Integrability and AdS/CFT correspondence

M. Beccaria, [1 2] G. F. De Angelis [1 2], and C. Ratti[1 2]

[1]Dipartimento di Fisica, Università del Salento, Italy

[2]Istituto Nazionale di Fisica Nucleare sez. di Lecce, Italy

The correspondence between quantum theories of gauge fields and string theories opens a new perspective in our understanding of fundamental interactions. The first and best understood example of the gauge/gravity duality is between the maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and free Type IIB superstrings on the $AdS_5 \times S^5$ background.

A fundamental new insight of the last years is that these theories might be described by an integrable model. Complex but exhaustive tools (Thermodynamic Bethe Ansatz, or equivalently Y-system) have been proposed to solve the full planar spectrum of both theories. In the limit of strong/weak coupling, substantial simplifications occur to the TBA/Y-system equations and the spectrum of particular sectors of the theories can be analytically solved. In the same regimes, other smart (and technically easier) approximations (semi-classical Lagrangian approaches or algebraic curve methods) have been proposed and provide useful crosschecks to TBA-inspired computations.

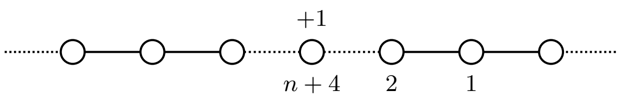
In 2011, our activity focused on two main points. 1) In the weak coupling regime, extend the case record over which we can apply the Y-system technique. In so doing, we produced new analytic results for the wrapping part of the spectrum of a large class of operators previously not yet studied in $\mathcal{N} = 4$ SYM theory [1], [2] as well as in less supersymmetric deformations of $\mathcal{N} = 4$ for which the associated integrable model is known [3]. 2) In the strong coupling regime, apply the algebraic curve method to get informations on the leading energy corrections of relevant string solutions [4]. Other papers discussing related issues can be found in [5].

In this note, we will give an insight of how the Y-system works at weak coupling by applying it to compute the spectrum of a special class of $\mathcal{N} = 4$ SYM operators, namely the 3-gluon operators.

3-gluon operators are twist 3 operators. Their general form can be written by inserting covariant and anti-covariant derivatives $\mathscr{D}$, $\bar{\mathscr{D}}$ into the half-BPS state Tr$\mathcal{Z}^3$ ($\mathcal{Z}$ being one of the three complex scalars of $\mathcal{N} = 4$ SYM theory):

$$\mathbb{O}_n^{3-\text{gluons}} = \text{Tr}\,(\mathscr{D}^{n+2}\bar{\mathscr{D}}^2\mathcal{Z}^3) + \cdots . \tag{1}$$

At strong coupling, these operators are dual to spinning string configurations with two spins $S_1 = n + m - \frac{1}{2}$ and $S_2 = m - \frac{1}{2}$ in AdS_5 and charge $J = L = 3$ in S^5. The distribution of the roots on the nodes of the $\mathfrak{psu}(2, 2|4)$ algebra is summarized in the Figure here below:



These operators are a set of operators out of the $\mathfrak{sl}(2)$ sector (note the roots out of the central node).

In general, the wrapping effects are expected to appear at weak coupling for these operators. The asymptotic part of their anomalous dimensions is already known as a closed formula in the spin parameter n. Up to this level, reciprocity holds for the asymptotic contributions. We show now how to derive from the Y-system equations precisely a similar closed formula for the leading wrapping corrections. This formula, together with the asymptotic results, completes the study of the anomalous dimensions for the 3-gluon operators up four loop. As a byproduct, we shall be able to test positively reciprocity as well as discuss the BFKL poles of the full four loop result. We also show that a very simple and natural modification of the twist-2 BFKL equation predicts the correct pole structure.

The Y-system is the following set of functional equations for the functions $Y_{a,s}(u)$ defined on the fat-hook of $\mathfrak{psu}(2, 2|4)$.

$$\frac{Y_{a,s}^+\,Y_{a,s}^-}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{a,s-1})}{(1 + Y_{a+1,s})(1 + Y_{a-1,s})}. \tag{2}$$

The anomalous dimension of a generic state is given through the Y's by the TBA formula

$$E \;\;=\;\; \sum_{\ell=0}^{\infty} g^{2\ell}\gamma_{\ell\text{-loop}} \tag{3}$$

$$
\begin{aligned}
&= \underbrace{\sum_i \epsilon_1(u_{4,i})}_{\text{asymptotic } \gamma^{\text{asy}}} \\
&+ \underbrace{\sum_{a\geq 1}\int_{\mathbb{R}} \frac{du}{2\pi i}\frac{\partial \epsilon_a^\star}{\partial u}\log(1+Y_{a,0}^\star(u)),}_{\text{wrapping W}} \\
\epsilon_a(u) &= a+\frac{2\,i\,g}{x^{[a]}}-\frac{2\,i\,g}{x^{[-a]}},
\end{aligned}
$$

where the star means evaluation in the mirror kinematics[1].

The relevant solutions to the Y-system are

$$
Y_{a,0}(u) \simeq \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L \frac{\phi^{[-a]}}{\phi^{[+a]}}\, T^{\rm L}_{a,1}\, T^{\rm R}_{a,1}, \tag{4}
$$

where $\frac{\phi^{[-a]}}{\phi^{[+a]}}$ is the fusion form factor and $T^{\rm L,R}_{a,1}$ are the transfer matrices of the antisymmetric rectangular representations of $\mathfrak{su}(2|2)_{\rm L,R}$. They can be explicitly computed by an appropriate generating functional.

At weak coupling, the formula (3) for the leading wrapping corrections takes the simpler form

$$
W=-\frac{1}{\pi}\sum_{a=1}^{\infty}\int_{\mathbb{R}} du\, Y^\star_{a,0} = -2i\sum_{a=1}^{\infty}\text{Res}_{u=\frac{ia}{2}}Y^\star_{a,0}. \tag{5}
$$

The formula (4) for the relevant Y functions gets simplified too. In particular, the Y's can be expressed as functions of the 1-loop Baxter polynomials Q_i's. Their expressions can be found in [1]. These expressions together with equations (4), (5) are in principle all what one needs to compute the leading wrapping corrections for the FRZ operators. The computation of the wrapping corrections is then straightforward.

In [1] we produced a list of results for the wrapping up to $n=70$. Then we were able to condensate these data in a closed formula that replicates and generalizes to any value of n the numerical results of the list. The formula is

$$
\begin{aligned}
W_n &= (r_{0,n}+r_{3,n}\,\zeta_3+r_{5,n}\,\zeta_5)\; g^8, \\
r_{5,n} &= 80\left(4\,S_1+\frac{2}{N+1}+4\right)\times \\
&\quad \left(-4(N+1)+\frac{1}{N+1}\right), \\
r_{3,n} &= 16\left(4\,S_1+\frac{2}{N+1}+4\right)\times \\
&\quad \left[8\,(N+1)\,S_2+8+\frac{2}{N+1}(2-S_2)\right. \\
&\quad \left. -\frac{2}{(N+1)^2}-\frac{1}{(N+1)^3}\right], \\
r_{0,n} &= 2\left(4\,S_1+\frac{2}{N+1}+4\right)\times \\
&\quad [16\,(N+1)\,(2\,S_{2,3}-S_5)+32\,S_3 \\
&\quad +\frac{4(S_5-2S_{2,3}+4S_3)}{N+1}+\frac{8(-S_3+2)}{(N+1)^2}+ \\
&\quad \left.+\frac{4(-S_3+4)}{(N+1)^3}-\frac{4(N+1)+1}{(N+1)^6}\right].
\end{aligned} \tag{6}
$$

Here $S_{a,b,\ldots}\equiv S_{a,b,\ldots}(N)$ and $N=n/2+1$[2]. Note that each of the rational coefficient r_i can be written as $r_{i,n}=\gamma_{\text{1-loop}}\widetilde{r}_{i,n}$, where $\gamma_{\text{1-loop}}=S_1+\frac{2}{N+1}+4$ is the one loop anomalous dimension.

The Ansatz (6) completes the four loop expression of the energy spectrum for the $\mathbb{O}^{\rm FRZ}_{n,2}$ operators, the other relevant contributions up to this order being the first four asymptotic orders.

Up to four loop, the asymptotic part of the spectrum shows the generalized Gribov-Lipatov reciprocity property. Formula (6) allows to check whether this property extends to the full four loop result. This is indeed the case: The large n expansion of $W_n/\gamma_{\text{1-loop}}$ reads

$$
\begin{aligned}
&\zeta_5\,\widetilde{r}_{5,n}+\zeta_3\,\widetilde{r}_{3,n}+\widetilde{r}_{0,n} = \\
&\quad \frac{\frac{32}{3}-32\zeta_3}{J^2}+\frac{\frac{232\zeta_3}{5}-\frac{352}{15}}{J^4}+\frac{\frac{4834}{105}-\frac{2344\zeta_3}{35}}{J^6} \\
&\quad +\frac{\frac{3544\zeta_3}{35}-\frac{83956}{945}}{J^8}+\frac{\frac{271768}{1485}-\frac{9512\zeta_3}{55}}{J^{10}}+ \\
&\quad \frac{\frac{1872392\zeta_3}{5005}-\frac{20053258}{45045}}{J^{12}}+\frac{\frac{87933002}{61425}-\frac{524872\zeta_3}{455}}{J^{14}}
\end{aligned} \tag{7}
$$

where we introduced the charge $J^2=N(N+2)$. All the odd powers of $1/J$ cancel proving that the reciprocity property does hold. It is remarkable that this property is a consequence of non-trivial cancellations of odd $1/J$ terms which are present in the expansion of each single coefficient $\widetilde{r}_{i,n}$. The presence of reciprocity is really appreciable, since it allows to predict a half of the large n expansion terms (expressed as functions of the same n) as combinations of the other half.

The Ansatz (6) allows also to study the BFKL poles of the full four loop result. In general, at ℓ-loop the analytic continuation of $\gamma_{\ell\text{-loop}}$ in the variable N around $N=-1$ is expected to behave at worst as $\omega^{-\ell}$, where ω is a small expansion parameter defined by $N=-1+\omega$. At four loops the asymptotic anomalous dimension $\gamma^{asy}_{\text{4-loop}}$ presents instead also poles in ω^{-k} with $k=(7,6,5)$. These poles get indeed compensated inside the full $\gamma_{\text{4-loop}}=\gamma^{asy}_{\text{4-loop}}+W$

[1] Shifted quantities are defined as $F^{[\pm a]}(u)=F\left(u\pm i\,\frac{a}{2}\right)$.

[2] We remind that n is an even positive integer. This means that N is an integer, $N\geq 2$.

where precisely the wrapping contribution (6) is included. The final expressions for the expansions of the first four $\gamma_{\ell\text{-loop}}$ are

$$\begin{aligned} \gamma_{1\text{-loop}} &= -\frac{4}{\omega} + \dots, \\ \gamma_{2\text{-loop}} &= \frac{8}{\omega^2} + \frac{4\pi^2}{3\omega} + \dots, \\ \gamma_{3\text{-loop}} &= \frac{\mathbf{0}}{\omega^3} - \frac{16\left(-3\zeta_3 + \pi^2 + 12\right)}{3\omega^2} + \dots, \\ \gamma_{4\text{-loop}} &= -\frac{32\,(1+2\zeta_3)}{\omega^4} + \frac{160\zeta_3}{\omega^3} + \dots. \end{aligned} \tag{8}$$

Strikingly, the leading poles can be reproduced by a BFKL-like equation that links ω to the full anomalous dimension γ

$$-\frac{\omega}{g^2} = \chi_1\left(\frac{\gamma}{2}\right), \quad \chi_1\,(z) = S_1(z) + S_1(z+1). \tag{9}$$

In fact, the weak coupling expansion of this equation reads precisely

$$\begin{aligned} \gamma &= \left(-\frac{4}{\omega} + \dots\right) g^2 + \left(\frac{8}{\omega^2} + \dots\right) g^4 \\ &+ \left(\frac{\mathbf{0}}{\omega^3} + \dots\right) g^6 + \left(-\frac{32\,(1+2\,\zeta_3)}{\omega^4} + \dots\right) g^8 \\ &+ \dots. \end{aligned} \tag{10}$$

REFERENCES

1. M. Beccaria, G. Macorini and C. Ratti, JHEP **1106**, 071 (2011) [arXiv:1105.3577 [hep-th]].
2. M. Beccaria, G. Macorini and C. Ratti, JHEP **1108**, 050 (2011) [arXiv:1107.2568 [hep-th]].
3. M. Beccaria and G. Macorini, JHEP **1106**, 004 (2011) [Erratum-ibid. **1201**, 112 (2012)] [arXiv:1104.0883 [hep-th]].
4. M. Beccaria and G. Macorini, JHEP **1110**, 040 (2011) [arXiv:1108.3480 [hep-th]].
5. M. Beccaria, C. Ratti and A. A. Tseytlin, arXiv:1201.5033 [hep-th].
   M. Beccaria and G. Macorini, arXiv:1201.0608 [hep-th].
   L. Huijse, J. Stat. Mech. **1104**, P04004 (2011) [arXiv:1102.1700 [cond-mat.stat-mech]].
   M. Beccaria, F. Levkovich-Maslyuk and G. Macorini, JHEP **1103**, 001 (2011) [arXiv:1012.2054 [hep-th]].
   M. Beccaria, M. Kreuzer and A. Puhm, JHEP **1101**, 077 (2011) [arXiv:1010.4564 [hep-th]].
   M. Beccaria and G. Macorini, Int. J. Mod. Phys. A **26**, 1229 (2011) [arXiv:1009.3428 [hep-th]].
   M. Beccaria, G. V. Dunne, G. Macorini, A. Tirziu and A. A. Tseytlin, J. Phys. A A **44**, 015404 (2011) [arXiv:1009.2318 [hep-th]].