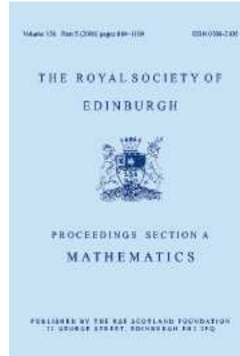
Proceedings of the Royal Society of Edinburgh: Section A Mathematics
http://journals.cambridge.org/PRM



The classifying space of the gauge group of an SO(3)-bundle over S

S. Bauer, M. C. Crabb and M. Spreafico

Proceedings of the Royal Society of Edinburgh: Section A Mathematics / Volume 131 / Issue 04 / August 2001, pp 767 - 783
DOI: 10.1017/S0308210500001098, Published online: 12 July 2007

Link to this article: http://journals.cambridge.org/abstract_S0308210500001098

How to cite this article:
S. Bauer, M. C. Crabb and M. Spreafico (2001). The classifying space of the gauge group of an SO(3)-bundle over S. Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 131, pp 767-783 doi:10.1017/S0308210500001098

Proceedings of the Royal Society of Edinburgh, **131A**, 767–783, *2001*

The classifying space of the gauge group of an SO(3)-bundle over S^2

S. Bauer, **M. C. Crabb** and **M. Spreafico**
Department of Mathematical Sciences, University of Aberdeen,
Aberdeen AB24 3UE, UK

(MS received 30 August 1999; accepted 14 September 2000)

Stable homotopy decompositions of the classifying spaces of the gauge groups of principal SO(3) and U(2)-bundles over the sphere S^2 are obtained using a fibrewise stable splitting theorem for the loop space of an unreduced suspension. The stable decomposition is related to a description of the integral cohomology ring.

1. Introduction

Principal SO(3)-bundles P over the sphere S^2 are classified, up to isomorphism, by $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$. The group SO(3) acts on itself by conjugation $\mathrm{Ad} : \mathrm{SO}(3) \to \mathrm{Aut}\,\mathrm{SO}(3)$. We shall call the space of sections of the associated bundle of groups $\mathrm{Ad}\,P = P \times_{\mathrm{SO}(3)} \mathrm{SO}(3)$ the gauge group $\mathcal{G}_P$.

Our main result gives the following description of the stable homotopy type of the classifying space $B\mathcal{G}_P$ as a wedge of Thom spaces over the infinite-dimensional complex projective space. (In the statement we use a subscript '+' to denote adjunction of a disjoint basepoint.)

Proposition 1.1. *Let P be a principal* SO(3)*-bundle over S^2. Then the classifying space of the gauge group $\mathcal{G}_P$ admits a stable decomposition,*

$$(B\mathcal{G}_P)_+ \simeq \begin{cases} B\mathrm{SO}(3)_+ \vee \displaystyle\bigvee_{k\geqslant 1\,\mathrm{odd}} B\mathrm{U}(1)^{kH} & \text{if } P \text{ is trivial},\\[2ex] \displaystyle\bigvee_{k\geqslant 0\,\mathrm{even}} B\mathrm{U}(1)^{kH} & \text{if } P \text{ is non-trivial},\end{cases}$$

where H is the complex Hopf line bundle over infinite-dimensional complex projective space $P(\mathbb{C}^\infty) = B\mathrm{U}(1)$.

We were led to this result by reading the paper [6] of Tsukuda, who showed there that the integral homology of $B\mathcal{G}_P$ is torsion-free when P is non-trivial, but left open the question of determining the precise ring structure. Restriction to a basepoint in S^2 gives a map $B\mathcal{G}_P \to B\mathrm{SO}(3)$ and so provides $H^*(B\mathcal{G}_P;\mathbb{Z})$ with the structure of an algebra over $H^*(B\mathrm{SO}(3);\mathbb{Z})$. A precise description of this algebra, in terms of generators that are specified in § 3, can be read off as follows from proposition 3.6, which computes the $\mathbb{Z}[\frac{1}{2}]$-cohomology.

Proposition 1.2. *Let P be a principal* $\mathrm{SO}(3)$*-bundle over S^2. If P is non-trivial, the integral cohomology ring $H^*(B\mathcal{G}_P;\mathbb{Z})$ is a free module over the polynomial ring* $\mathbb{Z}[p] = H^*(B\mathrm{SO}(3);\mathbb{Z})/\mathrm{torsion}$ *generated by the first Pontrjagin class $p \in H^4$, on generators $b_i \in H^{2i}$, $i \geqslant 0$, described in terms of a generator b of $H^2(B\mathcal{G}_P;\mathbb{Z})$ by the identities: $b_0 = 1$, $b_1 = b$ and, for $j \geqslant 1$,*

$$b_{2j} = \frac{(b^2 - 1^2p)(b^2 - 3^2p)(b^2 - 5^2p)\cdots(b^2 - (2j-1)^2p)}{(2j)!}, \qquad b_{2j+1} = \frac{b \cdot b_{2j}}{2j+1},$$

in the rational cohomology ring $H^*(B\mathcal{G}_P;\mathbb{Q}) = \mathbb{Q}[p,b]$.

In terms of the stable decomposition that we construct, the cohomology of the wedge summand $B\mathrm{U}(1)^{2jH}$ is the algebraic direct summand $\mathbb{Z}[p]b_{2j} \oplus \mathbb{Z}[p]b_{2j+1}$. There is a similar calculation when P is trivial.

Remark 1.3. We regard the stable splitting in proposition 1.1 primarily as a splitting of spectra. In fact, the proof shows that the stable splitting can be realized after a single suspension.

Remark 1.4. The individual summands appearing in the decomposition are stably indecomposable at the prime 2, with the exception of the $k = 0$ summand $B\mathrm{U}(1)_+$, which splits as $S^0 \vee B\mathrm{U}(1)$, and $B\mathrm{SO}(3)_+$, which splits as $S^0 \vee B\mathrm{SO}(3)$. A proof of indecomposability is outlined in § 6.

The classical stable splitting theorem of James and Milnor expresses the space $\Omega\Sigma F$ of loops on the (reduced) suspension of a connected pointed space F stably as a wedge of smash products $\bigwedge^k F$, $k \geqslant 1$. In § 2 we prove a version of the splitting theorem for the unreduced suspension $S^0 * F$, instead of ΣF, without reference to a basepoint in F. In this form the result is easily extended to a fibrewise splitting theorem, in which F is replaced by a fibre bundle, which will not, in general, have a section. This theorem is specialized to the case of sphere-bundles in § 3, which also contains calculations of the cohomology ring. The connection with the classifying space of the gauge group $\mathcal{G}_P$ is made in § 4. Standard results, for one of which we present a new proof in § 5, express the space $B\mathcal{G}_P$ as a fibrewise loop space with fibre $\Omega\mathrm{SO}(3)$. By regarding $\mathrm{SO}(3)$ as 3-dimensional real projective space and identifying the two components of $\Omega\mathrm{SO}(3)$ with the space of loops ΩS^3 and the space of paths from the North to the South Pole in S^3, respectively, we reduce propositions 1.1 and 1.2 to special cases of the results on sphere-bundles in § 3. The main result has other applications, such as to spaces of free loops and to equivariant splitting theorems, which will be treated elsewhere.

2. A stable splitting of the space of loops on an unreduced suspension

As explained in § 1, our first goal is to establish a stable James–Milnor splitting theorem for the space of loops on the join $S^0 * F$ (or unreduced suspension) of a connected space F. Our proofs require that a certain fibrewise pointed space constructed from F is locally trivial in the pointed sense. This is most easily guaranteed by requiring that F be a closed manifold. We assume, therefore, throughout this section that F is a connected closed (smooth) manifold.

The sphere $S^0 = \{\pm 1\}$ is contained as a subspace of $S^0 * F$, and we take $+1$ as the basepoint of $S^0 * F$. We write $\Gamma^0 F$ for the loop space $\Omega(S^0 * F)$ of continuous paths $[0,1] \to S^0 * F$ from $+1$ to $+1$ and $\Gamma^1 F$ for the space of paths from $+1$ to -1, with the indices $0, 1$ read as integers $(\bmod 2)$. Thus, if F is the circle S^1, $S^0 * F$ is the sphere S^2, $\Gamma^0 F$ consists of the loops at the North Pole $+1$ and $\Gamma^1 F$ consists of the paths from the North to the South Pole.

Let $\tau : S^0 * F \to S^0 * F$ denote the involution induced by the antipodal map -1 on S^0. Then concatenation of paths defines multiplication maps

$$\Gamma^i F \times \Gamma^j F \to \Gamma^{i+j} F, \qquad (\alpha, \beta) \mapsto \alpha \cdot \tau^i(\beta),$$

which give the disjoint union $\Gamma^0 F \sqcup \Gamma^1 F$ the structure of a homotopy-associative Hopf space.

If we choose a basepoint $* \in F$, we can form the (reduced) suspension ΣF, and the projection map $S^0 * F \to \Sigma F = (S^0 * F)/(S^0 * \{*\})$ is a pointed homotopy equivalence. The projection also induces homotopy equivalences $\Gamma^i F \to \Omega \Sigma F$, which together give an H-equivalence

$$\Gamma^0 F \sqcup \Gamma^1 F \to \Omega \Sigma F \ltimes \mathbb{Z}/2 \tag{2.1}$$

to the semidirect product defined by the antipodal involution on the Σ component of $\Omega \Sigma F$.

We shall effectively consider all possible basepoints of F by means of the following device. The trivial bundle $F \times F \to F$, projecting onto the first factor $(x, y) \mapsto x$, has a cross-section given by the diagonal map $x \mapsto (x, x)$, and this structure defines a pointed fibre bundle, or (locally trivial) bundle of pointed spaces, which we denote by $X = F \times F \to F$. The fibre over $x \in F$ is the pointed space $\{x\} \times F$ with basepoint (x, x). As one can see by considering the example $F = [0, 1]$, local triviality in the pointed sense does require proof, and it is here that we use the fact that F is a manifold without boundary. Local triviality is established in [4, (II.1.20) and (II.11.20)] by refining the classical proof that the diffeomorphism group of a connected manifold acts transitively on the manifold.

From F we can also form the trivial pointed fibre bundle $F \times F_+$, with fibre the pointed space F_+ obtained by adjoining a disjoint basepoint to F. There is a fibrewise pointed map $\pi : F \times F_+ \to X$, which restricts to the identity on the subspace $F \times F \subseteq F \times F_+$ and maps the basepoint $(x, +)$ in the fibre over $x \in F$ to (x, x). We also have a fibrewise pointed map $\theta : F \times F_+ \to F \times S^0$ given by $\theta(x, y) = (x, -1)$ for $x, y \in F$.

Lemma 2.1. *The stable sum*

$$\pi \vee \theta : F \times F_+ \to X \vee_F (F \times S^0)$$

over F is a fibrewise stable equivalence.

Proof. On a fibre the map $\pi \vee \theta$ restricts to the standard stable equivalence $F_+ \to F \vee S^0$ for a pointed space F. The result follows from a theorem of Dold which states that a fibrewise stable map which is an equivalence on each fibre is a fibrewise stable equivalence. Alternatively, one may reproduce the standard proof of the splitting, which splits the cofibre sequence $S^0 \to F_+ \to F$, over F and thus realize the stable equivalence after a single fibrewise suspension. □

Let $\rho : X \to F \times F_+$ denote the natural stable right inverse for π provided by the stable decomposition in lemma 2.1. We use this stable splitting first to show that the pointed fibre bundle X is stably trivial.

LEMMA 2.2. *Choose a basepoint $* \in F$ and let $q : F_+ \to F$ be the pointed map that restricts to the identity on F. Then the composition*

$$X \xrightarrow{\rho} F \times F_+ \xrightarrow{1\times q} F \times F$$

is a fibrewise stable equivalence over F.

Proof. The restriction to fibres over the basepoint $* \in F$ is the identity map (of pointed spaces) $F \to F$. Since the base F is connected and $X \to F$ is locally trivial, the result follows from Dold's theorem. ☐

DEFINITION 2.3. For $k \geqslant 0$, we can form the fibrewise smash product $\bigwedge_F^k X$ over F. We define

$$G^k(F) = (\textstyle\bigwedge_F^k X)/F$$

to be the pointed space obtained by collapsing the basepoint section F to a point. Thus $G^k(F)$ is the topological quotient of $F \times F^k$ by the subspace consisting of the points $(x, (y_1, \dots, y_k))$ with some y_i equal to x.

In particular, $G^0(F) = F_+$ and $G^1(F) = (F \times F)/\Delta(F)$ is the topological cofibre of the diagonal $\Delta : F \to F \times F$.

REMARK 2.4. A choice of basepoint in F allows us to use lemma 2.2 to obtain a stable equivalence

$$G^k(F) \simeq (F \times \textstyle\bigwedge^k F)/F \simeq F_+ \wedge (\bigwedge^k F) \simeq (F \vee S^0) \wedge (\bigwedge^k F) = \bigwedge^k F \vee \bigwedge^{k+1} F.$$

We shall next construct stable maps

$$\gamma_k : G^k(F) \to \begin{cases} (\Gamma^0 F)_+ & \text{for } k \geqslant 1 \text{ odd,} \\ (\Gamma^1 F)_+ & \text{for } k \geqslant 0 \text{ even.} \end{cases}$$

First, we have a map $\alpha : F \to \Gamma^1 F$ sending a point $x \in F$ to the path $\alpha_x : [0,1] \to S^0 * F$ from $+1$ to -1 through x,

$$\alpha_x(t) = [1 - 2t, x] \qquad (0 \leqslant t \leqslant 1).$$

Geometrically, when $F = S^1$, α_x is a great circle from the North to the South Pole through the point x on the equator. Using the multiplication in $\Gamma^0 F \sqcup \Gamma^1 F$ introduced above, we define maps

$$\tilde{\gamma}_k : F \times F^k \to \begin{cases} \Gamma^0 F & \text{for } k \text{ odd,} \\ \Gamma^1 F & \text{for } k \text{ even} \end{cases}$$

by

$$\tilde{\gamma}_k(x, (y_1, \dots, y_k)) = \alpha_x(\alpha_{y_1}(\alpha_{y_2}(\cdots(\alpha_{y_{k-1}}\alpha_{y_k}))\cdots)).$$

If k is odd, we get a closed path, and if k is even, a path from $+1$ to -1. Again in the geometric picture when $F = S^1$, $\tilde{\gamma}_k$ assigns to $(x, (y_1, \dots, y_k))$ a path passing up and down between the North and South Pole through the equatorial points x, $y_1, \dots, y_k$ in order.

The standard proof of the James–Milnor splitting, or the more general Snaith splitting (based on the stable splitting, for a pointed space F, of F_+ as $S^0 \vee F$), is due to Cohen [2]. There is a textbook account in [4, (II.14.27)]. We follow the same method to construct the stable map γ_k. The fibrewise k-fold smash product of the stable splitting $\rho : X \to F \times F_+$ of the projection $\pi : F \times F_+ \to X$ gives a fibrewise stable splitting over F,

$$\bigwedge^k \rho : \bigwedge_F^k X \to \bigwedge_F^k (F \times F_+) = F \times \bigwedge^k F_+,$$

of the projection $\bigwedge^k \pi$. By collapsing the basepoint section F to a point we obtain a stable splitting

$$G^k(F) = (\bigwedge_F^k X)/F \to (F \times (\bigwedge^k F_+))/F = (F \times F^k)_+ \tag{2.2}$$

of the projection map $F \times F^k \to G^k(F)$. The stable map γ_k is defined by composing this stable splitting with $(\tilde{\gamma}_k)_+$.

To state the basic stable decomposition result we let $\iota : S^0 \to (\Gamma^0 F)_+$ denote the map given by the inclusion of the basepoint (that is, the constant loop) in $\Gamma^0 F$.

Lemma 2.5. *The maps*

$$\iota \vee \bigvee \gamma_k : S^0 \vee \bigvee_{k \geqslant 1 \,\mathrm{odd}} G^k(F) \to (\Gamma^0 F)_+,$$

$$\bigvee \gamma_k : \bigvee_{k \geqslant 0 \,\mathrm{even}} G^k(F) \to (\Gamma^1 F)_+$$

are stable equivalences.

Proof. Since the spaces considered are of finite type, it suffices to check that the maps induce isomorphisms on homology H_* with $\mathbb{F}_p$-coefficients for each prime $p > 1$. To carry out the computation we again choose a basepoint $* \in F$ and let V denote the graded $\mathbb{F}_p$-vector space $\tilde{H}_*(F)$.

Now $\tilde{H}_*(F_+) = \mathbb{F}_p \oplus V$ and $\tilde{H}_*((F \times F^k)_+) = (\mathbb{F}_p \oplus V) \otimes \bigotimes^k (\mathbb{F}_p \oplus V)$. The direct summand $\tilde{H}_*(G^k(F))$ is included by the splitting map (2.2) as the algebraic summand $(\mathbb{F}_p \oplus V) \otimes \bigotimes^k V$. This follows from two observations: (i) that $\tilde{H}_*(G_k(F))$ projects isomorphically onto $\tilde{H}_*((F \times \bigwedge^k F)/F)$, by lemma 2.2, and (ii) that the summand is, by its construction, annihilated by any k-fold fibrewise smash product of factors 1 (the identity map) and θ,

$$1 \wedge \cdots \wedge \theta \wedge \cdots \wedge 1 : F \times (F_+ \wedge \cdots \wedge F_+ \cdots \wedge F_+) \to F \times (F_+ \wedge \cdots \wedge S^0 \cdots \wedge F_+),$$

in which at least one factor is θ.

We next recall the classical description of the $\mathbb{F}_p$-homology of $\Omega \Sigma F$. The James map $F \to \Omega \Sigma F$, for the pointed space F, induces an inclusion $V = \tilde{H}_*(F) \to \tilde{H}_*(\Omega \Sigma F)$. The Pontrjagin ring $H_*(\Omega \Sigma F)$ is the tensor algebra $T(V)$ on V and has the structure of a Hopf algebra (with antipode), in which the co-multiplication is determined by the map $\Delta : V \to V \otimes V$ induced by the diagonal inclusion $F \to F \wedge F$. We filter $H_*(\Omega\Sigma F) = T(V)$ by the ideals $I^k = \bigoplus_{l \geqslant k} \bigotimes^l V$. The antipode preserves the filtration and acts as $(-1)^k$ on I^k/I^{k+1}.

The homology of $\Gamma^i F$ is determined by the equivalence (2.1), $\Gamma^0 F \sqcup \Gamma^1 F \to \Omega\Sigma F \ltimes \mathbb{Z}/2$, which we use to identify $H_*(\Gamma^i F)$ with $T(V)$. By comparing $\alpha = \tilde{\gamma}_0$ with the James map, we see that $(\tilde{\gamma}_0)_* : H_*(F) \to H_*(\Gamma^1 F)$ is just the inclusion $\mathbb{F}_p \oplus V \to T(V)$, and its composition with the antipode $T(V) \to T(V)$ is given by $1 \oplus (-1) : \mathbb{F}_p \oplus V \to \mathbb{F}_p \oplus V$ modulo I^2. The effect of the maps $\tilde{\gamma}_k$, for $k \geqslant 1$, in homology is then prescribed by the Pontrjagin multiplication.

We conclude that γ_k maps $\tilde{H}_*(G_k(F))$ into I^k and induces an isomorphism

$$\tilde{H}_*(G_k(F)) \xrightarrow{\cong} I^k/I^{k+2}.$$

Thus, if we filter the homology of $\bigvee_{k \text{ even}} G^k(F)$ by the subspaces $\bigoplus_{k \geqslant 2l} \tilde{H}_*(G^k(F))$ and $H_*(\Gamma^1 F)$ by the subspaces I^{2l}, $\bigvee \gamma_k$ preserves the filtration and induces an isomorphism on the associated graded modules. Hence $\bigvee \gamma_k$ is a homology isomorphism. In the same way, we see that $\iota \vee \bigvee \gamma_k$ is a homology isomorphism, and this completes the proof of the lemma. □

As it stands, lemma 2.5 is weaker than the James–Milnor theorem, which by a choice of basepoint in F gives a finer decomposition. But it extends, with little more than notational changes, to the fibrewise theory. Consider a fibre bundle $M \to B$ over a finite complex B with fibre a connected closed (smooth) manifold. The constructions above may be carried through in the fibres to define a pointed fibre bundle $G^k_B(M)$ over B, with fibre at $b \in B$ the space $G^k(M_b)$ (where M_b is the fibre of M at b), for $k \geqslant 0$. Thus $G^0_B(M)$ is the bundle M_{+B} obtained by adjoining a disjoint basepoint to each fibre of M. Fibrewise mapping spaces $\Gamma^i_B M$ are produced by applying the Γ^i constructions to fibres; the topology is prescribed by requiring that the fibrewise spaces be locally trivial over B. Then we have fibrewise stable maps

$$\gamma_k : G^k_B(M) \to \begin{cases} (\Gamma^0_B M)_{+B} & \text{for } k \geqslant 1 \text{ odd}, \\ (\Gamma^1_B M)_{+B} & \text{for } k \geqslant 0 \text{ even}. \end{cases}$$

Proposition 2.6. *There are fibrewise stable equivalences*

$$\iota \vee \bigvee \gamma_k : (B \times S^0) \vee \bigvee_{k \geqslant 1 \,\mathrm{odd}}{}_B\, G^k_B(M) \to (\Gamma^0_B M)_{+B},$$

$$\bigvee \gamma_k : \bigvee_{k \geqslant 0 \,\mathrm{even}}{}_B\, G^k_B(M) \to (\Gamma^1_B M)_{+B}$$

over B.

Proof. This is immediate from lemma 2.5 and Dold's theorem. □

By taking quotients by the subspace B of basepoints in the fibres we pass from fibrewise pointed spaces to pointed spaces and obtain the following immediate corollary.

COROLLARY 2.7. *There are stable equivalences*

$$B_+ \vee \bigvee_{k\geqslant 1\,\mathrm{odd}} G_B^k(M)/B \to (\Gamma_B^0 M)_+,$$

$$\bigvee_{k\geqslant 0\,\mathrm{even}} G_B^k(M)/B \to (\Gamma_B^1 M)_+.$$

3. Real projective bundles

We suppose, to begin with, that W is a finite-dimensional real vector space (with Euclidean inner product) of dimension $n > 1$. Consider the double cover $S(\mathbb{R}\oplus W) \to P(\mathbb{R}\oplus W)$ from the unit sphere with basepoint $(1,0)$ to the n-dimensional real projective space with basepoint $[1,0]$. The loop space $\Omega P(\mathbb{R}\oplus W)$ has two components, which we label by subscripts in the fundamental group $\mathbb{Z}/2$. Loops in the 0-component lift to loops in the sphere, and loops in the 1-component lift to paths from the North Pole $(1,0)$ to the South Pole $(-1,0)$. Writing the sphere as the join $S^0 * S(W)$ we can thus make the identifications

$$\Omega_i P(\mathbb{R}\oplus W) = \Gamma^i S(W) \qquad (i \in \mathbb{Z}/2). \tag{3.1}$$

Under this correspondence, the basic map $\alpha : F \to \Gamma^1 F$ appears geometrically as

$$\alpha : S(W) \to \Omega_1 P(\mathbb{R}\oplus W), \qquad x \mapsto \alpha_x,$$

where

$$\alpha_x(t) = [\cos(\pi t), \sin(\pi t)x], \qquad 0 \leqslant t \leqslant 1.$$

The involution τ on $S^0 * S(W)$, given by $(-1,1) : S(\mathbb{R}\oplus W) \to S(\mathbb{R}\oplus W)$, lifts the map $(1,-1) : P(\mathbb{R}\oplus W) \to P(\mathbb{R}\oplus W)$ induced by the antipodal involution -1 on W. The map $\tilde{\gamma}_k$ thus transforms into the map

$$\tilde{\gamma}_k : S(W) \times S(W)^k \to \Omega P(\mathbb{R}\oplus W)$$

given by loop multiplication with alternating signs,

$$\tilde{\gamma}_k(x, (y_1, \ldots, y_k)) = \alpha_x \cdot (\alpha_{-y_1} \cdot (\alpha_{y_2}(\cdots (\alpha_{(-1)^{k-1}y_{k-1}} \cdot \alpha_{(-1)^k y_k})) \cdots)). \tag{3.2}$$

We shall need the following elementary observation.

LEMMA 3.1. *The map* $S(W) \to \Omega P(\mathbb{R}\oplus W)$ *taking* $x \in S(W)$ *to the product* $\alpha_x \cdot \alpha_{-x}$ *is homotopic to the constant map to the basepoint* $[1,0]$.

The spaces $G^k(S(W))$ admit the following interpretation. The tangent space η_x at a point $x \in S(W)$ consists of the vectors in W orthogonal to the unit vector x, and W can be written as the orthogonal direct sum $\mathbb{R}x \oplus \eta_x = W$. By stereographic projection the sphere $S(W)$ with basepoint x can be identified with the one-point compactification η_x^+ of η_x with basepoint ∞. In this way we can regard the fibrewise pointed space $X = F \times F \to F$ of § 2, when $F = S(W)$, as the fibrewise one-point compactification $\eta_{S(W)}^+$ of the tangent bundle η to the sphere. Then $\bigwedge_F^k X$ is the fibrewise one-point compactification $(\eta \oplus \cdots \oplus \eta)^+_{S(W)}$ of the k-fold direct sum, and hence

$$G^k(S(W)) = (\textstyle\bigwedge_F^k X)/F$$

is the Thom space $S(W)^{k\eta}$.

Now let ξ be a real vector bundle of dimension $n > 1$ over a finite complex B, with sphere-bundle $S(\xi)$. We write $P(\mathbb{R} \oplus \xi)$ for the projective bundle of the direct sum of a trivial line bundle $B \times \mathbb{R}$ and ξ; it is a pointed fibre bundle with basepoint $[1, 0]$ in each fibre. The sphere-bundle $S(\mathbb{R} \oplus \xi)$ is homeomorphic to the fibrewise join $(B \times S^0) *_B S(\xi)$. The fibrewise loop space $\Omega_B P(\mathbb{R} \oplus \xi)$, having the fibre $\Omega P(\mathbb{R} \oplus \xi_b)$ at $b \in B$ and topologized so as to be locally trivial, is the disjoint union

$$\begin{aligned} \Omega_B P(\mathbb{R} \oplus \xi) &= \Omega_{0B} P(\mathbb{R} \oplus \xi) \sqcup \Omega_{1B} P(\mathbb{R} \oplus \xi) \\ &= \Gamma_B^0 S(\xi) \sqcup \Gamma_B^1 S(\xi) \end{aligned} \tag{3.3}$$

of two bundles with connected fibres $\Omega_i P(\mathbb{R} \oplus \xi_b)$. The pullback of ξ to its sphere-bundle $S(\xi)$ splits as the direct sum of the trivial bundle $\mathbb{R}$ and the bundle η of tangents along the fibres. The pointed fibre bundle $G_B^k(S(\xi))$ can then be described as the fibrewise Thom space $S(\xi)_B^{k\eta}$, which is a bundle over B with fibre at b the Thom space of $k\eta_b$ over the sphere $S(\xi_b)$. Proposition 2.6 and corollary 2.7 specialize as follows.

Proposition 3.2. *There are fibrewise stable decompositions,*

$$(\Omega_{0B} P(\mathbb{R} \oplus \xi))_{+B} \simeq (B \times S^0) \vee_B \bigvee_{B\, k \,\mathrm{odd}} S(\xi)_B^{k\eta},$$

$$(\Omega_{1B} P(\mathbb{R} \oplus \xi))_{+B} \simeq \bigvee_{B\, k \,\mathrm{even}} S(\xi)_B^{k\eta},$$

over B.

Corollary 3.3. *There are stable decompositions*

$$(\Omega_{0B} P(\mathbb{R} \oplus \xi))_+ \simeq B_+ \vee \bigvee_{k \geqslant 1 \,\mathrm{odd}} S(\xi)^{k\eta},$$

$$(\Omega_{1B} P(\mathbb{R} \oplus \xi))_+ \simeq \bigvee_{k \geqslant 0 \,\mathrm{even}} S(\xi)^{k\eta}.$$

Remark 3.4. The components in the fibrewise decomposition are stably indecomposable if (and only if) the stable cohomotopy Euler class of ξ is non-zero. For the component $S(\xi)_B^{k\eta}$ is a fibrewise suspension of $S(\xi)_{+B}$, since $\mathbb{R} \oplus \eta = \xi$, and we have a fibrewise cofibre sequence

$$S(\xi)_{+B} \to B \times S^0 \to \xi_B^+,$$

in which the second map is the Euler class (see, for example, [4, (II.4)]). Suppose that we have a stable splitting $S(\xi)_{+B} \to Y \vee_B Z$, where Y and Z are locally homotopy trivial fibrewise pointed spaces over B ('pointed homotopy fibre bundles' in the terminology of [4]). We may assume that $\tilde{H}^*(Y_b; \mathbb{Z}) = \tilde{H}^*(S^0; \mathbb{Z})$, for each $b \in$

B. Then the inclusion of Y followed by the map to $B \times S^0$ must be an equivalence. This splits the sequence, and shows that the stable cohomotopy Euler class is zero.

We now turn to the computation of the cohomology of $\Omega_B P(\mathbb{R} \oplus \xi)$, with $\mathbb{Z}[\frac{1}{2}]$-coefficients, when ξ is of odd dimension $n = 2m + 1$ $(m \geqslant 1)$ and oriented, in terms of the splitting given in corollary 3.3. For the remainder of this section an unembellished 'H' denotes $\mathbb{Z}[\frac{1}{2}]$-cohomology.

LEMMA 3.5. *Under the hypotheses described above, the cohomology ring* $H^*(S(\xi))$ *is a free* $H^*(B)$*-module on the basis* $1, e(\eta)$*. The Euler class* $e(\eta)$ *of the fibrewise tangent bundle has square* $e(\eta)^2 = p_m(\xi)$*.*

Proof. By the Leray–Hirsch lemma, to prove that $1, e(\eta)$ is a basis, we can reduce to the elementary case that B is a point. For the $2m$-dimensional oriented bundle η, we have $e(\eta)^2 = p_m(\eta) = p_m(\xi) \cdot 1$, since $\mathbb{R} \oplus \eta$ is the pullback of ξ. □

We shall abbreviate the mth Pontrjagin class $p_m(\xi)$ to simply 'p'.

The stable splitting in corollary 3.3 gives the algebraic direct sum decomposition

$$
\begin{aligned}
H^*(\Omega_{0B} P(\mathbb{R} \oplus \xi)) &= H^*(B) \oplus \bigoplus_{k \geqslant 1 \,\mathrm{odd}} \tilde{H}^*(S(\xi)^{k\eta}), \\
H^*(\Omega_{1B} P(\mathbb{R} \oplus \xi)) &= \bigoplus_{k \geqslant 0 \,\mathrm{even}} \tilde{H}^*(S(\xi)^{k\eta}).
\end{aligned}
$$

This allows us to specify bases, as free $H^*(B)$-modules, a_i, $i \geqslant 0$, for the cohomology of $\Omega_{0B} P(\mathbb{R} \oplus \xi)$, and b_i, $i \geqslant 0$, for the cohomology of $\Omega_{1B} P(\mathbb{R} \oplus \xi)$ as follows. The class a_0 is the generator 1 of the first summand $H^*(B)$. For $k \geqslant 0$, $(-1)^l a_k$, $(-1)^l a_{k+1}$ if $k = 2l - 1$ is odd, and $(-1)^l b_k$, $(-1)^l b_{k+1}$ if $k = 2l$ is even, are the generators of $\tilde{H}^*(S(\xi)^{k\eta})$ corresponding to $1, e(\eta) \in H^*(S(\xi))$ under the Thom isomorphism for the oriented bundle $k\eta$. (The reason for the choice of signs will emerge in the proof of proposition 3.8.) The indexing is chosen so that a_i, b_i, for $i \geqslant 0$, are generators in dimension $2mi$.

PROPOSITION 3.6. *Let* ξ *be an oriented real vector bundle of odd dimension* $2m + 1$*, where* $m \geqslant 1$*, over a finite complex* B*. Then the* $H^*(B; \mathbb{Z}[\frac{1}{2}])$*-algebras* $H^*(\Omega_{0B} P(\mathbb{R} \oplus \xi); \mathbb{Z}[\frac{1}{2}])$ *and* $H^*(\Omega_{1B} P(\mathbb{R} \oplus \xi); \mathbb{Z}[\frac{1}{2}])$ *are freely generated as modules over* $H^*(B; \mathbb{Z}[\frac{1}{2}])$ *by the classes* a_i *and* b_i*, respectively, described above. The ring structure is determined in terms of* $a = a_1$ *and* $b = b_1$ *by the formulae*

$$
\begin{aligned}
(2j)!a_{2j} &= (a^2 - 0^2p)(a^2 - 2^2p)(a^2 - 4^2p) \cdots (a^2 - (2(j-1))^2p), \\
(2j+1)!a_{2j+1} &= a(a^2 - 2^2p)(a^2 - 4^2p)(a^2 - 6^2p) \cdots (a^2 - (2j)^2p), \\
(2j)!b_{2j} &= (b^2 - 1^2p)(b^2 - 3^2p)(b^2 - 5^2p) \cdots (b^2 - (2j-1)^2p), \\
(2j+1)!b_{2j+1} &= b(b^2 - 1^2p)(b^2 - 3^2p)(b^2 - 5^2p) \cdots (b^2 - (2j-1)^2p),
\end{aligned}
$$

modulo torsion (*and* $a_0 = 1$, $b_0 = 1$)*, where* $p = p_m(\xi)$*.*

Since $H^*(B\mathrm{SO}(2m); \mathbb{Z}[\frac{1}{2}])$ is torsion-free, the given relations completely determine the ring structure. The proof of proposition 3.6 will occupy the rest of the section.

We shall perform the calculation using fibrewise homology theory as described in [4, (II.15)] or [3]. To fix notation, suppose that X and Y are pointed fibre bundles over B with fibres of the homotopy type of CW-complexes, finite in the case of X, arbitrary in the case of Y. The *fibrewise cohomology groups* $H^i_B\{X;Y\}$, for $i \in \mathbb{Z}$, are defined as direct limits of sets of fibrewise pointed homotopy classes over B,

$$H^i_B\{X;Y\} = \varinjlim_j [\Sigma^j_B X; (B \times K_{j+i}) \wedge_B Y]_B,$$

where Σ_B denotes the fibrewise suspension and K_j is the Eilenberg–MacLane space $K(\mathbb{Z}[\frac{1}{2}], j)$. The graded group $H^*_B\{X;Y\}$ is a module over the cohomology ring $H^*(B)$ of the base. It is easy to identify $H^i_B\{X; B \times S^0\}$ with the ordinary (reduced) cohomology group $\tilde{H}^i(X/B)$. We refer to the less familiar group $H^i_B\{B \times S^0; Y\}$ as the *fibrewise homology* of Y; when B is a point, it is the usual homology $\tilde{H}_{-i}(Y)$ of the pointed space Y, but with a negative index. In favourable special cases a Leray–Hirsch argument can be used to identify the cohomology group $\tilde{H}^*(Y/B)$ with the $H^*(B)$-dual

$$\mathrm{Hom}_{H^*(B)}(H^*_B\{B \times S^0; Y\}, H^*(B))$$

of the fibrewise homology of Y over B.

This is the case in the present situation. The fibrewise homology group of $(\Omega_B P(\mathbb{R} \oplus \xi))_{+B}$ over B is a free graded $H^*(B)$-module, and the cohomology group $H^*(\Omega_B P(\mathbb{R} \oplus \xi))$ can be computed as its dual over $H^*(B)$. For the details of similar calculations the reader is referred to [4, (II.15.28)] and [3]. Let $t, u \in H^*_B\{B \times S^0; S(\xi)_{+B}\}$ be the $H^*(B)$-basis dual to the basis $1, e(\eta)$ of $H^*(S(\xi))$. Thus t has dimension 0 and u has negative dimension $-2m$. We use the same symbols t and u for the images of those classes under $\alpha = \tilde{\gamma}_0$ in the fibrewise homology of $\Omega_B P(\mathbb{R} \oplus \xi)$. The Pontrjagin multiplication given by loop multiplication μ and the co-multiplication given by the diagonal Δ

$$\Omega_B P(\mathbb{R} \oplus \xi) \xrightarrow{\Delta} \Omega_B P(\mathbb{R} \oplus \xi) \times_B \Omega_B P(\mathbb{R} \oplus \xi) \xrightarrow{\mu} \Omega_B P(\mathbb{R} \oplus \xi)$$

make $H^*_B\{B \times S^0; (\Omega_B P(\mathbb{R} \oplus \xi))_{+B}\}$ a Hopf algebra over $H^*(B)$.

Proposition 3.7. *The fibrewise homology group $H^*_B\{B \times S^0; (\Omega_B P(\mathbb{R} \oplus \xi))_{+B}\}$ is free over $H^*(B)$ on the basis u^i, tu^i $(i \geqslant 0)$. It has the structure of a Hopf algebra over $H^*(B)$ with co-multiplication and multiplication given by*

$$\Delta(t) = t \otimes t + pu \otimes u, \qquad \Delta(u) = u \otimes t + t \otimes u$$

and

$$t^2 = 1 + pu^2, \qquad tu = ut.$$

Proof. The verification that the $H^*(B)$-module is free on the given basis is reduced by the Leray–Hirsch lemma to a calculation on fibres which is just the classical description of the Pontrjagin ring of $\Omega \Sigma S^{2m}$.

We note that the generators t and u lie in the homology of the 1-component $\Omega_{1B} P(\mathbb{R} \oplus \xi)$. Thus $1, tu, u^2, tu^3, \ldots$ and $t, u, tu^2, u^3, \ldots$ are bases of the fibrewise homology of the 0-component and 1-component, respectively.

Since $\alpha = \tilde{\gamma}_0 : S(\xi) \to \Omega_B P(\mathbb{R} \oplus \xi)$ commutes with the diagonal maps, the comultiplication Δ is computed by the diagonal on $S(\xi)$ as the dual of the ring multiplication given by lemma 3.5.

To determine the multiplication we use the obvious fibrewise generalization of lemma 3.1. Let σ denote the antipodal involution -1 on ξ and also the maps that it induces on $S(\xi)$ and $P(\mathbb{R} \oplus \xi)$. Since the involution reverses the orientation of ξ, $\sigma(e(\eta)) = -e(\eta)$, and hence $\sigma(t) = t$ and $\sigma(u) = -u$. According to lemma 3.1, the composition $\mu \circ (1 \times \sigma) \circ \Delta \circ \alpha : S(\xi) \to \Omega_B P(\mathbb{R} \oplus \xi)$ is (homotopic to) the constant map to the basepoint in each fibre. Hence $\mu(1 \otimes \sigma)\Delta(t) = 1$, so that $t^2 - pu^2 = 1$, and $\mu(1 \otimes \sigma)\Delta(u) = 0$, so that $-tu + ut = 0$. $\square$

Next we relate the homology bases just defined to the cohomology bases a_i and b_i appearing in proposition 3.6.

PROPOSITION 3.8. *The cohomology ring $H^*(\Omega_B P(\mathbb{R} \oplus \xi))$ admits the structure of a Hopf algebra over $H^*(B)$ dual to the Hopf algebra $H^*_B\{B \times S^0; (\Omega_B P(\mathbb{R} \oplus \xi))_{+B}\}$ described in proposition 3.7. The bases $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$ of the* 0 *and* 1*-components are dual to the homology bases $1, tu, u^2, \dots$ and $t, u, tu^2, \dots$.*

Proof. For fixed $k \geqslant 0$, let us introduce the temporary abbreviations

$$L_k = H^*_B\{B \times S^0; S(\xi)^{k\eta}_B\},$$
$$M_k = H^*_B\{B \times S^0; (S(\xi) \times_B (S(\xi) \times_B \cdots \times_B S(\xi)))_{+B}\} \quad (1 + k \text{ factors } S(\xi)).$$

The $H^*(B)$-module L_k is free on classes x, y corresponding under the Thom isomorphism for $k\eta$ to t, u, while M_k, by the Künneth theorem, is the $(1 + k)$-fold tensor product $(H^*(B)t \oplus H^*(B)u) \otimes \bigotimes^k (H^*(B)t \oplus H^*(B)u)$ (over $H^*(B)$).

From the description (3.2) of $\tilde{\gamma}_k$, we see that the induced map

$$(\tilde{\gamma}_k)_* : M_k \to H^*_B\{B \times S^0; (\Omega_B P(\mathbb{R} \oplus \xi))_{+B}\}$$

is given by $1 \otimes (\sigma \otimes 1 \otimes \sigma \dots)$ followed by Pontrjagin multiplication.

We claim that the inclusion $i : L_k \to M_k$ given by the stable splitting (2.2) maps x to $t \otimes (u \otimes \cdots \otimes u)$ and y to $u \otimes (u \otimes \cdots \otimes u)$. Indeed, consideration of products involving the map θ, as in the proof of lemma 2.5, shows that $i(L_k)$ is contained in $(H^*(B)t \oplus H^*(B)u) \otimes (u \otimes \cdots \otimes u)$. Since $\sigma(x) = (-1)^k x$ and $\sigma(y) = (-1)^{k+1} y$, $i(x)$ and $i(y)$ must be as claimed, up to multiplication by elements of $H^0(B)$. It therefore suffices to check the assertion in the easy case that B is a point.

Finally, we see that the composition

$$L_k \to M_k \to H^*_B\{B \times S^0; (\Omega_B P(\mathbb{R} \oplus \xi))_{+B}\}$$

maps x to $(-1)^l tu^k$ and y to $(-1)^l u^{k+1}$ if $k = 2l - 1$ or $2l$, so that tu^k, u^{k+1} are dual to a_k, a_{k+1} if k is odd, and to b_k, b_{k+1} if k is even. $\square$

It is now a purely algebraic exercise to complete the proof of proposition 3.6. We pass to rational homology and make the change of variable from (t, u) to (s, v) where

$$t = s\cosh(\sqrt{p}v), \qquad \sqrt{p}u = \sinh(\sqrt{p}v),$$

so that $\cosh(\sqrt{p}v) = (1 + pu^2)^{1/2}$. An elementary calculation (using the addition formulae for sinh and cosh) shows that

$$s^2 = 1, \qquad \Delta s = s \otimes s, \qquad \Delta v = s \otimes v + v \otimes s.$$

The dual Hopf algebra, for each component, is just the familiar divided polynomial algebra. Writing $a = a_1$, $b = b_1$, we have dual bases $1, sv, v^2, sv^3, \ldots$ and $1, a, a^2/2!, \ldots$ for the homology and cohomology of the 0-component, and $s, v, sv^2, v^3, \ldots$ and $1, b, b^2/2!, \ldots$ for the 1-component.

To relate these bases to those appearing in proposition 3.8, let us introduce two power series

$$f(X) = \sum f_n(\lambda)X^n = \cosh(\lambda \sinh^{-1}(X))$$

and

$$g(X) = \sum g_n(\lambda)X^n = \sinh(\lambda \sinh^{-1}(X))$$

with coefficients in the polynomial ring $\mathbb{Q}[\lambda]$. Their derivatives are

$$f'(X) = \lambda(1 + X^2)^{-1/2}g(X), \qquad g'(X) = \lambda(1 + X^2)^{-1/2}f(X),$$

and $f(X)$ and $g(X)$ satisfy the differential equations

$$\begin{aligned}(1 + X^2)f''(X) + Xf'(X) - \lambda^2 f(X) = 0, \quad f(0) = 1, \quad f'(0) = 0,\\ (1 + X^2)g''(X) + Xg'(X) - \lambda^2 g(X) = 0, \quad g(0) = 0, \quad g'(0) = \lambda,\end{aligned}$$

from which follow the recurrence relations

$$\begin{aligned}(n + 2)(n + 1)f_{n+2}(\lambda) = (\lambda^2 - n^2)f_n(\lambda), \quad f_0(\lambda) = 1, \quad f_1(\lambda) = 0,\\ (n + 2)(n + 1)g_{n+2}(\lambda) = (\lambda^2 - n^2)g_n(\lambda), \quad g_0(\lambda) = 0, \quad g_1(\lambda) = \lambda.\end{aligned}$$

We obtain the explicit expansions

$$\begin{aligned}f(X) &= 1 + \frac{\lambda^2}{2!}X^2 + \frac{\lambda^2(\lambda^2 - 2^2)}{4!}X^4 + \cdots,\\ g(X) &= \lambda X + \frac{\lambda(\lambda^2 - 1^2)}{3!}X^3 + \frac{\lambda(\lambda^2 - 1^2)(\lambda^2 - 3^2)}{5!}X^5 + \cdots,\\ (1 + X^2)^{-1/2}g(X) &= \lambda X + \frac{\lambda(\lambda^2 - 2^2)}{3!}X^3 + \frac{\lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2)}{5!}X^5 + \cdots,\\ (1 + X^2)^{-1/2}f(X) &= 1 + \frac{(\lambda^2 - 1^2)}{2!}X^2 + \frac{(\lambda^2 - 1^2)(\lambda^2 - 3^2)}{4!}X^4 + \cdots.\end{aligned}$$

REMARK 3.9. The series for $\sinh(\lambda \sinh^{-1}(X))$ is well known. Indeed, substituting an odd integer $2r+1$ for λ we recover the formula, familiar from school mathematics, expressing $\sin((2r + 1)\theta)$ as a polynomial in $\sin(\theta)$.

We shall calculate in the formal power series ring $H^*(\Omega_B P(\mathbb{R}\oplus\xi); \mathbb{Q})[[X]]$. Using $\langle a, x\rangle \in H^*(B)$ for the evaluation of a fibrewise cohomology class a on a fibrewise homology class x, we have

$$\begin{aligned}
\sum_{i\text{ even}} a_i X^i &= \sum_{i,j\text{ even}} \langle a_i, v^j\rangle \frac{a^j}{j!} X^i \\
&= \sum_{j\text{ even}} \frac{1}{j!} (a p^{-1/2} \sinh^{-1}(\sqrt{p}X))^j \\
&= \cosh(a p^{-1/2} \sinh^{-1}(\sqrt{p}X)) \\
&= f(\sqrt{p}X),
\end{aligned}$$

where $\lambda = ap^{-1/2}$. The second equality follows from the fact that $\langle a_i, v^j\rangle$ is the coefficient of u^i in $(p^{-1/2}\sinh^{-1}(\sqrt{p}u))^j$. Similarly,

$$\begin{aligned}
\sum_{i\text{ odd}} a_i X^i &= \sum_{i,j\text{ odd}} \langle a_i, sv^j\rangle \frac{a^j}{j!} X^i \\
&= \sum_{j\text{ odd}} \frac{a^j}{j!} (1+pX^2)^{-1/2} \left(\frac{\sinh^{-1}(\sqrt{p}X)}{\sqrt{p}}\right)^j \\
&= (1+pX^2)^{-1/2} \sinh(a p^{-1/2} \sinh^{-1}(\sqrt{p}X)) \\
&= \lambda^{-1} f'(\sqrt{p}X),
\end{aligned}$$

because $\langle a_i, sv^j\rangle$ is the coefficient of tu^i in $(1+pu^2)^{-1/2} t(\sinh^{-1}(\sqrt{p}u)/\sqrt{p})^j$.

In the same way, with $\lambda = bp^{-1/2}$, we have

$$\sum_{i\text{ even}} b_i X^i = \lambda^{-1} g'(\sqrt{p}X), \qquad \sum_{i\text{ odd}} b_i X^i = g(\sqrt{p}X).$$

This completes the proof of proposition 3.6. □

4. The application to gauge groups

Recall that SO(3) is topologically a 3-dimensional real projective space. More precisely, if W is an oriented 3-dimensional Euclidean vector space, there is a natural identification

$$P(\mathbb{R} \oplus W) \to \mathrm{SO}(W)$$

from the real projective space to the special orthogonal group of W in which $[1,0]$ maps to the identity $1 \in \mathrm{SO}(W)$. Our results about stable homotopy type of the classifying spaces of gauge groups will be obtained by specializing the theory of § 3 to the case $n = 3$ of a 3-dimensional bundle ξ.

Let G be a compact connected Lie group. The components of the loop space ΩG and of the mapping space $\mathrm{map}(S^2, BG)$ are indexed by the fundamental group of G and labelled by a subscript $\alpha \in \pi_1(G)$.

Lemma 4.1. *Let $P \to S^2$ be a principal G-bundle, and let $\mathcal{G}_P$ be the group of sections of the bundle of groups $P \times_G G \to S^2$ associated to the adjoint action of G on itself. Then the classifying space $B\mathcal{G}_P$ can be realized as*

$$B\mathcal{G}_P \simeq \mathrm{map}_\alpha(S^2, BG) \simeq EG \times_G \Omega_\alpha G,$$

where $\alpha \in \pi_1(G)$ classifies the bundle P and the action of G on ΩG is by conjugation.

This description of the classifying space of the gauge group $\mathcal{G}_P$ is well known (see [1] for the first homotopy equivalence and, for example, [5] for the second). We present a new proof of the second equivalence in § 5.

The following cases can be handled by our methods.

(i) $G = \mathrm{SO}(3)$, $\alpha \in \mathbb{Z}/2$.

(ii) $G = \mathrm{SU}(2) = \mathrm{Spin}(3)$.

(iii) $G = \mathrm{U}(2)$, $\alpha \in \mathbb{Z}$.

(iv) $G = \mathrm{SO}(3) \times \mathrm{SO}(3)$, $\alpha \in \mathbb{Z}/2 \times \mathbb{Z}/2$.

(v) $G = \mathrm{SO}(4)$, $\alpha \in \mathbb{Z}/2$.

(vi) $G = \mathrm{Spin}(4)$.

We shall concentrate on the cases (i), (iii) and (v).

Proof of proposition 1.1. Consider an oriented 3-dimensional real vector bundle ξ, with Euclidean inner product, over a finite complex B. The bundle of groups $\mathrm{SO}(\xi)$, with fibre at $b \in B$ the special orthogonal group $\mathrm{SO}(\xi_b)$ of the fibre ξ_b, is naturally identified with the projective bundle $P(\mathbb{R}\oplus\xi)$. Thus proposition 3.2 gives a fibrewise stable decomposition of the fibrewise loop space $\Omega_B\mathrm{SO}(\xi)$ over B, and corollary 3.3 gives a stable decomposition of the space $\Omega_B\mathrm{SO}(\xi)$.

Applying this result to the universal vector bundle ξ of dimension 3 over $B\mathrm{SO}(3)$ (or, more precisely, to its restriction to finite skeleta) and noting that the sphere-bundle $S(\xi)$ is $B\mathrm{SO}(2) = B\mathrm{U}(1)$, we obtain the splitting in proposition 1.1. □

Proposition 4.2. *For $G = \mathrm{U}(2)$, $\alpha \in \mathbb{Z}$, there are stable splittings*

$$(EG \times_G \Omega_\alpha G)_+ \simeq \begin{cases} B\mathrm{U}(2)_+ \vee \displaystyle\bigvee_{k\geqslant 1\,\mathrm{odd}} (B\mathrm{U}(1)^{kH} \wedge B\mathrm{U}(1)_+) & \text{for } \alpha \text{ even},\\ \displaystyle\bigvee_{k\geqslant 0\,\mathrm{even}} (B\mathrm{U}(1)^{kH} \wedge B\mathrm{U}(1)_+) & \text{for } \alpha \text{ odd}.\end{cases}$$

Outline proof. The quotient map $\mathrm{U}(2) \to \mathrm{SO}(3)$ that factors out the centre of $\mathrm{U}(2)$ gives homotopy equivalences $\Omega_\alpha\mathrm{U}(2) \simeq \Omega_{\alpha(\mathrm{mod}\,2)}\mathrm{SO}(3)$. Associated to a complex vector bundle ζ of dimension 2, with Hermitian inner product, there is the 3-dimensional real vector bundle ξ, which is the space of skew-Hermitian endomorphisms of ζ. Taking ζ to be the universal bundle over $B\mathrm{U}(2)$, we can identify $S(\xi)$ with $B(\mathrm{U}(1) \times \mathrm{U}(1))$ and η with the Hopf bundle over the first factor. □

In order to state the next result, it is convenient to introduce some abbreviations. For $k \geqslant 0$ and $k_1, k_2 \geqslant 0$, we write

$$X(k) = B\mathrm{U}(2)^{kL}, \qquad Y(k_1,k_2) = (B\mathrm{U}(1) \times B\mathrm{U}(1))^{k_1(H_1\otimes H_2)\oplus k_2(H_1\otimes H_2^*)},$$

where L is the determinant bundle of the canonical two-dimensional complex vector bundle over $B\mathrm{U}(2)$ and H_1 and H_2 are the Hopf line bundles over the first and second factors of $B\mathrm{U}(1) \times B\mathrm{U}(1)$.

PROPOSITION 4.3. *For* $G = \mathrm{SO}(4)$, $\alpha \in \mathbb{Z}/2$, *the space* $(EG \times_G \Omega_\alpha G)_+$ *splits stably as*

$$\begin{aligned} &B\mathrm{SO}(4)_+ \vee \bigvee_{k\,\mathrm{odd}} (X(k) \vee X(k)) \vee \bigvee_{k_1,k_2\,\mathrm{odd}} Y(k_1,k_2) && \text{for } \alpha = 0, \\ &\bigvee_{k_1,k_2\,\mathrm{even}} Y(k_1,k_2) && \text{for } \alpha = 1, \end{aligned}$$

where the summands $X(k)$ *for* $k \geqslant 0$ *and* $Y(k_1,k_2)$ *for* $k_1, k_2 \geqslant 0$ *are as defined in the text above.*

Outline proof. This time we have an isomorphism $\mathrm{SO}(4)/\{\pm 1\} \to \mathrm{SO}(3) \times \mathrm{SO}(3)$ giving homotopy equivalences $\Omega_\alpha \mathrm{SO}(4) \to \Omega_\alpha \mathrm{SO}(3) \times \Omega_\alpha \mathrm{SO}(3)$. The homogeneous spaces $\mathrm{SO}(4)/(\mathrm{SO}(2) \times \mathrm{SO}(2))$ and $\mathrm{SO}(4)/\mathrm{U}(2)$ are $S^2 \times S^2$ and S^2, respectively. $\square$

Of course, the space of pointed maps $\mathrm{map}^*(S^2, BG) = \Omega G$ admits a finer stable splitting in each case, provided by the stable splitting of ΩS^3 as a wedge of even-dimensional spheres.

5. The fibrewise classifying space

Let G be a compact Lie group. Then G may be embedded as a closed subgroup of a contractible topological group E. For example, one may take $G \to E$ to be a faithful unitary representation $G \to U(H)$ of G on an infinite-dimensional separable Hilbert space H. Alternatively, one may use the general construction, due to Milgram and Steenrod and valid for any topological group G, which is described in [7]. The group G acts freely on the contractible space E, by right multiplication. We assume, as is the case for the two constructions mentioned, that the projection $E \to E/G$ is a principal G-bundle with base an ANR. Then the homogeneous space E/G, with basepoint the coset G/G of the identity, is a convenient model for the classifying space BG of G. The group G acts on itself and on E by conjugation and on E/G by left multiplication, preserving the basepoint.

The next proposition makes precise the informal statement that inner automorphisms of the group G act homotopically trivially on the (pointed) classifying space BG. For background and further details, see [5, § 3].

PROPOSITION 5.1. *There is a natural pointed fibre homotopy equivalence*

$$EG \times_G BG \to BG \times BG$$

from the fibrewise classifying space of the bundle of groups $EG \times_G G \to BG$ *to the trivial bundle* $BG \times BG \to BG$ *pointed by the diagonal map.*

Proof. Using the model E/G of the classifying space, we can write down an explicit trivialization. The space $E \times_G E/G$ is a quotient of $E \times E$. We map the class $[x, y]$, where $x, y \in E$, to $([x], [xy]) \in E/G \times E/G$. $\square$

The following corollary, in which map^* denotes the space of based maps, is obtained in [5, corollary 3.4] by applying the fibrewise pointed mapping space functor $\mathrm{map}^*_{BG}(BG \times F, -)$ to the equivalence in proposition 5.1.

Corollary 5.2. *Let Z be a finite pointed complex. Then there is a natural pointed fibre homotopy equivalence*

$$EG \times_G \operatorname{map}^*(Z, BG) \simeq \operatorname{map}(Z, BG)$$

over BG.

The second equivalence in lemma 4.1 follows from corollary 5.2 by taking $Z = S^2$ and identifying $\operatorname{map}^*(S^2, BG) = \Omega^2 BG$ with ΩG in the usual way. If we work with the model E/G for BG, a little care is necessary. Let Φ be the homotopy-fibre of the projection $E \to E/G$. Then we have G-equivariant maps $G \to \Phi$ and $\Omega(E/G) \to \Phi$ which are (non-equivariant) homotopy equivalences. These induce fibre homotopy equivalences

$$EG \times_G \Omega G \to EG \times_G \Omega\Phi \leftarrow EG \times_G \Omega^2(E/G).$$

6. Indecomposability

We shall show that the pointed spaces $B\mathrm{SO}(3)$ and $B\mathrm{U}(1)^{kH}$, for $k \geqslant 1$, are stably indecomposable at the prime 2. In this section 'H^*' will denote $\mathbb{F}_2$-cohomology and '$\mathcal{A}$' will be the mod 2 Steenrod algebra.

Let V be the subgroup of $\mathrm{SO}(3)$ consisting of the diagonal matrices; it is an elementary abelian 2-group, which we regard as an $\mathbb{F}_2$-vector space. Restriction gives an isomorphism from $H^*(B\mathrm{SO}(3))$ to the Dickson algebra of invariants in $H^*(BV)$ under the group $\mathrm{GL}(V)$ of automorphisms of V. If we write λ, μ, ν for the non-zero elements of the dual vector space V^*, then we have

$$H^*(BV)^{\mathrm{GL}(V)} = \mathbb{F}_2[\alpha, \beta], \quad \text{where } \alpha = \lambda\mu + \mu\nu + \nu\lambda,\ \beta = \lambda\mu\nu.$$

Since $H^*(BV)$ is injective in the category of unstable $\mathcal{A}$-modules, every $\mathcal{A}$-module endomorphism of $H^*(B\mathrm{SO}(3)) = H^*(BV)^{\mathrm{GL}(V)}$ extends to an $\mathcal{A}$-module endomorphism of $H^*(BV)$. Given the well-known description of the ring of $\mathcal{A}$-module endomorphisms of $H^*(BV)$ as $\mathbb{F}_2[\mathrm{End}(V^*)]$, it is an elementary exercise to deduce the structure of the endomorphism ring for $B\mathrm{SO}(3)$.

Lemma 6.1. *The ring of $\mathcal{A}$-module endomorphisms of $H^*(BV)^{\mathrm{GL}(V)} = \mathbb{F}_2[\alpha, \beta]$ is of dimension 3 over $\mathbb{F}_2$, generated by 1, e, n, with $e^2 = e$, $n^2 = 0$, $en = e = en$, where $e(1) = 1$, $e(\alpha^i\beta^j) = 0$ if $2i + 3j > 0$, $n(\alpha^i\beta^j) = 0$ if $j > 0$, $n(\alpha^i) = (\lambda^i + \mu^i + \nu^i)^2$.*

It follows that there are no idempotents in the ring of $\mathcal{A}$-module endomorphisms of $\tilde{H}^*(B\mathrm{SO}(3))$ other than 0 and 1. Hence $B\mathrm{SO}(3)$ is stably indecomposable at 2.

A similar argument can be used to show that the only $\mathcal{A}$-module endomorphisms of $\tilde{H}^*(B\mathrm{U}(1)^{kH})$, for $k \geqslant 1$, are 0 and 1. But this is easily seen directly, by computing the Steenrod squares in terms of binomial coefficients.

Acknowledgments

We are grateful to John Hubbuck and Lionel Schwartz for conversations on stable indecomposability. S.B. and M.S. were supported, respectively, by the EPSRC and an EC Marie Curie fellowship.

References

1 M. F. Atiyah and R. Bott. The Yang–Mills equations over Riemann surfaces. *Proc. R. Soc. Lond.* A **308** (1982), 523–615.

2 R. L. Cohen. Stable proofs of stable splittings. *Math. Proc. Camb. Phil. Soc.* **88** (1980), 149–151.

3 M. C. Crabb. Fibrewise homology. *Glasgow Math. J.* **43** (2001), 199–208.

4 M. C. Crabb and I. M. James. *Fibrewise homotopy theory* (Springer, 1998).

5 M. C. Crabb and W. A. Sutherland. Counting homotopy types of gauge groups. *Proc. Lond. Math. Soc.* **81** (2000), 747–768.

6 S. Tsukuda. On the cohomology of the classifying space of a certain gauge group. *Proc. R. Soc. Edinb.* A **127** (1997), 407–409.

7 R. A. Piccinini and M. Spreafico. The Milgram–Steenrod construction of classifying spaces for topological groups. *Expo. Math.* **16** (1998), 97–130.

(*Issued 17 August 2001*)