Number Theory

Dirichlet Series and Gamma Function Associated with Rational Functions

by

Mauro SPREAFICO

Presented by Jerzy KACZOROWSKI

Summary. We investigate zeta regularized products of rational functions. As an application, we obtain the asymptotic expansion of the Euler Gamma function associated with a rational function.

1. Introduction. Let \( r(z) = \frac{p_h(z)}{q_k(z)} \) be a rational function of \( z \), where \( p_h \) and \( q_k \) are monic polynomials with real coefficients of degree \( h \) and \( k \), respectively, and \( c \neq 0 \) is a real number. Factoring \( r(z) \) into the product

\[
\prod_{n=1}^{\infty} \frac{p_h(n)}{q_k(n)}
\]

converges if \( c = 1, h = k, a_1 + \cdots + a_h - b_1 - \cdots - b_k = 0 \), and assuming that no factor in the denominator vanishes. If this is the case, it is a result of Euler that

\[
\prod_{n=1}^{\infty} \frac{p_h(n)}{q_k(n)} = \frac{\Gamma(1 + b_1) \cdots \Gamma(1 + b_k)}{\Gamma(1 + a_1) \cdots \Gamma(1 + a_h)}.
\]

M. Eie [4, Main theorem II] proved that this result generalizes to zeta regularized products when \( q_k(z) = 1 \) and \( |a_i| < 1 \). Recall that if \( \Lambda = \{\lambda_n\}_{n=1}^{\infty} \) is a sequence of complex numbers with a unique accumulation point at infini-

2010 Mathematics Subject Classification: Primary 11M06.

Key words and phrases: rational function, Riemann zeta function, zeta regularized product.

DOI: 10.4064/ba58-3-1
ity and genus $g$ (see for example [2, 7.5], or [6, Section 2] for the definition), and if $\Lambda$ is contained in some suitable sector of the complex plane, then the zeta regularization of the infinite product

$$\prod_{n=1}^{\infty} \lambda_n$$

is by definition $e^{-\zeta'(0, \Lambda)}$, where the zeta function associated to $\Lambda$ is defined by the Dirichlet series

$$\zeta(s, \Lambda) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

when Re$(s) > g$, and by analytic continuation elsewhere, and where by $\zeta'(0, \Lambda)$ we mean the finite part of $\zeta(s, \Lambda)$ if $\zeta(s, \Lambda)$ has a pole at $s = 0$ (we refer to [6] for details). If the unique accumulation point of $\Lambda$ is zero, then we define the associated zeta function by $\zeta(s, \Lambda) = \zeta(-s, 1/\Lambda)$.

If $\Lambda = \{cp_h(n)/q_k(n)\}_{n=1}^{\infty}$, we denote by $\zeta(s, cp_h/q_k)$ the associated zeta function, and we call it the zeta function associated with a rational function. The polynomial zeta function $\zeta(s, p_k)$ has been studied in the cited work of Eie. Subsequently, the construction has been generalized by studying multiple polynomial zeta functions in [5], and introducing polynomial multiplicity in [3].

In this note, we extend this construction to the case of rational functions. Our first result is the following proposition, which also gives an elementary proof of Main Theorem II in [4].

**Proposition 1.** Let $c \neq 0$, and $a_1, \ldots, a_h$ and $b_1, \ldots, b_k$ be complex numbers with $|a_j| < 1$, $|b_j| < 1$, and $h \neq k$. Then the zeta regularization of the infinite product

$$\prod_{n=1}^{\infty} \frac{cp_h(n)}{q_k(n)} = \prod_{n=1}^{\infty} \frac{c(n + a_1) \cdots (n + a_h)}{(n + b_1) \cdots (n + b_k)}$$

is

$$e^{-\zeta'(0, cp_h/q_k)} = (2\pi)^{h-k} \frac{c}{h-k} \frac{a_1 \cdots a_h - b_1 \cdots b_k}{1 + a_1 \cdots a_h} \frac{1}{\Gamma(1 + a_1) \cdots \Gamma(1 + a_h)} \frac{1}{\Gamma(1 + b_1) \cdots \Gamma(1 + b_k)}.$$

As a second result, we present the following natural application of Proposition 1. Define the Euler Gamma function associated with the rational function $r(z) = cp_h(z)/q_k(z)$, with $h > k > 0$, to be the Weierstrass product

$$\Gamma(z, cp_h/q_k) = \prod_{n=1}^{\infty} \frac{e^{z/r(n)}}{1 + z/r(n)},$$

where we put $g = 1$ if $h = k + 1$, and $g = 0$ otherwise (note that $g$ is the genus of the sequence $\Lambda$). Then we have the following asymptotic expansion,
where the notation $\text{Res}_z f(s)$ denotes the coefficient of the term $(s - s_0)^{-1}$ in the Laurent expansion of $f(s)$ at $s = s_0$ (see for example [1, p. 420]).

**Proposition 2.** For large $z$ with $|\arg(z)| < \pi$,

$$
\log \Gamma(z, cp_h/q_k) = \begin{cases}
\frac{\pi c^{-1} - 1}{\sin \frac{\pi c}{c-k}} z^{1-\frac{1}{c-k}} & \text{if } h > k + 1, \\
\frac{1}{c} z \log z + \left( \text{Res}_z \zeta(s, cp_h/q_k) - \frac{1}{c} \right) z & \text{if } h = k + 1, \\
\left( \frac{1}{2} + \frac{a_1 + \cdots + a_h - b_1 - \cdots - b_k}{h-k} \right) \log z \\
- \left( \frac{1}{2} + \frac{a_1 + \cdots + a_h - b_1 - \cdots - b_k}{h-k} \right) \log c \\
+ \frac{h-k}{2} \log 2\pi + \log \frac{\Gamma(1+b_1) \cdots \Gamma(1+b_k)}{\Gamma(1+a_1) \cdots \Gamma(1+a_h)} + o(1)
\end{cases}
$$

The proofs of these propositions are presented in the next two sections.

2. The proof of Proposition 1. Expanding the powers of the binomials we obtain, for large $\text{Re}(s)$,

$$
c^s \zeta(s, cp_h/q_k) = \sum_{j,l=0}^{\infty} \binom{-s}{j} \binom{s}{l} a^j b^l \zeta((h-k)s + |j| + |l|),
$$

where $\zeta(s)$ is the Riemann zeta function, we use the multi-indices $j = (j_1, \ldots, j_h)$, $l = (l_1, \ldots, l_k)$, and $|j| = j_1 + \cdots + j_h$, $|l| = l_1 + \cdots + l_k$.

$$
a_j^j = a_1^{j_1} \cdots a_h^{j_h}, \quad b_k^k = b_1^{k_1} \cdots b_k^{k_k}, \quad \beta_j^j = \frac{\beta_1^1}{j_1^1} \cdots \frac{\beta_h^h}{j_h^h}, \quad \gamma_j^j = \frac{\gamma_1^1}{j_1^1} \cdots \frac{\gamma_h^h}{j_h^h}.
$$

Thus,

$$
c^s \zeta(s, cp_h/q_k) = \zeta((h-k)s) + \sum_{a=1}^{h} \sum_{j_a=1}^{\infty} \binom{-s}{j_a} a^{j_a} \zeta((h-k)s + j_a) + \sum_{\beta=1}^{k} \sum_{l_{\beta}=1}^{\infty} \binom{s}{l_{\beta}} b^{l_{\beta}} \zeta((h-k)s + l_{\beta}) + \varphi(s),
$$

where $\varphi(s)$ has a zero of degree 2 at $s = 0$. Isolating the singular terms, we obtain

$$
(1) \quad c^s \zeta(s, cp_h/q_k) = \zeta((h-k)s) - s \zeta((h-k)s + 1) \left( \sum_{a=1}^{h} a - \sum_{\beta=1}^{k} b_{\beta} \right) + \sum_{a=1}^{h} \sum_{j_a=2}^{\infty} \binom{-s}{j_a} a^{j_a} \zeta((h-k)s + j_a) + \sum_{\beta=1}^{k} \sum_{l_{\beta}=2}^{\infty} \binom{s}{l_{\beta}} b_{\beta}^{l_{\beta}} \zeta((h-k)s + l_{\beta}) + \varphi(s).
$$
The analytic continuation at $s = 0$ of the function on the right side of (1) is given by that of the Riemann zeta function, and is regular at $s = 0$. In particular,

$$\frac{d}{ds} \left|_{s=0}^{\infty} \sum_{j_\alpha=2}^\infty \left( -s \right)^j \zeta((h - k)s + j_\alpha)a_{j_\alpha} = \sum_{j_\alpha=2}^\infty \frac{(-1)^{j_\alpha}}{j_\alpha} \zeta(j_\alpha)a_{j_\alpha} \right. = \log \Gamma(1 + a_\alpha) + \gamma a_\alpha,$$

$$\frac{d}{ds} \left|_{s=0}^{\infty} \sum_{l_\beta=2}^\infty \left( s \right)^l \zeta((h - k)s + l_\beta)b_{l_\beta} = -\sum_{l_\beta=2}^\infty \frac{(-1)^{l_\beta}}{l_\beta} \zeta(l_\beta)b_{l_\beta} \right. = -\log \Gamma(1 + b_\beta) - \gamma b_\beta.$$

This gives the expansions near $s = 0$ of the different terms:

$$-s\zeta((h - k)s + 1) \left( \sum_{a=1}^h a_\alpha - \sum_{\beta=1}^k b_\beta \right) = -\frac{1}{h - k} \left( \sum_{a=1}^h a_\alpha - \sum_{\beta=1}^k b_\beta \right)s + O(s^2),$$

$$\sum_{a=1}^h \sum_{j_\alpha=2}^\infty \left( -s \right)^j \alpha_{j_\alpha} \zeta((h - k)s + j_\alpha) = \left( \log \prod_{a=1}^h \Gamma(1 + a_\alpha) + \gamma \sum_{a=1}^h a_\alpha \right)s + O(s^2),$$

$$\sum_{\beta=1}^k \sum_{l_\beta=2}^\infty \left( s \right)^l \beta_{l_\beta} \zeta((h - k)s + l_\beta) = -\left( \log \prod_{\beta=1}^k \Gamma(1 + b_\beta) + \gamma \sum_{\beta=1}^k b_\beta \right)s + O(s^2),$$

and hence

$$\zeta(0, p_h/q_k) = \zeta(0) - \frac{1}{h - k} \left( \sum_{a=1}^h a_\alpha - \sum_{\beta=1}^k b_\beta \right),$$

$$\zeta(0, p_h/q_k) = (h - k)\zeta'(0) + \log \frac{\prod_{a=1}^h \Gamma(1 + a_\alpha)}{\prod_{\beta=1}^k \Gamma(1 + b_\beta)}.$$
for large \( z \) with \( |\arg(z)| < \pi \) and \( \alpha_0 > \alpha_1 > \cdots > \alpha_J \). Observe that our sequence \( A \) is a simply regular sequence of spectral type with genus \( g \), where \( g = 1 \) if \( h = k + 1 \) and \( g = 0 \) otherwise. Indeed, \( A \) is a sequence of spectral type by Lemma 2.5 in [6], and is simply regular by Proposition 2.11 in [7], since the possible poles of the zeta function \( \zeta(s, A) \) are at most simple. Moreover, by the same proposition the possible poles of the zeta function are located at \( s = \alpha_j \), and by Remark 2.9 in [7], \( \alpha_0 < g + 1 \). Now, using the expansion given in the previous section, it is easy to see that \( \zeta(s, A) \) has at most one simple pole on the positive part of the real axis, and this pole is at \( s = 1 \) if \( g = 1 \), and at \( s = 1/(h - k) \) otherwise. It follows that the unique possible positive value of the \( \alpha_i \) is either \( \alpha_0 = 1 \), if \( g = 1 \), or \( \alpha_0 = \frac{1}{h-k} \), if \( g = 0 \); and that \( \alpha_1 = 0 \) for any \( g \). Also note that \( \log F(z, A) = -\log \Gamma(z, A) \).

This means that

\[
\log \Gamma(z, A) = \sum_{j=0}^{g} a_{j,1} z^j \log z + \sum_{j=0}^{1} a_{\alpha_j,0} z^{\alpha_j} + o(1).
\]

The values of \( a_{x,k} \)'s can be calculated explicitly as follows (see Propositions 2.11 and 2.14 in [7], and Proposition 2.6 in [6]):

\[
a_{0,0} = -\text{Res}_{s=0} \zeta'(s, A),
\]

\[
a_{0,1} = -\text{Res}_{s=0} \zeta(s, A),
\]

\[
a_{\alpha_0,0} = \begin{cases} \frac{1}{h-k} \frac{1}{k-h} & g = 0, \\ \text{Res}_{s=1} \zeta(s, A) - \text{Res}_{s=1} \zeta(s, A) = \text{Res}_{s=1} \zeta(s, A) - \frac{1}{c}, & g = 1, \end{cases}
\]

\[
a_{\alpha_0,1} = \begin{cases} \frac{1}{h-k}, & g = 0, \\ \text{Res}_{s=1} \zeta(s, A) = \frac{1}{c}, & g = 1. \end{cases}
\]

Applying Proposition 1, we are done.

4. Remarks. In this section we investigate the case \( h = k \). Before, we discuss multiplicativity of zeta regularization. This appears in the interpretation of the infinite product \( \prod \Lambda_n \) as the determinant of the infinite diagonal matrix \( \Lambda \) with entries \( \lambda_n \). Namely, we set

\[
\det \zeta \Lambda = e^{-\zeta'(0, A)}.
\]
Then it is natural to ask: given two infinite diagonal matrices $A_1$ and $A_2$, is $\det \zeta A_1 A_2 = \det \zeta A_1 \det \zeta A_2$? Multiplicativity of determinants clearly corresponds to additivity of the derivative at zero of the associated zeta functions. Now, it is clear that $\zeta(0, cA) = \zeta(0, A)$ for any $c \neq 0$, and that

$$\zeta'(0, cA) = -\zeta(0, A) \log c + \zeta'(0, A).$$

Restricting to the case where the $\lambda_n$ are rational functions of $n$, it follows from Proposition 1 and the formula

$$\zeta(0, p_h) = -\frac{1}{2} - \frac{1}{h}(a_1 + \cdots + a_h)$$

that $\zeta'(0, c_1 c_2 p_{1,h_1} p_{2,h_2})$ is not equal to $\zeta'(0, c_1 p_{1,h_1}) + \zeta'(0, c_2 p_{2,h_2})$ (however, note this is the case when $h_1 = h_2$ and $p_{1,h_1} = p_{2,h_2}$). Therefore, we further restrict to monic polynomials, and in this case it is easy to see that (if $h_j \neq k_j$)

$$\zeta'(0, p_{1,h_1} p_{2,h_2} q_{1,k_1} q_{2,k_2}) = \zeta'(0, p_{1,h_1} / q_{1,k_1}) + \zeta'(0, p_{2,h_2} / q_{2,k_2}).$$

Thus we consider the case $k = h$ only for monic polynomials. It is clear that a zeta function for the sequence \( \{p_h(n)/q_h(n)\}_{n=1}^{\infty} \) cannot be defined, since the exponent of convergence is not finite. However, we can introduce the following regularization of the infinite product: we define the regularized product

$$\prod_{n=1}^{\infty} \frac{R_{p_h(n)}}{R_{q_h(n)}} = \prod_{n=1}^{\infty} \frac{R_{(n + a_1) \cdots (n + a_h)}}{R_{(n + b_1) \cdots (n + b_h)}} = e^{a_1 + \cdots + a_h - b_1 - \cdots - b_h} \prod_{n=1}^{\infty} \frac{(n + a_1) \cdots (n + a_h)}{(n + b_1) \cdots (n + b_h)} e^{-(a_1 + \cdots + a_h - b_1 - \cdots - b_h)/n}.$$

It is then easy to see that the product on the right side converges, as desired, to

$$\frac{\Gamma'(1 + b_1) \cdots \Gamma'(1 + b_h)}{\Gamma'(1 + a_1) \cdots \Gamma'(1 + a_h)} = \frac{e^{-\zeta'(0, p_h)}}{e^{-\zeta'(0, q_h)}}.$$

We conclude by observing that the method described in this note can be used to obtain a formula for the derivative at zero of the zeta function studied by Dąbrowski in [3], where some polynomial multiplicity has been introduced.

**Acknowledgments.** We thank the referee for his comments and remarks to strengthen the presentation of the results in this note.

**References**


Mauro Spreafico
ICMC, Universidade de São Paulo
São Carlos, Brazil
E-mail: mauros@icmc.usp.br

Received October 15, 2010;
received in final form November 1, 2010