Perpetuants–A Lost Treasure

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The Theorem of EMIL STROH from 1890

Let $P_{n,g}$ denote the space of *perpetuants* of degree n and weight g. Then

$$\sum_{g=0}^{\infty} \dim(P_{n,g})\, x^g = \begin{cases} \dfrac{x^{2^{n-1}-1}}{(1-x^2)(1-x^3)\cdots(1-x^n)} & \text{for } n > 2, \\ \dfrac{x^2}{(1-x^2)} & \text{for } n = 2, \\ 1 & \text{for } n = 1. \end{cases}$$

Note that in the series

$$\sum_{g=0}^{\infty} p_g(n) x^g = \frac{1}{(1-x^2)(1-x^3)\cdots(1-x^n)}$$

the integer $p_g(n)$ counts the *number of ways in which the integer g can be written as a sum of integers* $2, 3, \ldots, n$. This is hard to compute!

This formula was conjectured by MACMAHON in 1884.

Classical Invariant Theory of Binary Forms

A homogeneous form of degree q in two variables x, y is classically called a *binary quantic* or q-antic:

$$\boxed{f(x,y) = \sum_{i=0}^{q} \binom{q}{i} a_i \, x^{q-i} y^i}$$

For $q = 2, 3, 4, 5, \cdots$ we have a binary *quadratic, cubic, quartic, quintic, etc.*

- This is a $q+1$-dimensional vector space $R_q := \mathbb{C}[x,y]_q$.
- The polynomial functions over R_q are $\mathbb{C}[a_0, a_1, a_2, \ldots, a_q]$.

The binomial coefficients are introduced to simplify some of the formulas.

Invariants

Obvious action of $\mathrm{GL}_2(\mathbb{C})$ on R_q by substitution, hence an action on the polynomial functions $\mathcal{O}(R_q) := \mathbb{C}[a_0, a_1, \ldots, a_q]$:

$$\boxed{(g \cdot F)(f) := F(g^{-1} f)}$$

The *invariants under* $\mathrm{SL}_2(\mathbb{C})$ of a general binary quantic are thus polynomials in the variables $a_0, a_1, \ldots, a_q$, called *binary invariants*:

$$\boxed{\mathcal{O}(R_q)^{\mathrm{SL}_2} := \{F \in \mathbb{C}[a_0, a_1, \ldots, a_q] \mid g \cdot F = F \text{ for all } g \in \mathrm{SL}_2\}}$$

There is a *bigrading* by defining the *weight* of a_i to be i:

$$\text{weight of } (\prod_{j=0}^{q} a_j^{h_j}) = \sum_{j=1}^{q} j \cdot h_j.$$

Definition

A polynomial with terms all of the same weight is called *isobaric*.

Invariants

One easily sees that for a homogeneous isobaric invariant of a binary q-form there is a relation between weight and degree:

$$g = \frac{q \cdot n}{2} \quad \text{where } g \text{ is the weight and } n \text{ the degree.}$$

As an example, the *discriminant* of the cubic ($q = 3$),

$$D = 3a_1^2a_2^2 + 6a_0a_1a_2a_3 - 4a_1^3a_3 - 4a_0a_2^3 - a_0^2a_3^2,$$

is of degree 4 and weight $6 = \dfrac{3 \cdot 4}{2}$, and it generates the algebra of invariants of the cubic.

Some Results

Classically, generators for invariants of binary q-forms were known only for forms of degree $q = 2, 3, 4, 5, 6$ and 8 (with partial results by von Gall for degree 7).

Now, with the help of computers, a few other cases have been analyzed, see the thesis of Mihaela Popoviciu (2014).

Example

- $q = 7$: There are 30 generators in degrees $4, 8(3), 12(6), 14(4), 16(2), 18(9), 20, 22(2), 26, 30$.
- $q = 9$: There are 92 generators in degrees $4(2), 8(5), 10(5), 12(14), 14(17), 16(21), 18(25), 20(2), 22$.
- $q = 10$: There are 106 generators.

Hopeless to go much further!

Covariants and U-Invariants

The classical theory is developed using *covariants*, or in modern terms *U-invariants*, where

$$U := \left\{ \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \mid \lambda \in \mathbb{C} \right\} \subset \mathsf{SL}_2(\mathbb{C}),$$

acting as

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \lambda y \\ y \end{bmatrix}.$$

This is equivalent, by setting $y = 1$, to replace binary forms by *polynomials* in x

$$\boxed{p(x) = \sum_{i=0}^{q} \binom{q}{i} a_i x^{q-i}}$$

with the action by translation $x \mapsto x + \lambda$.

U-Invariants

Denote the ring of U-invariants of binary q-forms by

$$\boxed{S(q) := \mathcal{O}(R_q)^U = \mathbb{C}[a_0, a_1, a_2, \ldots, a_q]^U}$$

- $S(q) = \bigoplus S(q)_{n,g}$ decomposes into a direct sum of homogeneous (of degree n) and isobaric (of weight g) components.
- Inside $S(q)$ the ring of invariants under SL_2 is the direct sum of the homogeneous and isobaric components with $2g = q \cdot n$.

For a covariant C of degree n and weight g the positive integer $\boxed{m := q \cdot n - 2g}$ is called the *order of C*. The meaning is that C defines an SL_2-equivariant homogeneous morphism $\phi_C : R_q \to R_m$ of degree n.

Example. The identity $\mathrm{id}: R_q \to R_q$ has order q, degree $n = 1$ and thus weight $g = 0$, hence corresponds to the U-invariant a_0.

Examples

Example: binary cubic

The algebra $\mathcal{O}(R_3)^U$ of U-invariants of the binary cubic is generated by the following 4 covariants:

$$D := 9a_0^2a_3^2 - 18a_0a_1a_2a_3 + 8a_0a_2^3 + 6a_1^3a_3 - 3a_1^2a_2^2 \quad \text{(the discriminant)},$$
$$A := a_0 \quad \text{(degree 1, weight 0, order 3)},$$
$$H := -a_1^2 + 2a_0a_2 \quad \text{(degree 2, weight 2, order 2)},$$
$$T := a_1^3 - 3a_0a_1a_2 + 3a_0^2a_3 \quad \text{(degree 3, weight 3, order 3)},$$

related by the syzygy $H^3 = A^2\,D - T^2$ of degree 6 and weight 6.

Example: binary quadratic

With the notation above we have $\mathcal{O}(R_2)^U = \mathbb{C}[A, H]$.

Some Results

Classically, the covariants were known up to the sextic (with some incomplete computations by SYLVESTER (1879) and VON GALL (1888) for the septic).

Using heavy computations with computers (plus some Cohen-Macauly-properties) one can go a little further.

Theorem (MIHAELE POPOVICIU 2013)

- *The covariants of the septic ($q = 7$) are generated by 147 covariants of degree ≤ 30 and order ≤ 15.*
- *The covariants of the octavic ($q = 8$) are generated by 69 covariants of degree ≤ 12 and order ≤ 18.*

So far, no hope to go much further!!

Also, there is no general pattern for the number of generators, or the system of parameters, or the HILBERT-series, etc.

Thus STROH's formula really comes as a surprise!

An Important Inclusion

The linear operator $\dfrac{d}{dx}$ maps R_{q+1} surjectively onto R_q, commuting with U, and thus defines a U-equivariant inclusion

$$\mathcal{O}(R_q) \hookrightarrow \mathcal{O}(R_{q+1}). \tag{$*$}$$

Thus a U-invariant of a q-antic is also a U-invariant of an m-antic for any $m \geq q$.

Using divided powers $x^{[i]} := \dfrac{x^i}{i!}$ and setting

$$p(x) = \sum_{i=0}^{q} a_i \, x^{[q-i]}$$

one immediatly sees that the map $(*)$ becomes the canonical inclusion $\mathbb{C}[a_0, \ldots, a_q] \subset \mathbb{C}[a_0, \ldots, a_q, a_{q+1}]$.

The Action of U

Using divided powers it is also easy to describe the action of U:

$$\lambda \cdot a_k = \sum_{j+h=k} \lambda^{[h]} a_j = \sum_{j=0}^{k} \lambda^{[k-j]} a_j.$$

For instance,

$$\lambda \cdot a_0 = a_0, \quad \lambda \cdot a_1 = \lambda a_0 + a_1, \quad \lambda \cdot a_2 = \lambda^{[2]} a_0 + \lambda a_1 + a_2,$$
$$\lambda \cdot a_3 = \lambda^{[3]} a_0 + \lambda^{[2]} a_1 + \lambda a_2 + a_3, \quad \ldots .$$

Theorem (Cayley)

A polynomial $F \in \mathbb{C}[a_0, a_1, \ldots]$ is a U-invariant iff it satisfies

$$\boldsymbol{D} F := \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} F(a_0, a_1, \ldots) = 0, \quad \boldsymbol{D} := \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i}.$$

Indecomposable U-Invariants

$$S(q) = \mathcal{O}(R_q)^U = \mathbb{C}[a_0, a_1, \ldots, a_q]^U$$

As usual, a homogeneous U-invariant $F \in S(q)$ is called *indecomposable* if it cannot be expressed by lower degree invariants.

Facts known classically

- An indecomposable $F \in S(q)$ might become decomposable in $S(q+1)$. E.g., the discriminant $D \in S(3)$ is decomposable in $S(4)$:

$$\boxed{\begin{array}{c} D = 3HB + a_0 C: \quad H := -a_1^2 + 2a_0a_2, \; B := 2a_0a_4 - 2a_1a_3 + a_2^2, \\ C := 2a_2^3 - 6a_1a_2a_3 + 9a_0a_3^2 + 6a_1^2a_4 - 12a_0a_2a_4. \end{array}}$$

- In general, a minimal set of generators of $S(q)$ cannot be completed to a minimal set of generators of $S(q+1)$.

Perpetuants

Definition

A *perpetuant* is a homogeneous indecomposable U-invariant $F \in S(q)$ which remains indecomposable in all $S(m)$, $m \geq q$.

Let $I_q \subset S(q)$ denote the homogeneous maximal ideal. Then I_q^2 are the *decomposable invariants*, and a minimal set of generators is a set of homogeneous elements giving a basis of I_q/I_q^2.

The map $I_q/I_q^2 \to I_{q+1}/I_{q+1}^2$ needs not to be injective!

A *perpetuant* thus gives an element in I_q/I_q^2 which *lives forever*, that is it remains nonzero in all I_m/I_m^2, $m \geq q$.

Perpetuants

This shows that to describe perpetuants is related to describe minimal sets of generators for the graded algebra

$$\boxed{S := \bigcup_q S(q) = \mathbb{C}[a_0, a_1, a_2, \ldots]^U}$$

In other words, denoting by $I = (a_0, 2a_0a_2 - a_1^2, \ldots) \subset S$ the homogeneous maximal ideal of S, we want to describe I/I^2.

This space decomposes into a direct sum

$$I/I^2 = \bigoplus_{n,g \in \mathbb{N}} P_{n,g}$$

where $P_{n,g}$ is the image of the elements in I of degree n and weight g. (Note that $P_{n,g}$ is finite dimensional!)

$I \subset \mathbb{C}[a_0, a_1, \ldots]^U$ homog. max. ideal,
$I/I^2 = \bigoplus_{n,g\in\mathbb{N}} P_{n,g}$

Now STROH*'s formula gives the dimension of these spaces:*

$$\sum_{g=0}^{\infty} \dim(P_{n,g})\, x^g = \begin{cases} \dfrac{x^{2^{n-1}-1}}{(1-x^2)(1-x^3)\cdots(1-x^n)} & \text{for } n > 2, \\ \dfrac{x^2}{(1-x^2)} & \text{for } n = 2, \\ 1 & \text{for } n = 1. \end{cases}$$

Final Goal

Construct subspaces $\tilde{P}_{n,g} \subset S$ of U-invariants of degree n and weight g which project bijectively onto $P_{n,g}$.

Main ingrediens of the proof:

- *Umbral calculus or symbolic method from classical invariant theory.* This comes via the map $\boldsymbol{E}$ and STROH's "Potenziante" (see below).
- *A duality between symmetric polynomials and homogeneous isobaric functions.* This is somewhat disguised in STROH's work.

Umbral Calculus

We define a linear map $\boldsymbol{E}$ from the space of polynomials in auxiliary variables $\alpha_1, \ldots, \alpha_n$ (*the umbrae*) to the space of polynomials of degree n in the variables $a_0, a_1, a_2, \ldots$,

$$\boldsymbol{E}\colon \mathbb{C}[\alpha_1, \ldots, \alpha_n] \to \mathbb{C}[a_0, a_1, a_2, \ldots]_n\,,\ \alpha_1^{[r_1]} \cdots \alpha_n^{[r_n]} \mapsto a_{r_1} \cdots a_{r_n}.$$

E.g.

$$\boldsymbol{E}(\alpha_1^{[3]}\alpha_2^{[2]}) = \boldsymbol{E}(\alpha_3^{[3]}\alpha_1^{[2]}) = a_0^{n-2}a_2a_3 \quad\text{and}\quad \boldsymbol{E}(\alpha_i^{[2]}\alpha_j^{[2]}) = a_0^{n-2}a_2^2.$$

$$\boldsymbol{E}\colon \alpha_1^{[r_1]}\cdots\alpha_n^{[r_n]} \mapsto a_{r_1}\cdots a_{r_n}$$

Properties of $\boldsymbol{E}$

1. A homogeneous polynomial of degree g in $\alpha_1, \dots, \alpha_n$ is mapped by $\boldsymbol{E}$ to a homogeneous and isobaric polynomial in $a_0, a_1, \dots$ of weight g and degree n.
2. The map $\boldsymbol{E}$ commutes with the permutation action on the α_i.
3. Basic formula:

$$\boldsymbol{E} \circ \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} \circ \boldsymbol{E} = \boldsymbol{D} \circ \boldsymbol{E}.$$

Note that he map $\boldsymbol{E}$ in **not** a homomorphism. But if $f(\alpha_1, \dots, \alpha_h)$ and $g(\alpha_{h+1}, \dots, \alpha_n)$ are in *disjoint variables*, then we have

$$\boldsymbol{E}(f(\alpha_1, \dots, \alpha_h) \cdot g(\alpha_{h+1}, \dots, \alpha_n)) \\ = \boldsymbol{E}(f(\alpha_1, \dots, \alpha_h)) \cdot \boldsymbol{E}(g(\alpha_{h+1}, \dots, \alpha_n)).$$

The Potenziate of STROH and Duality

This is the following object:

$$\pi_{n,g} := \boldsymbol{E}\left((\sum_{j=1}^{n} \lambda_j \alpha_j)^{[g]} \right) = \sum_{\substack{r_1,\ldots,r_n \in \mathbb{N} \\ r_1+\cdots+r_n=g}} \lambda_1^{r_1} \cdots \lambda_n^{r_n}\, a_{r_1} \cdots a_{r_n}$$

where the $\alpha_1 \ldots, \alpha_n$ are the umbrae and $\lambda_1, \ldots, \lambda_n$ some *dual* variables. The Potenziate $\pi_{n,g}$ lives in the tensor product

$$\boxed{\pi_{n,g} \in \Sigma_{n,g} \otimes \mathbb{C}[a]_{n,g}}$$

where

$\Sigma_{n,g} \subset \mathbb{C}[\lambda_1, \ldots, \lambda_n]^{S_n}$: homogeneous symmetric polys of degree g,

$\mathbb{C}[a]_{n,g} \subset \mathbb{C}[a_1, a_2, \ldots]$: homogeneous isobaric polys of degree n and weight g.

This tensor $\pi_{n,g}$ is a so-called dualizing tensor!

Dualizing Tensors and Duality

Given two finite dimensional vector spaces U, W and denoting by $U^\vee, W^\vee$ their duals one has canonical isomorphisms

$$U \otimes W \simeq \mathrm{Hom}(U^\vee, W) \simeq \mathrm{Hom}(W^\vee, U) \simeq \mathrm{Bil}(U^\vee \times W^\vee).$$

Definition

A *dualizing tensor* $\pi \in U \otimes W$ is an element which corresponds, under these isomorphisms, to an isomorphism $U^\vee \simeq W$ (or $W^\vee \simeq U$).

Remarks

- If $\pi = \sum_{i=1}^k u_i \otimes w_i$ is a dualizing tensor and $u_1, \ldots, u_k$ a basis of U, then $w_1, \ldots, w_k$ is a basis of W.
- For a subspace $V \subseteq U$ we have the *orthogonal* space $V^\perp \subset W$, and the dualizing tensor π defines a dualizing tensor $\bar{\pi} \in V \otimes W/V^\perp$.

STROH's Potenziate is a Dualizing Tensor!

$$\pi_{n,g} := \boldsymbol{E}\left((\sum_{j=1}^{n} \lambda_j \alpha_j)^{[g]} \right) = \sum_{\substack{r_1,\dots,r_n \in \mathbb{N} \\ r_1+\dots+r_n=g}} \lambda_1^{r_1} \cdots \lambda_n^{r_n}\, a_{r_1} \cdots a_{r_n} \in \Sigma_{n,g} \otimes \mathbb{C}[a]_{n,g}$$

Taking as a basis for $\Sigma_{n,g}$ the *monomial sums*

$$m_{h_1,\dots,h_n}(\lambda) := \sum_{S_n\text{-orbit}} \sigma(\lambda_1^{h_1} \cdots \lambda_n^{h_n}),\ h_1 \geq h_2 \geq \cdots \geq h_n,\ \sum_i h_i = g,$$

we find

$$\pi_{n,g} = \boldsymbol{E}\left((\sum_{r=1}^{n} \lambda_r \alpha_r)^{[g]} \right) = \sum_{\substack{h_1 \geq \dots \geq h_n \\ h_1+\dots+h_n=g}} m_{h_1,\dots,h_n}(\lambda) a_{h_1} a_{h_2} \cdots a_{h_n},$$

proving that $\pi_{n,g}$ is a dualizing tensor in $\Sigma_{n,g} \otimes \mathbb{C}[a]_{n,g}$.

Examples

$\boxed{n = 2,\ g = 4}$

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2)^{[4]} =$$
$$= \lambda_1^4\alpha_1^{[4]} + \lambda_2^4\alpha_2^{[4]} + (\lambda_1^3\lambda_2\alpha_1^{[3]}\alpha_2 + \lambda_2^3\lambda_1\alpha_2^{[3]}\alpha_1) + \lambda_1^2\lambda_2^2\alpha_1^{[2]}\alpha_2^{[2]}.$$

Applying $\boldsymbol{E}$ this gives

$$(\lambda_1^4 + \lambda_2^4)\ a_0^3a_4 + (\lambda_1^3\lambda_2 + \lambda_2^3\lambda_1)\ a_0^2a_1a_3 + (\lambda_1^2\lambda_2^2)\ a_2^2 =$$
$$= m_{4,0}(\lambda)\ a_0^3a_4 + m_{3,1}(\lambda)\ a_0^2a_1a_3 + m_{2,2}(\lambda)\ a_2^2.$$

$\boxed{n = g = 3}$

$$\boldsymbol{E}((\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3)^{[3]}) =$$
$$= m_{3,0,0}(\lambda)\ a_0^2a_3 + m_{2,1,0}(\lambda)\ a_0a_1a_2 + m_{1,1,1}(\lambda)\ a_1^3.$$

Elementary Symmetric Polynomials

Another basis for $\Sigma_{n,g}$ are the monomials $e_1^{k_1} \cdots e_n^{k_n}$ in the *elementary symmetric polynomials* e_i, with $\sum_j jk_j = g$,

$$m_{h_1,\dots,h_n} = \sum_{k_1,\dots,k_n} \beta_{h_1,\dots,h_n,k_1,\dots,k_n} \, e_1^{k_1} \cdots e_n^{k_n}$$

where the β are computable integers.

$$\pi_{n,g} = \sum_{k_1+2k_2+\cdots+nk_n=g} e_1^{k_1} \dots e_n^{k_n} \, \tilde{U}_{k_1,\dots,k_n},$$

where the polynomials $\tilde{U}_{k_1,\dots,k_n} \in \mathbb{C}[a]_{n,g}$ are given by

$$\tilde{U}_{k_1,\dots,k_n} = \sum_{h_1,\dots,h_n} \beta_{h_1,\dots,h_n,k_1,\dots,k_n} \, a_{h_1} \cdots a_{h_n},$$

A Basis for the U-invariants

Recall the basic formula
$\boldsymbol{D} \circ \boldsymbol{E} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} \circ \boldsymbol{E} = \boldsymbol{E} \circ \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i}$

Using $\pi_{n,g} = \boldsymbol{E}\left((\sum_{j=1}^{n} \lambda_j \alpha_j)^{[g]}\right)$ one gets

$$\boxed{\boldsymbol{D}\pi_{n,g} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} \pi_{n,g} = (\sum_{i=1}^{n} \lambda_i)\, \pi_{n,g-1} = e_1\, \pi_{n,g-1}}$$

In our duality between polynomials in a_i and symmetric functions in λ_i, the transpose of the operator $\boldsymbol{D}$ is multiplication by $e_1 = \sum_{i=1}^{n} \lambda_i$.

Recall that the kernel of $\boldsymbol{D} = \sum_i a_{i-1} \frac{\partial}{\partial a_i}$ are the U-invariants!

$$\pi_{n,g} = \sum_{k_1+2k_2+\cdots+nk_n=g} e_1^{k_1} \dots e_n^{k_n} \, \tilde{U}_{k_1,\dots,k_n}$$

$$\boldsymbol{D}\pi_{n,g} = \sum_{\sum i\cdot k_i = g} e_1^{k_1} \dots e_n^{k_n} \, \boldsymbol{D}\tilde{U}_{k_1,\dots,k_n} = \sum_{\sum i\cdot j_i = g-1} e_1^{j_1+1} \dots e_n^{j_n} \, \tilde{U}_{j_1,\dots,j_n}$$

which implies

$$\boxed{\boldsymbol{D}\tilde{U}_{k_1,\dots,k_n} = \begin{cases} 0 & \text{if } k_1 = 0, \\ \tilde{U}_{k_1-1,\dots,k_n} & \text{if } k_1 > 0. \end{cases}}$$

Theorem

1. *The elements* $U_{k_2,\dots,k_n} := \tilde{U}_{0,k_2,\dots,k_n}$ *form a basis of the space* $S_{n,g}$ *of* U*-invariants of degree* n *and weight* g*.*
2. *The subspace* $S_{n,g} \subset \mathbb{C}[a]_{n,g}$ *of* U*-invariants is orthogonal to* $\Sigma_{n,g} \cap (e_1)$*, and thus is dual to the quotient*

$$\overline{\Sigma}_{n,g} = \Sigma_{n,g}/(\Sigma_{n,g} \cap (e_1)) \subset \mathbb{C}[\lambda]^{S_n}/(\sum \lambda_i) = \mathbb{C}[e_2, \dots, e_n].$$

This already gives the following series for the dimensions of the subspaces $S_{n,g} \subset S$ of the U-invariants:

Corollary

$$\sum_{g=0}^{\infty} \dim(S_{n,g})\, x^g = \sum_{g=0}^{\infty} \dim(\overline{\Sigma}_{n,g})\, x^g = \frac{1}{(1-x^2)(1-x^3)\cdots(1-x^n)}$$

Example: The case $n = 2$

In degree 2 there is a unique decomposable U-invariant, namely a_0^2 which has weight 0. It follows from the Corollary that there is a perpetuant of degree 2 for every even weight:

$$\sum_{g=0}^{\infty} \dim(P_{2,g})\, x^g = \frac{x^2}{1-x^2}.$$

We can construct these perpetuants by calculating the dualizing tensor (STROH's potenziate) in $\overline{\Sigma}_{2,g} \otimes S_{2,g}$:

$$
\begin{aligned}
\bar{\pi}_{2,g} = \boldsymbol{E}\left((\bar{\lambda}_1\alpha_1 + \bar{\lambda}_2\alpha_2)^{[g]}\right) &= \bar{\lambda}_1^g \boldsymbol{E}\left((\alpha_1 - \alpha_2)^{[g]}\right) \\
&= \bar{\lambda}_1^g \boldsymbol{E}\left(\sum_{j=0}^{g} \alpha_1^{[j]}(-\alpha_2)^{[g-j]}\right) \\
&= \bar{\lambda}_1^g \sum_{j=0}^{g} (-1)^{g-j} a_j a_{g-j}
\end{aligned}
$$

Here we use that $\bar{\lambda}_1 + \bar{\lambda}_2 = 0$!

giving the well-known irreducible quadratic U-invariants (for even g)

$$
\boxed{2a_0a_2 - a_1^2, \quad 2a_0a_4 - 2a_1a_3 + a_2^2, \quad 2a_0a_6 - 2a_1a_5 + 2a_2a_4 - a_3^2, \ \ldots}
$$

Decomposable U-Invariants

$$\overline{\Sigma}_n := \bigoplus_g \overline{\Sigma}_{n,g} = \mathbb{C}[\lambda]^{S_n}/(\sum \lambda_i) = \mathbb{C}[e_2, \ldots, e_n]$$

For $h \leq \frac{n}{2}$ define

$$p_{n,h} := \prod_{1 \leq j_1 < j_2 < \cdots < j_h \leq n} (\bar{\lambda}_{j_1} + \bar{\lambda}_{j_2} + \cdots + \bar{\lambda}_{j_h}) \in \overline{\Sigma}_n = \mathbb{C}[\lambda]^{S_n}$$

Lemma

The subspace $S_{n,g,h} \subset S_{n,g}$ of U-invariants admitting a decomposition with a factor of degree h is orthogonal to $\overline{\Sigma}_{n,g} \cap (p_{n,h})$.

Main Lemma

The subspace of decomposable U-invariants is orthogonal to $\overline{\Sigma}_{n,g} \cap (q_n)$ where $q_n = p_{1,n} \cdots p_{m,n}$, $m := \lfloor \frac{n}{2} \rfloor$. Hence, by duality,

$$\dim P_{n,g} = \dim \overline{\Sigma}_{n,g} \cap (q_n).$$

Moreover, $\deg q_n = 2^{n-1} - 1$.

Proof of STROH's Formula

$$\dim P_{n,g} = \dim \overline{\Sigma}_{n,g} \cap (q_n), \;\deg q_n = 2^{n-1} - 1$$

Since $\overline{\Sigma}_{n,g} \cap (q_n) = q_n \cdot \overline{\Sigma}_{n,g-2^{n-1}+1}$ for $g \geq 2^{n-1} - 1$ (and $= 0$ otherwise) we get

$$\dim P_{n,g} = \dim \overline{\Sigma}_{n,g} \cap (q_n) = \begin{cases} \dim \overline{\Sigma}_{n,g-(2^{n-1}-1)} & \text{for } g \geq 2^{n-1} - 1, \\ 0 & \text{for } g < 2^{n-1} - 1. \end{cases}$$

Hence

$$\sum_{g=0}^{\infty} \dim(P_{n,g})\, x^g = \frac{x^{2^{n-1}-1}}{(1-x^2)(1-x^3)\cdots(1-x^n)} \qquad \text{qed.}$$

A Basis for the Perpetuants

Recall $\pi_{n,g} = \sum_{\sum i \cdot k_i = g} e_1^{k_1} \dots e_n^{k_n} \tilde{U}_{k_1,\dots,k_n}$

Since $\boldsymbol{D}\tilde{U}_{k_1,\dots,k_n} = 0$ if and only if $k_1 = 0$, we got a basis for the U-invariants $\mathbb{C}[a]^U_{n,g}$:

$$\boxed{U_{k_2,\dots,k_n} := \tilde{U}_{0,k_2,\dots,k_n}, \quad \sum_{i=2}^n ik_i = g}$$

Now we use the partial order

$$(t_2,\dots,t_n) \succeq (s_2,\dots,s_n) \iff t_i \geq s_i \text{ for all } i.$$

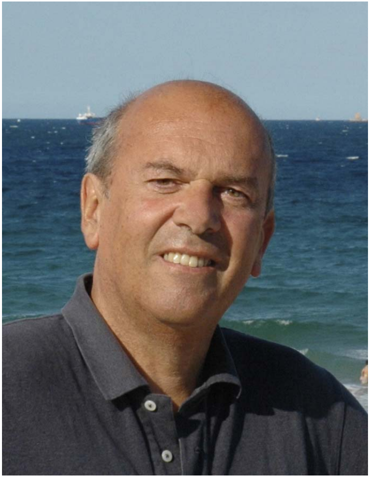
Main Theorem

The elements $U_{\boldsymbol{k}} = U_{k_2,\dots,k_n} = \tilde{U}_{0,k_2,\dots,k_n}$ with $\sum_{i=2}^n i \cdot k_i = g$ and

$$\boldsymbol{k} \succeq (0, 2^{n-4}, 2^{n-5}, \dots 4, 2, 1, 1)$$

form a basis of a space of perpetuants of degree $n > 3$ and weight g.

Thank you for your attention!



Happy birthday, Corrado!!