First return time probability in correlated stationary signals

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The statistics of extreme values and return intervals of extreme events is object of a renewed and large attention in the recent literature [1–14]. Many studies have concerned with long-term correlated time series [3,4,7–9,15]. In particular, Bunde et al. [3,4] and Kantz et al. [8,9] investigated the effect of long-term correlations on the statistics of return times of extreme values above a given threshold. With the exception of Ref. [9], these studies were based on numerical analyses performed on time series of different nature and they pointed out the existence of stretched exponential distributions of return times [3,4,8]. However, as reported by other authors, a stretched exponential distribution of return times does not seem a general and universal property necessarily associated with the existence of long-term correlations [9,10,12,13].

Actually, here, we do not directly study the distribution of return times above a given threshold, but a closely related problem: the distribution of first return times at the threshold [17]. We notice that, though related, there is a difference between these two distributions. However, we expect that this difference does not affect significantly the asymptotic form of the PDFs. We consider explicitly two cases: a stationary signal with exponential decay of the correlations and a stationary signal with long-term correlations. The aim is to provide analytical expressions for the PDF of first return times.

We consider the simplest model of stationary correlated stochastic process on discrete times $t_m = \{\Delta t, 2\Delta t, \dots, m\Delta t\}$ with discrete and equally spaced states, x, separated by a distance Δx, i.e. the Ornstein-Uhlenbeck process given by:

$$x_{m+1} = \Delta x \left[\frac{x_m - k x_m \Delta t + \sqrt{D \Delta t}\xi}{\Delta x} \right] \tag{1}$$

with [...] denoting the integer part while ξ is a Gaussian zero mean and unitary variance noise. If $\Delta t \ll 1/k$ and $\Delta x \ll \sqrt{D/k}$ we can suppose that the dynamics of this model is not so different from the continuous case. This condition allows us to use the solution of the continuous Fokker-Planck equation:

$$\partial_t \rho(x,t) = k\partial_x(x\rho(x,t)) + \frac{1}{2}\sigma^2 \partial_x^2 \rho(x,t) \tag{2}$$

with the initial condition:

$$\rho(x,0) = \begin{cases} 0 & \text{if} \quad |x-L| > \Delta x/2 \\ 1/\Delta x & \text{if} \quad |x-L| < \Delta x/2 \end{cases} \tag{3}$$

where L is the level at which we want to compute the first return time statistics. Let us call $P_m(L)$ the probability that x_m is in the state L at time $m\Delta t$ and define as $\Phi_m(L)$ the probability to have the first return in the state L after the time $t = m\Delta t$, thus meaning that at time $m\Delta t$ for *the first time* $|x_m - L| < \Delta x/2$ given that for $m = 0$, $|x_0 - L| < \Delta x/2$. According to Ref. [16], we assume that the condition of having a return after the time $j\Delta t$ does not change the formal dependence of the probability to stay in L after a time $(m-j)\Delta t$. This implies the following relation between $P_n(L)$ and $\Phi_m(L)$:

$$P_m(L) = \delta_{m,0} + \sum_{j=1}^{m} \Phi_j(L) P_{m-j}(L). \tag{4}$$

By defining the characteristic functions:

$$P(L,z) = \sum_{m=0}^{\infty} z^m P_m(L); \Phi(L,z) = \sum_{m=1}^{\infty} z^m \Phi_m(L) \tag{5}$$

it is easy to derive the equation:

$$\Phi(L,z) = 1 - \frac{1}{P(L,z)}. \tag{6}$$

By making use of the solution of Eq.(2) reported in Ref.[18], after some algebra (see for details Refs. [19]) we are able to obtain the following expression:

$$\Phi(y,z) = 1 - \left\{ 1 + b' \sum_{n=0}^{\infty} \frac{c_n z \exp(-kn\Delta t)}{1 - z\exp(-kn\Delta t)} \right\}^{-1} \tag{7}$$

with: $y = L\sqrt{k/D}$, $c_n = H_n^2(y)/(2^n n!)$, $b' = \Delta x\,(k/\pi D)^{1/2} \exp(-y^2)$ and $H_n(x)$ the physical Hermite polynomials.

Let us now consider a continuous time process. If $\psi(t)$ is the probability density function of the time intervals between two consecutive changes of value of x_m, following [16] we can write the first return time $\bar{\Phi}(y,t)$ as:

$$\bar{\Phi}(y,t) = \sum_{n=1}^{\infty} \Phi_n(y) \psi_n(t) \tag{8}$$

where $\psi_n(t)$ is the probability for the occurrence of n-th jump exactly at time t. By taking equally spaced changes, $\psi(t) = \delta(t - \Delta t)$, by some calculations reported in Refs. [19], we obtain:

$$\bar{\Phi}(y,t) \simeq \sum_{i=1}^{n} \tilde{c}_i \exp(-\alpha_i t). \quad (9)$$

where the expressions of $\tilde{c}_i$ and α_i are given in Refs. [19]. This development converges very fastly and, with typical values of parameters, we have a very reliable approximation also for low values of n [19]. Fig.1 shows the good agreement obtained with $n = 5$ for different values of y.

Now we consider a process with power law decaying correlation function. To this purpose we use the subordination approach and we suppose that the average distance between consecutive changes in x_m is always Δt but that its variance diverges due to the fact that the variable Δt is distributed according to a power law tail with exponent $\mu \in]2, 3]$. As a example we take:

$$\psi(t) = (\mu-1)\frac{[(\mu-2)\Delta t]^{\mu-1}}{[(\mu-2)\Delta t + t]^{\mu}} = (\mu-1)\frac{T^{\mu-1}}{(T+t)^{\mu}}$$

with $T \equiv (\mu - 2)\Delta t$ and $\langle t \rangle = \Delta t$ with $\langle \ldots \rangle$ denoting the average value. After some algebra (see Ref. [19]) we obtain:

$$\Phi(y,t) = \frac{(\mu-2)(\mu-1)T^{\mu-2}}{b(y)} \cdot \left[\frac{1}{t^{\mu}} + \left(\frac{1}{b(y)} + \sum_{n=1}^{\infty} \frac{c_n}{n}\right) \frac{\mu}{t^{\mu+1}}\right]$$

It must be noted that besides the asymptotic behavior with tail exponent μ, we have a transient with exponent $\mu+1$. A discussion of these results and of their implications can be found in [19,20].

Here we summarize them as follows: we have obtained analytical expressions of the first return time probability density function for two stationary correlated model processes. We have tested these expressions by comparing them with numerical simulations and we have found a good agreement, suggesting that the approximations performed to get the analytical expressions are reasonable. Furthermore we have shown that in a renewal model with power law tail of correlations, the PDF of the first return times is itself a power law and not a stretched exponential.

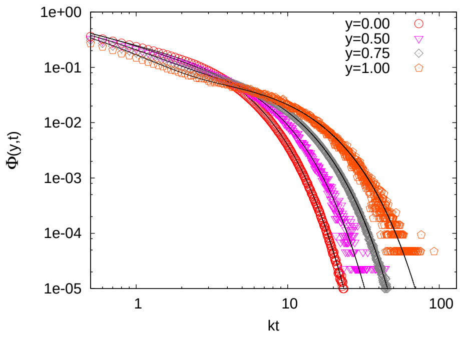

Figure 1. First return time PDFs for different values of $y = L\sqrt{k/D}$. The continuous lines are calculated by numerical inversion of Eq.(9) with $n = 5$, while symbols show the results of simulatiions.

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