Spatiotemporal Structure of Dissipative Solitons in the Cubic–Quintic Ginzburg–Landau Equation

Spatiotemporal Structure of Dissipative Solitons in the CGLE

Stefan C. Mancas, Roy S. Choudhury

Department of Mathematics
University of Central Florida

Nonlinear Physics. Theory and Experiment. IV
Outline

1. Introduction
   - Work on Pulsating Solitons
   - CGLE
   - Numerical Simulations

2. The Generalized Variational Formulation
   - Plane Pulsating Soliton
   - Snaking Soliton

3. Future Work
   - Chaotic Soliton
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Numerical simulations of CGLE

\[ \partial_t A = \epsilon A + (b_1 + \nu c_1) \partial_x^2 A - (b_3 - \nu c_3)|A|^2 A - (b_5 - \nu c_5)|A|^4 A \]

Canonical equation governing the weakly nonlinear behavior of dissipative systems

5 new classes viz. pulsating, creeping, snaking, erupting and chaotic solitons

Interesting bifurcations sequences as the parameters in CGLE are varied
Simulations by Akhmediev

Figure: Plain Pulsating, Snaking, Creeping Soliton
Figure: Exploding, Chaotic Soliton
Figure: Plain Pulsating, Snaking Soliton?
Figure: Creeping \& Chaotic Soliton
Characteristics of 5 new classes

- Not stationary in time
- Don’t exist as stable structures in Hamiltonian systems
- Envelopes exhibit complicated temporal dynamics
- Unique to dissipative systems
Period Doubling

Figure: Plain Pulsating, Period Doubling, Quadrupling Soliton

\[ b_3 = -0.75, \ b_3 = -0.785, \ b_3 = -0.793 \]
Characterization of Variational Formulation

- Detailed comparisons with Nail’s and Divo’s numerics
- Trial functions generalized from Kaup, Malomed
- Instead of stable fixed points (stationary solitons or plain pulses) use dynamical systems theory to focus on more complex attractors viz. periodic, quasiperiodic, and chaotic ones
- Periodic evolution of the trial function parameters on stable periodic attractors $\Rightarrow$ solitons whose amplitudes are periodic or time dependent
- Chaotic evolution of the trial function parameters $\Rightarrow$ chaotic solitary waves
Formulation

- Lagrangian for the cubic–quintic CGLE

\[ L = r^* \left[ \partial_t A - \epsilon A - (b_1 + \nu c_1) \partial_x^2 A + (b_3 - \nu c_3) |A|^2 A + (b_5 - \nu c_5) |A|^4 A \right] \]
\[ + r \left[ \partial_t A^* - \epsilon A^* - (b_1 - \nu c_1) \partial_x^2 A^* + (b_3 + \nu c_3) |A|^2 A^* + (b_5 + \nu c_5) |A|^4 A^* \right] \]

- General ansatz for \( A(t) \) and \( r(t) \)

\[ A(x, t) = A_1(t) e^{-\sigma_1(t)^2[x - \phi_1(t)]^2} e^{i\alpha_1(t)} \]
\[ r(x, t) = e^{-\sigma_2(t)^2[x - \phi_2(t)]^2} e^{i\alpha_2(t)} e^{i\beta(t)x^2} \]

- The effective (averaged) Lagrangian is

\[ L_{EFF} = \int_{-\infty}^{\infty} L \, dx \]

- Simplifying assumptions for various classes
Plane Pulsating Soliton – No Chirp, No Phase

- Speed is always zero $\Rightarrow \phi_1(t) = \phi_2(t) = 0$
- No chirp $\Rightarrow \beta(t) = 0$
- Width $\Rightarrow \sigma_1(t) = \sigma_2(t) = \sigma(t)$
- No Phase $\Rightarrow \alpha_1(t) = \alpha_2(t) = 0$
Varying $\sigma(t)$, $A_1(t)$ in $L_{\text{EFF}}$ we have

$$\frac{\partial L_{\text{EFF}}}{\partial \sigma(t)} - \frac{d}{dt} \left( \frac{\partial L_{\text{EFF}}}{\partial \dot{\sigma}(t)} \right) =$$

$$= - \frac{\sqrt{\pi}}{6\sigma^2(t)} \left[ 2A_1^3(t)(\sqrt{6}b_5A_1^2(t) + 3b_3) ight]$$

$$- 6\sqrt{2}A_1(t)(\epsilon + b_1\sigma^2(t)) + 3\sqrt{2}\dot{A}_1(t) \right] = 0$$

$$\frac{\partial L_{\text{EFF}}}{\partial A_1(t)} - \frac{d}{dt} \left( \frac{\partial L_{\text{EFF}}}{\partial \dot{A}_1(t)} \right) =$$

$$= \frac{\sqrt{\pi}}{6\sigma^2(t)} \left[ 2A_1^2(t)\sigma(t)(5\sqrt{6}b_5A_1^2(t) + 9b_3) ight]$$

$$- 6\sqrt{2}\sigma(t)(\epsilon - b_1\sigma^2(t)) + 3\sqrt{2}\dot{\sigma}(t) \right] = 0$$
Solving for $\dot{A}_1(t)$ and $\dot{\sigma}(t)$

$$\dot{A}_1(t) = -\frac{1}{3} A_1(t) \left( -6\epsilon + 3\sqrt{2} b_3 A_1^2(t) + 2\sqrt{3} b_5 A_1^4(t) - 6 b_1 \sigma^2(t) \right)$$

$$\dot{\sigma}(t) = -\frac{1}{3} \sigma(t) \left( -6\epsilon + 9\sqrt{2} b_3 A_1^2(t) + 10\sqrt{3} b_5 A_1^4(t) + 6 b_1 \sigma^2(t) \right)$$

Looking at the characteristic polynomial of the Jacobian Matrix

$$\lambda^2 + \delta_1 \lambda + \delta_2 = 0$$

Hopf Bifurcation occurs when

$$\delta_1 = 0 \text{ and } \delta_2 > 0$$

Which actually never happens since $\delta_2 < 0$ at all fixed points
Plane Pulsating Soliton – No Chirp and Phase

- No Speed $\Rightarrow \phi_1(t) = \phi_2(t) = 0$
- No chirp $\Rightarrow \beta(t) = 0$
- Width $\Rightarrow \sigma_1(t) = \sigma_2(t) = \sigma(t)$
- Rescale the Phase $\Rightarrow \alpha_1(t) = \alpha(t), \alpha_2(t) = 0$
Vary $\sigma(t), A_1(t)$ and $\alpha(t)$ in $L_{EFF}$

$$\frac{\partial L_{EFF}}{\partial \dot{\star}(t)} - \frac{d}{dt} \left( \frac{\partial L_{EFF}}{\partial \dot{\star}(t)} \right) = 0$$

Solving for $\dot{\star}(t)$

$$\dot{A}_1(t) = f_1[A_1(t), \sigma(t), \alpha(t)]$$
$$\dot{\sigma}(t) = f_2[A_1(t), \sigma(t), \alpha(t)]$$
$$\dot{\alpha}(t) = f_3[A_1(t), \sigma(t), \alpha(t)]$$
Stability of Fixed Points

- Characteristic polynomial of Jacobian matrix:
  \[ \lambda^3 + \delta_1 \lambda^2 + \delta_2 \lambda + \delta_3 = 0 \]

- Routh–Hurwitz conditions imply FP is stable for:
  \[ \delta_1 > 0, \quad \delta_3 > 0, \quad \delta_1 \delta_2 - \delta_3 > 0 \]

- Hopf bifurcation at \( \delta_1 \delta_2 - \delta_3 = 0 \) implies onset of periodic solutions for \( A_1(t), \sigma(t) \) and \( \alpha(t) \) which may be stable or unstable.

- If a trajectory is confined to a closed, bounded region with no fixed points then the trajectory must approach a closed orbit.

- Periodic oscillations of \( A_1(t), \sigma(t) \) and \( \alpha(t) \) correspond to a spatiotemporal pulsating soliton structure of \( |A(x, t)| \).
Results

Figure: Evolution of the periodic orbit for $\epsilon = -0.3454$, $b_3 = -0.1878$, $b_3 = -0.1594$, $b_3 = -0.1469$
Figure: The time evolution for $A_1(t)$ leading to a stable periodic orbit as $b_3$ varies
Figure: Power spectral density as a function of frequency
Figure: The Plain Pulsating Soliton for $b_1 = 0.08$, $c_1 = 0.5$, $c_3 = 1$, $b_5 = 0.1$, $c_5 = -0.1$
**Figure:** Evolution of the periodic orbit for $b_1 = 0.0401$, $b_1 = 0.1278$
Figure: The time evolution for $A_1(t), \sigma(t)$ and $\alpha(t)$ leading to a stable periodic orbit as $b_1$ varies.
Figure: Power spectral density as a function of frequency for $b_1 = 0.0401$, $b_1 = 0.1278$
Figure: The Plain Pulsating Soliton for $b_1 = 0.0401, b_1 = 0.1278$
Plane Pulsating Soliton – Chirp and Phase

- No Speed $\Rightarrow \phi_1(t) = \phi_2(t) = 0$
- Width $\Rightarrow \sigma_1(t) = \sigma_2(t) = \sigma(t)$
- Chirp $\Rightarrow \beta(t) = \sigma^2(t)$
- Rescale the Phase $\Rightarrow \alpha_1(t) = \alpha(t), \alpha_2(t) = 0$
Snaking Soliton

- Rescale the Speed ⇒ \( \phi_1(t) = \phi(t), \phi_2(t) = 0 \)
- Vary \( \sigma(t), A_1(t) \) and \( \phi(t) \) in \( L_{EFF} \)
- The resulting E – L equations

\[
\frac{\partial L_{EFF}}{\partial \dot{x}(t)} - \frac{d}{dt} \left( \frac{\partial L_{EFF}}{\partial \dot{x}(t)} \right) = 0
\]

- Similar with the Plane Pulsating Soliton (3-D)
- Regimes of supercritical Hopf bifurcations identified by Multiple Scales analysis then yield stable periodic solutions for \( A_1(t), \sigma(t), \phi(t) \)
Figure: Evolution of the periodic orbit and time evolution
Figure: Snaking Soliton
Chaotic Soliton

- Investigated within the same formulation as the snake by looking for chaotic attractors
- Chaotic regimes when: (no simple attractors)
  - all fixed points are unstable
  - there are no stable periodic orbits. Sometimes sufficient to check: No Hopf bifurcation
  - no attractors at infinity
- Other ways to chaos (??): 
  - subcritical Hopf bifurcation
  - repeated period doubling
  - bifurcations of periodic solutions (???)
Creeping Soliton

- Procedure not completely clear
- Additional constant speed condition $\dot{\phi}(t) = \nu$ is imposed
- Needs further work
- May need invariants of EL equations à la Kaup suggestions
Erupting Soliton

- Procedure not clear at all
- Open question
Thank you, any questions?