recall that a commutator brings in an additional Q^3 factor in overall power counting), we find

$$\vec{q} \cdot \vec{j}^{(-2)} = [T^{(-1)}, \rho^{(-3)}]$$

$$\vec{q}\cdot\vec{j}^{\,(-1)} = [v^{(0)}, \rho^{(-3)}]$$

$$\vec{q}\cdot\vec{j}^{\,(0)} = \cancel{[T^{(-1)}, \rho^{(-1)}]} \quad \sim 1/m^3 \text{ (relativistic corrections)}$$

$$\vec{q}\cdot\vec{j}^{\,(1)} = \underbrace{\cancel{[v^{(0)}, \rho^{(-1)}]}}_{\sim 1/m} + [v^{(2)}, \rho^{(-3)}]$$

loop correction and
contact current have no $1/m$
dependence

The relations above have been verified explicitly [see Pastore et al. (2008)].

Finally, we note that, even after regularization, the v, $v_{\gamma L}$, and $v_{\gamma T}$ have power law behavior in momentum space. Their Fourier transforms (i.e., the configuration space versio[n] of these operators) are singular, and must be regularized before they can be used in practical calculations. This is accomplished in practice by multiplying the operators by an appropriate cutoff

$$C_\Lambda(\text{momenta}).$$

Of course, inclusion of this cutoff spoils the current conservation relations above as well as the Fierz equivalence between (apparently) different operators, established earlier.