# Reductions of the Skyrme - Faddeev model

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In classical nonlinear field equations certain classes of localized solutions are interpreted as particle-like excitations, when they are related to the existence of a topological index, which prevents the decaying into a superposition of elementary wave-like particles. For this reason such solutions are known as topological solitons. A great variety of field models admitting topological solitons have been studied (monopoles, skyrmions, instantons and vortices), playing an important role in High Energy Physics [1,24], General Relativity [4], as well as in Condensed Matter Physics [7]. A special interest deserves a 3D static nonlinear sigma model, called *Skyrme - Faddeev model* [21], the free energy given by

$$S_{SF} = \int d^3x \left[\frac{1}{4}\rho^2 \left(\partial_k \vec{\phi}\right)^2 + H_{ik}^2\right], \qquad H_{ik} = \vec{\phi} \cdot \left[\partial_i \vec{\phi} \times \partial_k \vec{\phi}\right], \quad \vec{\phi} \cdot \vec{\phi} = 1,$$

which it was proved [24] to be a special subcase of the pure quantum $SU(2)$-Yang-Mills theory in the infrared limit. Imposing the boundary conditions $\lim_{|\mathbf{x}| \to \infty} \vec{\phi} = (0,0,1)$ the homotopy group of the theory results $\pi_3\left(S^2\right) = \mathbb{Z}$ and one can conclude that all solutions of the E.-L. equations are labelled by the integer Hopf index

$$Q = \frac{1}{16\pi^2} \int d^3x \, \varepsilon_{ikl} a_i \partial_k a_l,$$

where $a_k = -\frac{\varepsilon_{ijk}}{4\pi} \int_{S^3} \frac{\left(x_i - x_i'\right) H_{ij}\left(x'\right)}{\left[\left(x_m - x_m'\right)^2\right]^{3/2}} d^3x'$, for sufficiently rapid decreasing $H_{ik}$ at infinity. It provides the linking number of the pre-images of two independent points on the target $S^2$. Any spherical symmetric solution has $Q = 0$. Numerical calculations [32] have produced a comprehensive analysis of topological solitons with $1 \leq Q \leq 16$, proving the existence of local energy minima of knotted toroidal shape, possibly many times tangled ($Q = 7$ corresponds to a trifoil knot). Global analytical considerations [35] have shown the bounded from below by $S_{SF} \geq C\ \pi^2\ \rho\ |Q|^{3/4}$. Its main consequence is that vortices of higher topological charge are metastable configurations. Finally the characteristic size of a generic but stable perturbation is $\frac{1}{\rho} \leq R_{knot} \leq \frac{\sqrt{2}}{\rho}$.

All those global analytic and numerical studies, however, do not clarify nor exploit the symmetries hidden in the systems. This is because such a structure has not been sufficiently investigated. Thus, our aim is to reveal which are the symmetries of the systems, continuous and discrete, conditional and generalized and we look for hydrodynamical reductions and how they can be used in order to extract informations about the integrability of the system. Here we report the results contained in the article [36].

We analyze the 4-dimensional relativistic generalization of the Skyrme - Faddeev model in the space endowed with the pseudo-riemannian metric $\operatorname{diag}(g_i) = (+,-,-,-)$. We adopt the stereographic complex variables $S^2 \to \mathbb{C}$, namely $\phi_1 = \frac{w+\bar{w}}{w\bar{w}+1}, \phi_2 = -\frac{i(w-\bar{w})}{w\bar{w}+1}, \phi_3 = \frac{1-w\bar{w}}{w\bar{w}+1}$. The Lagrangian density is

$$\mathcal{L}_w = \frac{\sum_{i=0}^3 g_i \partial_i w\ \partial_i \bar{w}}{8\pi^2 \left(1 + w\bar{w}\right)^2} + \lambda \frac{\sum_{i,j=0, i<j}^3 g_i\ g_j \left(\partial_i w\ \partial_j \bar{w} - \partial_j w\ \partial_i \bar{w}\right)^2}{16\pi^2 \left(1 + w\bar{w}\right)^4}.$$

where $\lambda = \frac{16}{\rho^2} > 0$ is the scaling parameter. Now, from the point symmetries analysis of the Skyrme - Faddeev model, the symmetry group reduces to

$$\left(\mathbb{R}^4 \rtimes SO(3,1)\right) \otimes SO(3)_g .$$

The infinitesimal generators of the algebra group are

$$\mathbf{t}_i = \partial_i, \qquad \mathbf{r}_{i,j} = x^i \partial_j - g_i g_j x^j \partial_i,$$
$$\mathbf{w}_0 = -w\partial_w + \bar{w}\partial_{\bar{w}}, \ \mathbf{w}_\pm = \tfrac{w^2 \pm 1}{2}\partial_w \pm \tfrac{\bar{w}^2 \pm 1}{2}\partial_{\bar{w}},$$

with the commutation relations

$$[\mathbf{w}_+, \mathbf{w}_-] = \mathbf{w}_0, \quad [\mathbf{w}_0, \mathbf{w}_\pm] = \mp \frac{\mathbf{w}_\mp}{2},$$

providing the non singular 3-parameter linear fractional trasformation for $w$. One can prove also that all those symmetries are variational symmetries for the model [41], then the Nöther theorem provides the conserved currents: the 3-component linear and angular momenta and the triplet of the gauge charges. They all involve new $\lambda$ contributions. All the above quantities can be easily computed for the special solution $w = e^{ip_jx_j}$ with $\sum_i g_jp_j = 0$. But, because of the nonlinearity, one cannot infer simple conclusions for other solutions. To simplify the problem one proceeds to a symmetry reduction [42]. This procedure is algorithmic and it can performed by classifing the symmetry algebra $so(3,1) \uplus \left(Span\{\mathbf{t}_i\} \oplus so(3)_g\right)$. It provides a list of representatives of the different conjugacy classes up to dimension 3, which posses orbits of co-dimensions $\leq 3$ in space-time, precisely i) 1-dimensional splitting s.a. : $\mathbf{t}_0$, $\mathbf{r}_{12}$, $\mathbf{t}_3$, $\mathbf{r}_{12} + \alpha\mathbf{w}_0$, $\alpha \in \mathbb{R}$; ii) 1-dimensional non splitting s.a. : $\mathbf{r}_{12} \pm \mathbf{t}_3$; iii) 2-dimensional s.a. : $\{\mathbf{t}_0, \mathbf{t}_3\}$, $\{\mathbf{t}_3, \mathbf{t}_1\}$, $\{\mathbf{t}_0, \mathbf{r}_{12}\}$, $\{\mathbf{t}_0, \mathbf{r}_{12} \pm \mathbf{t}_3\}$ and $\{\mathbf{t}_0, \mathbf{r}_{12} + \alpha\mathbf{w}_0\}$ for any $\alpha \in \mathbb{R}$ ; iv) 3-dimensional splitting s.a. : $so(3)_{rot}$ , $\{\mathbf{t}_0, \mathbf{t}_3, \mathbf{t}_1\}$. The reduction with respect $\mathbf{t}_0$ leads to static solutions, thus further space reduction are 2- or 3- dimensional. Pure 1- and 2- dimensional space reductions lead to solutions with infinite energy. Moreover, in 2 dimensions the nonlinear-$\sigma$ models is completely integrable [43]. The reduction induced by spacial $so(3)$ has the 2D spheres as generic orbits. But also the fields $w, \bar{w}$ enjoy of the same symmetry algebra, then one can establish an isomorphism between the two set of rotations, leading to the so-called hedghog solutions [1], expressed by $w = e^{i\phi}\tan\left(\frac{\theta}{2}\right)$, or equivalently as $\vec{\phi} = \frac{\vec{r}}{r}$.

Finally, let us consider the special case $\{\mathbf{t}_0, \mathbf{r}_{12} + \alpha\mathbf{w}_0\}$ for $\alpha \in \mathbb{R}$. The invariants are $r, \theta, we^{i\alpha\phi}, \bar{w}e^{-i\alpha\phi}$. Any invariant solution is of the form $w_\alpha = e^{-i\alpha\phi}W(r,\theta)$ for a certain function $W$, which can be put as $W = \tan(\psi(\theta)) + i\cot(\chi(r))\sec(\psi(\theta))$. Continuity and differentiability in $\theta = 0, \pi$ imply $\psi(\theta) = m\,\theta$ for $m \in \mathbb{Z}$. Continuity of the azimuthal rotations implies $\alpha = n \in \mathbb{Z}$. Then $\vec{\phi}\cdot\vec{\sigma} = U\,\sigma_3\,U^\dagger$, where

$$\begin{aligned} U &= \exp\left[i\chi(r)\,\vec{\nu}(\vartheta,\varphi)\cdot\vec{\sigma}\right], \\ \vec{\nu} &= (\sin m\vartheta\,\cos n\varphi, \sin m\vartheta\,\sin n\varphi, \cos m\vartheta). \end{aligned}$$

This is the group theoretical derivation of the well known axisymmetric ansatz for skyrmions [28,35]. Since the map $\vec{\nu} : \mathbb{S}^2 \to \mathbb{S}^2$, covers $m\,n$ times the sphere, the corresponding Hopf charge is $Q = m\,n$. In the case $m = n = 1$, one obtains the energy functional for the static configurations

$$E = \int_0^\infty \left(r^2\chi'^2 + 2\sin^2\chi\left(\lambda\chi'^2 + 1\right) + \lambda\frac{\sin^4\chi}{r^2}\right) dr,$$

leading to the equation

$$\left(r^2 + 2\lambda\sin^2\chi\right)\chi'' + 2r\chi' + \sin 2\chi\left(\lambda\chi'^2 - 1 - \lambda\frac{\sin^2\chi}{r^2}\right) = 0,$$

with the supplementary boundary conditions $\chi \to \pi$ for $r \to 0$. The above boundary problem does not admit analytic solutions. Setting the value of the scaling parameter $\lambda = 1$, we have found a new approximation

$$\begin{aligned} \chi(r) &= 2\arcsin(g(r)), \\ g(r) &= \frac{1 + a_1r + a_2r^2}{1 + a_1r + b_2r^2 + b_3r^3 + b_4r^4}, \end{aligned}$$

$a_1 = 0.216$, $a_2 = 0.230$, $b_2 = 0.752$, $b_3 = -0.018$, $b_4 = 0.302$, which are determined by a minimization of $E[\chi]$, differing by $\approx 10^{-2}\%$ from the numerical calculations. The above expression are sufficiently simple to be manipulated into adiabatic interactions of two of such excitations. The

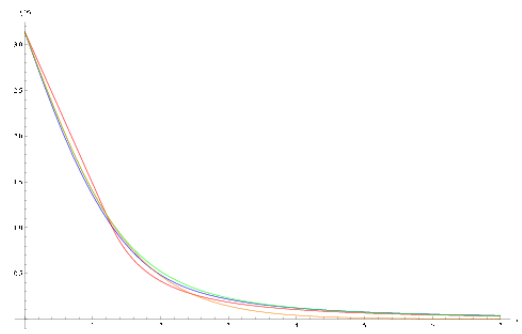

Figure 1. Blu: numerical solution, Green: our approximation, Red: quasi-linear approximation, Orange: Atiyah-Manton ansatz.

idea of compensating the space rotation with a simultaneous rotation in the field can be applied also to discrete (Platonic) subgroups of the space rotations [1]. They are described by the Klein Polynomials [44] in the complex plane, specifying all rational maps in the Riemannian sphere invariant under linear fractional transformations. The new ansatz for maps $\vec{\nu} : \mathbb{S}^2 \to \mathbb{S}^2$ and field is

$$\vec{\nu}_R = \frac{1}{1 + |R|^2}\left(R + \bar{R}, i\left(\bar{R} - R\right), 1 - |R|^2\right),$$

$$w(r,z,\bar{z}) = \frac{(1 - |R|^2) + i(1 + |R|^2)\cot\chi(r)}{2\bar{R}},$$

leading to the radial energy for $\chi(r)$

$$E = \int_0^\infty \left(\mathcal{I}r^2\chi'^2 + 2\sin^2\chi\left(\lambda\mathcal{B}_1\chi'^2 + \mathcal{B}_2\right) + \lambda\mathcal{J}\frac{\sin^4\chi}{r^2}\right) dr,$$

and to the she equation of motion as aboe except for the coefficients $\mathcal{I}$, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{J}$, which are completely defined in terms of $R$ dans its derivatives. In particular one proves the relation $\mathcal{B}_1 = \mathcal{B}_2 = N$ (the baryonic number) and we built the following table:

| $R(z)$ | $N$ | $\mathcal{I}$ | $\mathcal{J}$ |
|---|---|---|---|
| $z$ | 1 | 1 | 1 |
| $z^2$ | 2 | 0.644 | 3.956 |
| $\frac{z^3-\sqrt{3}iz}{\sqrt{3}iz^2-1}$ | 3 | 1 | 13.577 |
| $\frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$ | 4 | 1.172 | 25.709 |
| $\frac{z^7-7z^5-7z^2-1}{z^7+7z^5-7z^2+1}$ | 7 | 1 | 60.868 |

Note that $R(z) = z$ corresponds to the usual axisymmetric hedgehog ansatz discussed in the previous section.

The work is part of the project entitled "Vortices, Topological Solitons and their Excitations" as joint call Consortium EINSTEIN - Russian Foundation for Basic Researches. The work has been partially supported by the INFN - Sezione of Lecce under the project LE41.

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