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On traveling round without feeling it and uncurling curves

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We discuss an elementary example of how, in a strong gravitational field, the basic kinematical concepts of acceleration and circular motion seem to have paradoxical properties. This allows an insight into the physical significance of space-time curvature without the use of difficult mathematical formalism.

Our intuition about the motion of macroscopic bodies is based on Newtonian physics. This is why some properties of fast motions and strong gravitational fields appear paradoxical to us. The so-called twin paradox is well known. Here we present and discuss another striking paradox of this type. We follow the tradition of many textbooks and articles on relativity (e.g., Ref. 1) that use free-falling elevators or accelerating spacecraft as an illustrative device.

A spacecraft can stay at a fixed distance from a spherical celestial body by using its engines to balance the gravitational attraction. There is, however, another possibility: by moving around the body along a circular orbit a centrifugal force is introduced, and less help is therefore needed from the engine to overcome the pull of gravity. The orbital velocity can even be such that the gravitational and centrifugal forces are equal. On such a free orbit the engine must obviously be switched off.

For orbital velocities smaller than the free one the engines must point down in order to reduce the attraction of gravity; in the opposite case, the engines will point up in order to reduce the effect of the centrifugal force. In both cases the thrust of the engine is correlated with the orbital velocity: the bigger the difference between the orbital speed and the free velocity, the stronger the engine power has to be to prevent the spacecraft from leaving the orbit. All that is rather obvious and everybody would certainly agree that Fig. 1 makes sense and describes a general situation.

Figure 2 shows spacecraft with exactly the same engines working with exactly the same thrust, but moving with different orbital speeds at a fixed distance from the central body. Is this possible? Could it be that orbiting spacecraft which use the same engine thrust circle with different orbital speeds? In other words: is it possible that acceleration on a given circular orbit does not depend on angular velocity? However paradoxical it might sound, the answer is "yes."

The situation described in Fig. 2 is possible on a circular orbit at which the velocity of free motion equals the velocity...
of light. Such orbits exist in a gravitational field strong enough to force photons, or the "particles of light," to turn around on a closed path.\textsuperscript{2,3} By light we mean here not only visible light, but electromagnetic radiation in general.

Before explaining how spacecraft can circle on a close photon orbit with different velocities but the same engine thrust, let us first comment on bending light rays by gravitating bodies. Such bending was predicted by Albert Einstein in his General Theory of Relativity. Its observational confirmation was the first triumph of the theory. Even the relatively weak gravitation of the Sun bends the paths of light arriving from distant stars to such an extent that during the solar eclipse, stars that should remain invisible behind the solar disk are observed due to this effect (see Fig. 3).\textsuperscript{3,4}

The Theory of Relativity describes light rays as trajectories of photons of a given energy mass. Newton's theory applies only to velocities much smaller than velocity of light \( c \). Since photons always move with that velocity, Newton's theory cannot accurately describe their trajectories. For the sake of simplicity however, we shall use Newtonian reasoning to find a rough formula for the radius of the circular photon orbit. The correct, relativistic formula differs from the Newtonian one only by a numerical factor.\textsuperscript{3,4}

Centrifugal acceleration on a circular orbit with radius \( R \) depends only on orbital speed \( V \) and equals \( a_c = V^2 / r \). Gravitational acceleration, according to Newton's law of gravity, is proportional to central mass \( M \) and inversely proportional to the square of the radius \( a_g = -GM/r^2 \). Here \( G \) is the gravitational constant. For the free motion, centrifugal and gravitational accelerations must be of equal value and opposite sign, so that the total acceleration, which is the sum of the two, vanishes. From the condition \( a_c = -a_g \) one obtains the radius of the orbit in terms of orbital speed \( r = GM/V^2 \). The correct relativistic calculation for \( v = c \) differs by a factor of 3, so the formula for the radius of the closed photon orbit reads

\[
r_c = 3GM/c^2 = 4.4 \left( M/M_\odot \right) \text{km},
\]

where we measure the mass of the central body in units of solar mass \( M_\odot \). For the Sun itself \( M = M_\odot \), so this radius is well inside the solar core. This reflects the relative weakness of solar gravity. For the close photon orbit to exist the whole gravitating mass must be concentrated in a volume with a radius smaller than \( r_c \). This is the case of black holes (nonrotating black holes have radii \( r_H = 2GM/c^2 \)) and, maybe, some extremely compact neutron stars.

We shall now explain why, on the closed photon orbit, all the spacecraft move with the same engine thrust or, to put it another way, why they all have the same accelerations. One can obtain this result from a simple but technical calculation, which uses the formalism of the General Theory of Relativity. It is presented in the Appendix. We will now show it by a reasoning that gives more insight into the problems of relativistic kinematics.

Let us then consider the case of spacecraft moving at a distance \( r_c = 3GM/c^2 \) from a black hole. In order to observe the motion of his colleagues, a spacecraft pilot must use light signals—either in a passive way or actively, by
radar ranging, say. The light rays in the direction of his and his colleagues’ motion follow exactly the same path as the spacecraft: they must move along the circular photon orbit. Thus the pilot observes no change in the direction of light which comes to him from the other spacecraft. Instead, he observes that the other spacecraft move, with respect to him, along a straight line (Fig. 4). This is of course true for any pilot in any spacecraft orbiting in orbit \( r = R_c \). Since each spacecraft moves with a different but constant orbital speed, an observer sees his colleagues’ spacecraft moving not only along straight lines but also with constant velocity.

Each observer therefore sees the other spacecraft in the photon orbit as moving along straight lines with constant velocities. This means, clearly, that their relative acceleration is equal to zero.

On the other hand, the pilot of the spacecraft knows that he is accelerating, since his engine must work to keep him in orbit. No spacecraft can be in free motion, since this would correspond to moving with the velocity of light. He can measure the acceleration of his vehicle by measuring the thrust of the engine: the acceleration is proportional to the thrust. Moreover, all the pilots must obtain the same result. If it were not the same, their relative acceleration would not vanish, and this would contradict the observation of relative motion.

Thus all the spacecraft on the photon orbit have the same acceleration (and therefore use engines with the same thrust), which means that, contrary to our intuition, Fig. 2 describes a perfectly reasonable physical situation. One can also understand this result by noting that the path of light provides us with a physical model of a straight line. Centrifugal acceleration acts only when an object moves along a curved path. Thus, all the spacecraft moving on the circular photon orbit do not feel the centrifugal acceleration: they travel round without feeling it because the curve is uncurved. They feel only gravitational force which, since they are all at the same distance from the center, is the same for all of them. The engines must overcome the gravity with no help from (nonexisting) centrifugal acceleration. Thus all the engines must work with exactly the same thrust.

In the discussion above it was crucial that the photons moved on the circular orbit in both directions, otherwise light sent by spacecraft A and reflected by spacecraft B would not come back to A. Photons move both ways only if the central black hole does not rotate. If it does, there are two separate circular photon orbits: one which is closer to the hole and has photons circling in the same direction as the hole, and a more distant orbit with counterrotating photons (see Fig. 5). The locations of these orbits depend on the angular momentum of the hole.

The circular photon orbit is unstable: a small perturba-

![Fig. 4. Light signals sent by observers moving along the closed photon orbit follow exactly the same path as the spacecraft. With respect to them the light seems to follow a straight line.](image)

\[ R_c = \frac{3GM}{c^2} \]

Nonrotating Black Hole

Rotating Black Hole

![Fig. 5. In the case of a rotating black hole there are two closed photon orbits, one for photons circling in the same direction as the hole, a second one for counterrotating photons.](image)

The situation will kick photons out of it. The situation we discuss cannot therefore be realized. However, it helps to understand an important concept in the Theory of Relativity by putting it in an apparently paradoxical context. There are no paradoxes in physics, but only in our attempts to understand physical ideas by using inadequate reasoning or false intuition. In the above story, the physical situation, the closed circular photon orbit, can be correctly described by the General Theory of Relativity. It seems paradoxical only because many of us still have physical intuitions based on a three-hundred-year-old Newtonian picture of Nature.

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APPENDIX

For those familiar with the General Theory of Relativity we give here a formal calculation of the acceleration.\(^5\)

In a static, axially symmetric space-time the metric tensor written in "natural" coordinates \( t, \phi, r, \theta \) is independent of time \( t \) and the azimuthal angle \( \phi \). One has therefore (see e.g., Ref. 4):

\[
ds^2 = g_{tt}(r, \theta) dt^2 + g_{\phi\phi}(r, \theta) d\phi^2 + g_{rr}(r, \theta) dr^2 + g_{\theta\theta}(r, \theta) d\theta^2. \tag{A1}
\]

The circular orbits are described by the condition \( r = \text{const} \) and for the circular motion the nonvanishing components of velocity on these orbits are \( u^t \) and \( u^\phi \) (or \( u_i \) and \( u_\phi \)). The angular velocity \( \Omega \) and the specific angular momentum \( l \) are defined by

\[
\Omega = \frac{u^\phi}{u^t}, \quad l = -u_\phi/u_t. \tag{A2}
\]

The general formula for the acceleration \( a_i = u^k u_{i;k} \) takes in the metric (A1), the form:

\[
a_i = \frac{1}{2} \left[ \left( \frac{\partial g_{tt}}{\partial r} \right) + \Omega^2 \left( \frac{\partial g_{\phi\phi}}{\partial r} \right) \right] \frac{2}{g_{tt} + \Omega^2 g_{\phi\phi}}. \tag{A3}
\]

All the other components of acceleration are zero if \( \Omega = \text{const} \). The difference between the accelerations of two observers moving on the same circular orbits with different
angular velocities $\Omega$ and $\Omega + \delta \Omega$ (assuming $\delta \Omega \ll \Omega$) is

$$\delta a_r = \frac{\Omega (\partial R^2 / \partial r)}{(1 - \Omega^2 R^2)} \delta \Omega,$$

(A4)

where $R = -g_{\phi \phi} / g_{tt}$. We see that the acceleration is the same for all the observers (with $\Omega \neq \pm 1/R$) on a given orbit if and only if $\partial R / \partial r = 0$ on this orbit. Note, that on a free photon orbit we have $u' u_i = 0$ and $a_i = 0$. Therefore $\Omega = \pm 1/R$ and $\partial R / \partial r = 0$ there. Thus, we have formally proved that observers (e.g., spacecraft) moving on a circular photon orbit with different angular velocities have exactly the same acceleration. It is not equal to the acceleration of the photons (which is zero) because of the assumption $\Omega \neq \pm 1/R$ one must make to derive formulas (3) and (4).

In the special case of the nonrotating black hole with the mass $M$:

$$g_{tt} = [1 - (2GM / c^2 r)] c^2, \quad g_{\phi \phi} = r^2 \sin^2 \theta$$

(A5)

and the condition $\partial R / \partial r = 0$ reduces to the formula for the radius of the photon orbit quoted in the paper:

$$r_c = 3GM / c^2.$$

The acceleration formula (3) gives

$$a_r = c^2 / 6GM,$$

which is independent of the angular velocity.

One could have the impression that these results are not covariant, that they are an artifact of the special coordinate system used. This is not the case. It is easy to express all the formulas presented here in terms of two Killing vector fields, $\gamma'$ and $\xi'$, which exist in the space-time with the metric (A1) because of time and axial symmetries. These two Killing vectors in the special coordinates connected with (A1) have the form:

$$\gamma' = \delta'(t), \quad \xi' = \delta'(\phi),$$

with $\delta'(k)$ being the Kronecker delta.

The formulas for angular momentum and angular velocity read:

$$\Omega = -\frac{(\xi' u_i)}{(\eta' u_k) (\eta' \xi_k)} \frac{\Omega^2 (\xi' \xi_k)}{(\xi' \xi_k)},$$

and the acceleration formulas take the form:

$$a_i = \frac{\Omega \nabla_i R^2}{(1 - \Omega^2 R^2)}, \quad R^2 = -\frac{(\xi' \xi_i)}{(\eta' \eta_k)}.$$

On the photon orbit one has $\nabla_i R = 0$ and the acceleration

$$a_i = \frac{\Omega \nabla_i (\xi' \xi_k)}{2 (\xi' \xi_k)},$$

does not depend on the angular velocity. This proves our point.

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**Chaotic dynamics of a bouncing ball**

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An undergraduate experiment is described that illustrates the period doubling route to chaos in a simple dissipative mechanical system, a bouncing ball subject to repeated impacts with a vibrating table.

I. INTRODUCTION

The period doubling route to chaos has now been observed in an impressive number of experimental systems. Electrical, optical, hydrodynamic, chemical, and biological systems can all exhibit period doubling instabilities. A few recent articles in this Journal deal with the chaotic dynamics of nonlinear systems, but recent discoveries in nonlinear dynamics are still not well known at the undergraduate level.

We have developed an undergraduate experiment that illustrates many of the ideas and methods used in describing nonlinear dissipative dynamical systems. The experiment consists of two parts. In the first part the students explore the "quadratic map" on a microcomputer. Many aspects of the quadratic map are common to a large class of systems showing chaotic behavior. By studying the quadratic map, the students are introduced to the basic notions of deterministic randomness (i.e., chaos), subharmonic bifurcations, strange attractors, and the like.

These ideas are immediately applied to a simple mechanical system in the second part of the lab. The students ex-