# FUNDAMENTAL DOMAIN AND CELLULAR DECOMPOSITION OF TETRAHEDRAL SPHERICAL SPACE FORMS

L.L. FÊMINA, A.P.T. GALVES, O. MANZOLI NETO, AND M. SPREAFICO

Abstract. Given a free isometric action of the binary tetrahedral group on a $(4n-1)$-dimensional sphere we obtain an explicit finite cellular decomposition of the sphere, equivariant with respect to the group action. A cell decomposition of the correspondent spherical space form and an explicit description of the associated cellular chain complex of modules over the integral group ring of the fundamental group of the space form follows. In particular, the construction provides a simple explicit 4-periodic free resolution for the binary tetrahedral group.

**Keywords:** binary tetrahedral group, spherical space forms, fundamental domain, cellular decomposition

## 1. Introduction

The spherical space forms are obtained as the quotient space of the spheres by fixed-point free actions of finite groups; the family of such groups is explicitly known. These spaces are complete Riemannian manifolds of constant positive curvature.

The spherical space forms problem splits into two problems, that of describing the groups which can occur, and that of describing the ways in which a given group can act upon the sphere in question. This problem was actually solved by means of the results of several authors using group theory and representation theory. We refer the interested reader to the book of Wolf [8], which gives a complete classification of the spherical space forms.

In [7], Tomoda and Zvengrowski studied the cohomology ring of the tridimensional spherical space forms. The basic idea is to produce an explicit resolution for the fundamental group $\pi$ of these spaces. This is indeed a long standing problem in algebraic topology. Also in [7], they present an explicit resolution for the binary tetrahedral group. The basic idea of the present work is to obtain an explicit resolution for the binary tetrahedral group using geometry. In the last section, we will prove the equivalence of our resolution with the one presented in [7].

Our approach is based on the original idea of Swan in [6]. Let a finite group $\pi$ act freely on a sphere $S^n$. Then, in order to obtain a resolution for $\pi$, it is sufficient to obtain a $\pi$-equivariant CW decomposition of $S^n$. Of course, the main problem in applying this approach is computational, and this is the reason why, after it was successfully exploited for the cyclic groups, it was somehow abandoned. In [4], we studied the generalized quaternionic groups, and in [2], we studied the split metacyclic groups. In particular, we followed the clever geometric setting introduced by M.M. Cohen in [1, Ch. 9]. In that work, Cohen considered the cyclic

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groups, and a sophisticated description of the cellular complex is obtained using the join decomposition of a sphere into spheres of lower dimension, i.e., the method used to obtain the decomposition is essentially geometric. We refer to that book for all the basic details of the construction.

In [2] and [4], the ideas of Cohen were used, and after some substantial improving of his technique, it was possible to obtain a cellular decomposition of the sphere $S^n$, equivariant with respect to the actions of the groups studied. We show in this work that this technique is in fact powerful enough to deal with another class of spherical space forms: the binary tetrahedral spherical space forms. The main advantage of using a geometric approach is that it is likely to be extended to tackle all the other groups of the family $P_{8\cdot 3^k}$. There is work in progress in this direction.

## 2. Preliminaries and notations

2.1. **Binary tetrahedral group.** We denote by $P_{24}$ the binary tetrahedral group with presentation [9, 2.2]

$$P_{24} = \langle x, y, z \mid x^2 = (xy)^2 = y^2,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ xyx^{-1} = y^{-1},\ z^3 = x^4 = 1\rangle.$$

This group also has the following semidirect product structure see [7, Section 5.1]. Let $Q_8 = \langle x, y \mid x^2 = y^2; yxy^{-1} = x^{-1}\rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ be the quaternionic group, and $C_3 = \{1, z, z^2\}$ the cyclic group of order 3. Consider the homomorphism $\varphi : C_3 \to \mathrm{Aut}(Q_8)$, defined by $\varphi_z(x) = zxz^{-1} = y$ and $\varphi_z(y) = zyz^{-1} = xy$ (writing $Q_8 = \{\pm 1, \pm, i, \pm, j, \pm k\}$, $\varphi$ reads: $\varphi_z(i) = j$, $\varphi_z(j) = k$, $\varphi_z(k) = i$). Then,

$$P_{24} \cong Q_8 \rtimes_\varphi C_3,$$

and we have the split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & Q_8 \arrow[r, "\iota"] & P_{24} \arrow[r, "p"] & C_3 \arrow[r] \arrow[l, bend left, "s"] & 1, \end{tikzcd}$$

where $\iota$ is the inclusion onto the normal subgroup generated by $\iota(x) = x$ and $\iota(y) = y$ ($\iota(i) = x$, $\iota(j) = y$, $\iota(k) = xy$), $p(z) = z$, the generator of $C_3$ , $p(x) = 1 = p(y)$, and the splitting map is $s(z) = z$. Note also that $(P_{24})_{\mathrm{ab}} = C_3$.

**Remark 2.1.** *The order of the binary tetrahedral group is 24 and the elements of $P_{24}$ can be written as follows:*

$$P_{24} = \{1, x, x^2, x^3, y, xy, x^2y, x^3y, z, zx, zx^2, zx^3, zy, zxy, zx^2y, zx^3y, z^2, z^2x, z^2x^2,$$
$$z^2x^3, z^2y, z^2xy, z^2x^2y, z^2x^3y\}.$$

2.2. **Free actions on spheres.** The group $P_{24}$ is a group of type III, according to the table of Theorem 6.1.11 of J. Wolf [8], with $A = 1$, $B = z$, $P = x$, $Q = y$, $m = 1$, $n = 3$, $r = 1$, $d = 1$, and satisfies all of the three conditions given below the table (where $d$ is defined in [8, Theorem 5.5.1]).

Then, according to Stepanov [5, Lemma 2.15], the binary tetrahedral group has only one irreducible complex representation $\alpha$ without fixed points $(\alpha(g)(p) = p$

implies $g = 1 \in P_{24}$), explicitly given by

$$\alpha(x) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \alpha(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \alpha(z) = -\frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}.$$

This representation gives a free action of $P_{24}$ on $S^3 \subset \mathbb{C}^2$. Next, consider actions of $P_{24}$ on the spheres $S^{4n-1}$, $n \geq 1$. By [8, 7.4], all the possible actions are direct sums of $\alpha$; then define the representation

$$\alpha = \alpha_1 \oplus \cdots \oplus \alpha_n : P_{24} \to U(2n, \mathbb{C}),$$

where $\alpha_j = \alpha : P_{24} \to U(2, \mathbb{C})$, $j = 1, 2, ..., n$.

**Definition 2.1.** *Let $P_{24}$ be the binary tetrahedral group. The quotient space of the action $\alpha$ of $P_{24}$ over $S^{4n-1}$, $n \geq 1$,*

$$\mathscr{P}_{4n-1} = \frac{S^{4n-1}}{\alpha(P_{24})}$$

*is called a tetrahedral spherical space form.*

2.3. **Curved join.** Let $w \in \mathbb{C}, |w| = 1$ and $w = (\cos\phi, \sin\phi) \in \mathbb{R}^2$, for some $\phi \in [0, 2\pi)$ (we identify $\mathbb{C}$ here with $\mathbb{R}^2$ via $x + iy = (x, y)$). Given two points $w_1 = (\cos\phi_1, \sin\phi_1)$ and $w_2 = (\cos\phi_2, \sin\phi_2)$, $\phi_1, \phi_2 \in [0, 2\pi)$, with $(w_1, w_2) \in \mathbb{C} \times \mathbb{C} = \mathbb{R}^4$, the vectors $\overrightarrow{w_1} = (\cos\phi_1, \sin\phi_1, 0, 0)$ and $\overrightarrow{w_2} = (0, 0, \cos\phi_2, \sin\phi_2)$ are orthogonal and then we can take the smallest unitary geodesic arc from $w_1$ to $w_2$. We denote this arc by $w_1 * w_2 = [w_1, w_2]$. The arc $[w_1, w_2]$ can be written explicitly as

$$\{(\cos t \cos\phi_1, \cos t \sin\phi_1, \sin t \cos\phi_2, \sin t \sin\phi_2) : 0 \leq t \leq \pi/2\}.$$

For any two subsets $Z_1$ and $Z_2$, with $W_1 \times W_2 \subset S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$, we define their *curved join* by

$$W_1 * W_2 = \bigcup_{w_1 \in W_1, w_2 \in W_2} w_1 * w_2. \tag{2.1}$$

For example: $S^1 * S^1 = S^3$. This process generalizes as follows: identifying $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ and given the standard orthonormal basis $\{e_1, \ldots, e_{2m}\}$ of $\mathbb{R}^{2m}$, for each $1 \leq r \neq s \leq m$, denote by $\Pi_{r,s}$ the plane generated by $\{e_r, e_s\}$. Suppose $\Pi_{r_1,s_1} \cap \Pi_{r_2,s_2} = \{0\}$. Let $W_1$ and $W_2$ be subsets of the unit circles of $\Pi_{r_1,s_1}$ and $\Pi_{r_2,s_2}$, respectively. Then, the curved join $W_1 * W_2$ is well defined by equation (2.1). In particular, we denote by $\Sigma_l$ the unit circle lying in the $l$-th complex hyperplane of $\mathbb{C}^{2n}$. Then, we have an equality of the iterated curved join

$$S^{4n-1} = \Sigma_1 * \cdots * \Sigma_{2n}.$$

We can represent the two half spheres $S^1 * S^1_{\pm}$ (where $S^1_{\pm}$ denotes the north/south hemisphere of $S^1$) as in Figure 1, where the framing is given by the geodesic lines joining the end points of the basic vectors $e_j$ (see [1, Section 26] for details).

It is clear that the curved join is homeomorphic to the usual join:

$$J(X, Y) = (X \times I \times Y)/\sim,$$

where $(x, t, y) \sim (x', t', y')$ if and only if $t = 0 = t'$ and $y = y'$, or $t = 1 = t'$ and $x = x'$. However, the usual join $J(X, Y)$ and the curved join $X * Y$ are

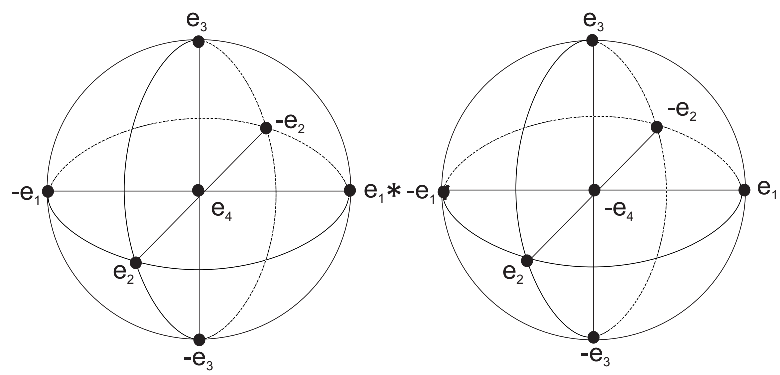



FIGURE 1

not isometric. The metric of the curved join is the metric of the sphere, and the segments are segments of the geodesic. In particular, this is fundamental when we describe the natural action appearing in the definition of the spherical space forms. More precisely, let $G$ be a finite group acting freely and orthogonally on a sphere $S^n$, let $h$ be a positive integer. Then, there is a natural action of $G$ on $S^{h(n+1)-1}$ defined by

$$
\begin{aligned}
(S^{h(n+1)-1} \subset (\mathbb{R}^{n+1})^h) \times G \to S^{h(n+1)-1} \subset (\mathbb{R}^{n+1})^h, \\
((x_1, \ldots, x_h), g) \mapsto (gx_1, \ldots, gx_h).
\end{aligned} \tag{2.2}
$$

This action coincides with the action

$$
\begin{aligned}
(S^{h(n+1)-1} = S^{(h-1)(n+1)-1} * S^n) \times G \to S^{h(n+1)-1} = S^{(h-1)(n+1)-1} * S^n, \\
((x, t, y), g) \mapsto (gx, t, gy),
\end{aligned}
$$

where the join is the curved join.

## 3. The tridimensional case

In this section we present the two important technical results of this work. In the first subsection we describe the fundamental domain $\mathcal{F}$ for the action of $P_{24}$ via the representation $\alpha$ on the tridimensional sphere $S^3$, in the second subsection we achieve a CW decomposition of $\mathcal{F}$ by first obtaining an equivariant CW decomposition of $S^3$.

### 3.1. The fundamental domain.

In order to deal with the fundamental domain, we will use the approach and the notation introduced in [4].

**Definition 3.1.** *Let $G$ be a finite group acting on a space $\mathbf{X}$. A fundamental domain of the action of $G$ on $\mathbf{X}$ is a connected closed subset $\mathcal{F}$ of $\mathbf{X}$ such that $\mathbf{X} = \bigcup_g g\mathcal{F}$ and $g\mathcal{F} \cap g'\mathcal{F}$ has empty interior, that we denote by Int, for all $g \neq g' \in G$.*

**Remark 3.1.** *We denote by $\mathcal{F}_{|G|,4n-1}$ the fundamental domain, where $|G|$ is the order of the group $G$ and $4n-1$ is the dimension of the sphere.*

The first important difference with respect to the case of the quaternion group studied in [4] and the case of the split metacyclic group studied in [2] is that in the present case the fundamental domain is not the curved join of two arcs of geodesic, one in $\Pi_{1,2}$ and another in $\Pi_{3,4}$. This is due to the fact that when the elements of the group with generator $z$ act on the elements of the canonical basis of $\mathbb{R}^4$ the resulting points have all coordinates different from zero.

The alternative approach introduced for the group $P_{24}$ is to take the point $a = \left(0, 0, \dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$ as a base point, because when we act by the 24 elements of the group on $a$ the results are elements in the six planes, namely, $\Pi_{1,2}$, $\Pi_{1,3}$, $\Pi_{1,4}$, $\Pi_{2,3}$, $\Pi_{2,4}$ and $\Pi_{3,4}$. So with this base point we can find the fundamental domain.

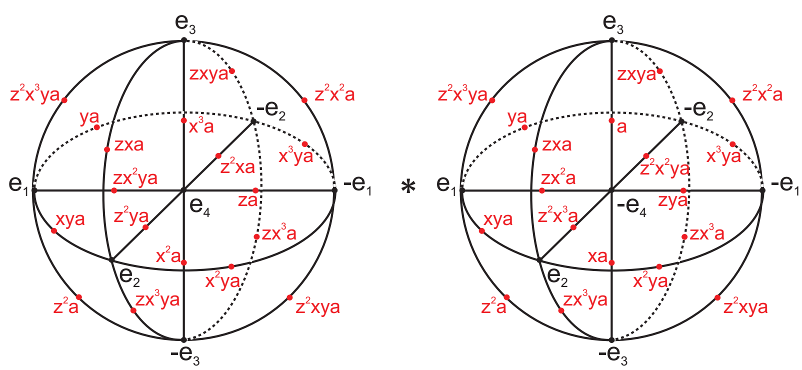



Figure 2

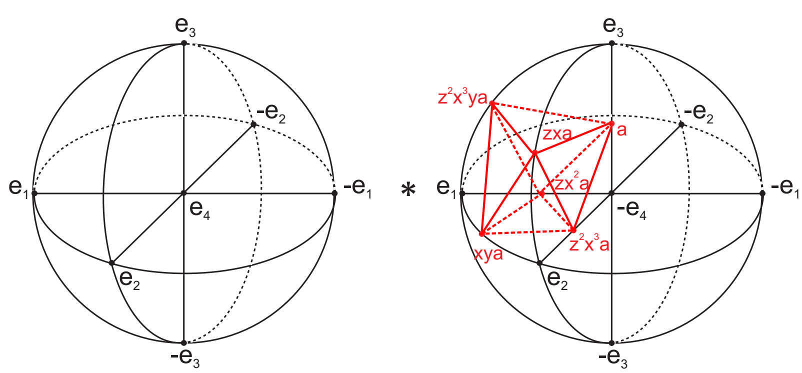



Figure 3

We find that the action of the elements of the group $P_{24}$ on the point $a=\left(0,0,\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ are:

$$\begin{aligned}
1a&=\left(0,0,\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right), & xa&=\left(0,0,-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\\
x^2a&=\left(0,0,-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right), & x^3a&=\left(0,0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\\
ya&=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0,0\right), & xya&=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0,0\right),\\
x^2ya&=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0,0\right), & x^3ya&=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0,0\right),\\
za&=\left(-\frac{\sqrt{2}}{2},0,0,\frac{\sqrt{2}}{2}\right), & zxa&=\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\\
zx^2a&=\left(\frac{\sqrt{2}}{2},0,0,-\frac{\sqrt{2}}{2}\right), & zx^3a&=\left(0,-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0\right),\\
zya&=\left(-\frac{\sqrt{2}}{2},0,0,-\frac{\sqrt{2}}{2}\right), & zxya&=\left(0,-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\\
zx^2ya&=\left(\frac{\sqrt{2}}{2},0,0,\frac{\sqrt{2}}{2}\right), & zx^3ya&=\left(0,\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0\right),\\
z^2a&=\left(\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},0\right), & z^2xa&=\left(0,-\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right),\\
z^2x^2a&=\left(-\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2},0\right), & z^2x^3a&=\left(0,\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2}\right),\\
z^2ya&=\left(0,\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right), & z^2xya&=\left(-\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},0\right),\\
z^2x^2ya&=\left(0,-\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2}\right), & z^2x^3ya&=\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2},0\right).
\end{aligned}$$

These points are displayed in Figure 2. Note that they are always in the middle of the geodesic arc joining the basis points.

Next consider the sixteenth of the sphere $S^3=S^1*S^1$, namely the tetrahedron spanned by $e_1,e_1,e_3$ and $e_{-4}$, see Figure 3 and Figure 4

The main purpose of this section is to prove the following result:

**Proposition 3.1.** *A fundamental domain for the action of the group $P_{24}$ on $S^3$ via the representation $\alpha$ is $\mathcal{F}_{24,3}=a*xya*zx^2a*z^2x^3a\cup a*zxa*xya*z^2x^3a\cup a*zx^2a*z^2x^3ya*xya\cup a*zxa*z^2x^3ya*xya$, displayed in Figure 3 and in Figure 4.*

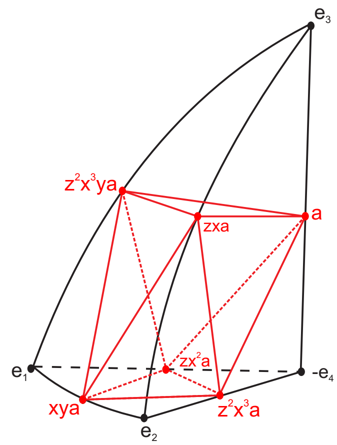



FIGURE 4

The proof of Proposition 3.1 follows by the next five lemmas. The Lemmas 3.1, 3.2, 3.3 and 3.4 assure that $g\mathcal{F}_{24,3} \cap g'\mathcal{F}_{24,3}$ has empty interior. The Lemma 3.5 together with the previous lemmas assures that $S^3 = \bigcup_{g \in P_{24}} g\mathcal{F}_{24,3}$, i.e, that the action of $P_{24}$ on $\mathcal{F}_{24,3}$ covers all of $S^3$. The proof of the first lemma is clear.

**Lemma 3.1.** *Consider $P * Q$ and $P' * Q'$, two subsets in $S^1 * S^1$, then $Int(P * Q) \cap Int(P' * Q') \neq \emptyset$ if and only if $Int(P) \cap Int(P') \neq \emptyset$ and $Int(Q) \cap Int(Q') \neq \emptyset$.*

We saw that the group $P_{24} \simeq Q_8 \rtimes_\varphi C_3$. In [4], we showed that the fundamental domain for the action of $Q_8$ on $S^3$ is the following join that we denote by $\mathcal{F}_{8,3} = A_1 * A_2$, where $A_1$ is the arc of $\Sigma_1$ of length $\frac{\pi}{2}$ starting at $e_2$, and $A_2$ is the arc of $\Sigma_2$ of length $\pi$ starting at $e_3$, displayed in Figure 5.

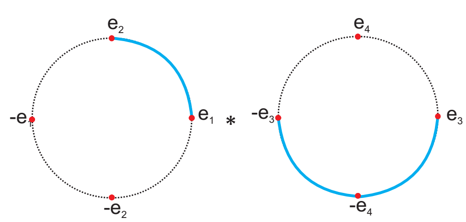



FIGURE 5

We describe now the action of the group generators $x$ and $y$ in the representation $\alpha$. Let $\beta_j = z_j * w_j$ be an arc of $\Sigma_j$, and let $R(\theta)$ denote the rotation of angle $\theta$, then:

$$\alpha(x)(\beta_1 * \beta_2) = R\left(\frac{\pi}{2}\right)(\beta_1) * R\left(-\frac{\pi}{2}\right)(\beta_2)$$
$$\alpha(y)(\beta_1 * \beta_2) = R(\pi)(\beta_2) * \beta_1.$$

**Remark 3.2.** $\alpha(x^\gamma)(\beta_1 * \beta_2) = R\left(\gamma\frac{\pi}{2}\right)(\beta_1) * R\left(-\gamma\frac{\pi}{2}\right)(\beta_2)$.

Since the fundamental domain $\mathcal{F}_{24,3}$ is contained in $\mathcal{F}_{8,3}$ we have the following lemmas:

**Lemma 3.2.** *The rotations on the two circles $\Sigma_j$, $j = 1, 2$, determined by the action of the elements of group $Q_8$ on $\mathcal{F}_{8,3}$, satisfy the following bounds:*

(1) *for* $\gamma = 1, 2, 3$, $\alpha(x^\gamma)(\beta_1 * \beta_2) = R\left(\gamma\frac{\pi}{2}\right)(\beta_1) * R\left(-\gamma\frac{\pi}{2}\right)(\beta_2)$, *with* $0 \leq \gamma\dfrac{\pi}{2} < 2\pi$, *i.e. the rotations determined by $x^\gamma$ on $\Sigma_1$ and $\Sigma_2$ are less than $2\pi$;*

(2) $\alpha(y)(\beta_1 * \beta_2) = R(\pi)(\beta_2) * \beta_1$, *i.e. the rotations determined by $y$ on $\Sigma_1$ and $\Sigma_2$ are less than $2\pi$;*

(3) *for* $\gamma = 1, 2, 3$, $\alpha(x^\gamma y)(\beta_1 * \beta_2) = R\left(\gamma\frac{\pi}{2} + \pi\right)(\beta_2) * R\left(-\gamma\frac{\pi}{2}\right)(\beta_1)$, *with* $0 \leq \gamma\dfrac{\pi}{2} + \pi < 4\pi$ *and* $0 \leq -\gamma\dfrac{\pi}{2} < 2\pi$, *i.e. the rotation determined by $x^\gamma y$ on $\Sigma_1$ is less than $4\pi$ and on $\Sigma_2$ is less than $2\pi$.*

*Proof.* This follows by lengthy direct verification, using the minimal rotations of each element of the group. See [3] for more details. □

**Lemma 3.3.** *The following statements are true:*

(1) *for* $\gamma = 1, 2, 3$, $Int(\alpha(x^\gamma)(\mathfrak{F}_{8,3})) \cap Int(\mathfrak{F}_{8,3}) = \emptyset$;

(2) $Int(\alpha(y)(\mathfrak{F}_{8,3})) \cap Int(\mathfrak{F}_{8,3}) = \emptyset$;

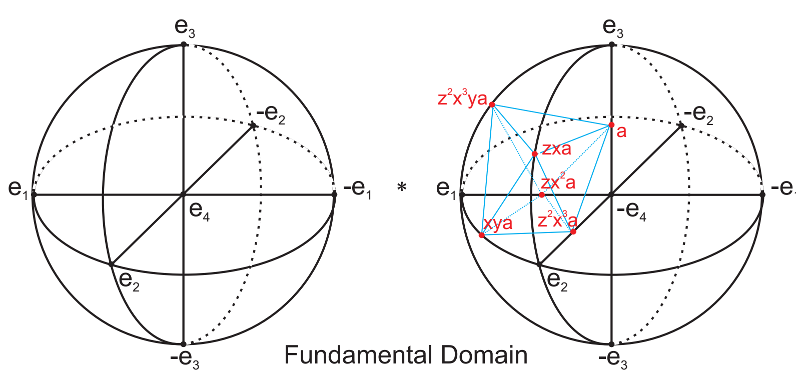
(3) *for* $\gamma = 1, 2, 3$, $Int(\alpha(x^\gamma y)(\mathfrak{F}_{8,3})) \cap Int(\mathfrak{F}_{8,3}) = \emptyset$.

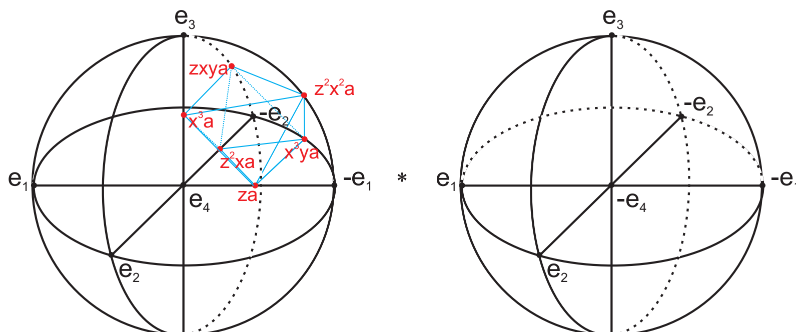
*Proof.* The method of proving these facts is always the same: we first show that the condition that the intersection of the interiors is empty in each of (1), (2), and (3) is equivalent to some equation in $\mathbb{Z}$, and then that such equation has no solution, by contradiction. These calculations are simple so we omit the proof. See [3] for more details. □

**Lemma 3.4.** *For* $\rho = 1, 2$, $\gamma = 0, 1, 2, 3$ *and* $\sigma = 0, 1$,

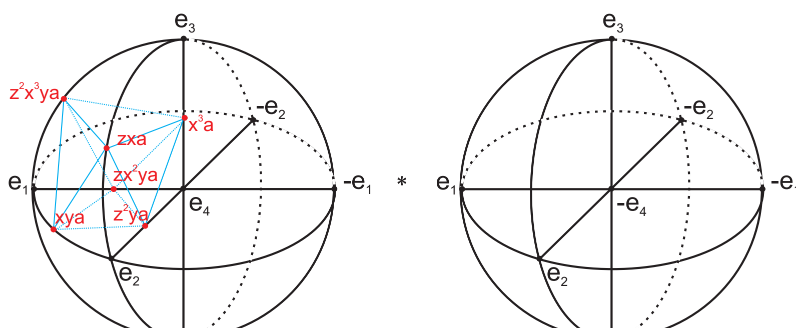
$$Int(\alpha(z^\rho x^\gamma y^\sigma)(\mathfrak{F}_{24,3})) \cap Int(\mathfrak{F}_{24,3}) = \emptyset.$$

*Proof.* This proof is by direct determination of the intersection set, using the explicit description and proceeding case by case. We present some figures to give an idea. See [3] for all details.

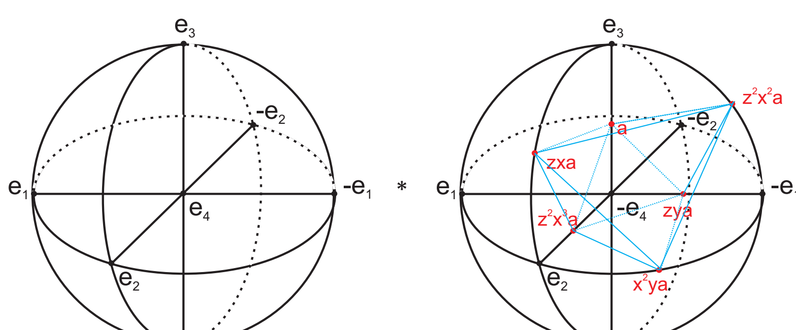




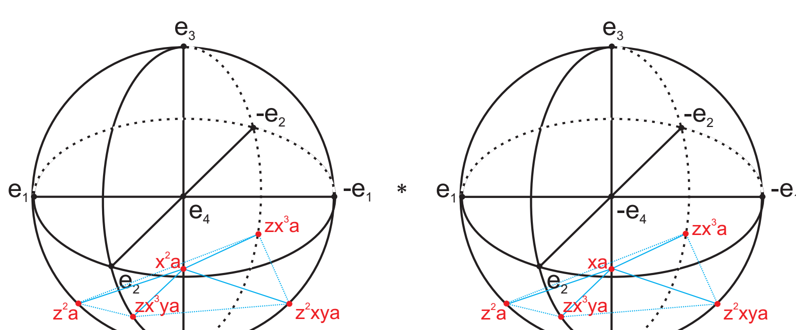

Action of the element z



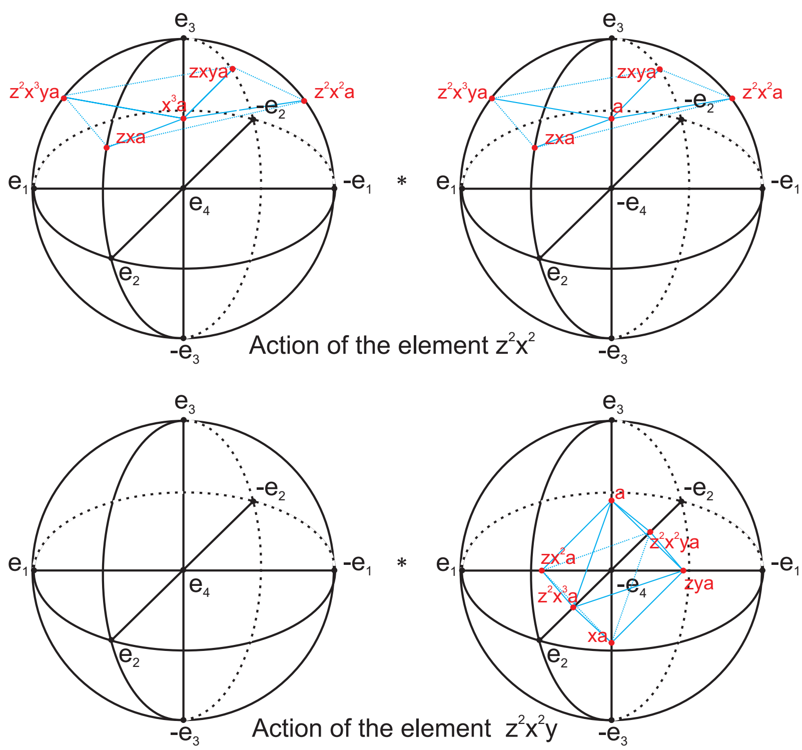

Action of the element zx




Action of the element zy




Action of the element $z^2$




Action of the element $z^2x^2$

Action of the element $z^2x^2y$

□

**Lemma 3.5.** *The image of $\mathcal{F}_{24,3}$ given by the action of the group $P_{24}$ has volume equal to* $\dfrac{V_{S^3}}{|P_{24}|} = \dfrac{V_{S^3}}{24}$*, where $V_{S^3}$ denotes the volume of the sphere $S^3$.*

*Proof.* It is clear that the volume of the tetrahedron with vertices $e_1, e_2, e_3$ and $-e_4$ is $\frac{V_{S^3}}{16}$. Now denote by $A$ the volume of the tetrahedron with vertices $zxa$, $a$, $e_3$ and $z^2x^3ya$. Note that the edges of this tetrahedron have length $\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{\pi}{3}$ and $\dfrac{\pi}{3}$ (see Figure 6), which are the same lengths as the edges of the tetrahedra with the vertices

$$xya, zx^2a, z^2x^3ya, e_1$$
$$xya, e_2, z^2x^3a, zxa$$
$$z^2x^3a, -e_4, a, zx^2a,$$

Thus each of these tetrahedra will also have volume equal to $A$.

Now we will find the volume of the solid $P$ with vertices $z^2x^3ya, xya, zxa, z^2x^3a,$ $a, zx^2a$. Tracing the three diagonals of length $\dfrac{\pi}{2}$ inside the solid $P$, we obtain the intersection point $O$ of these diagonals, with coordinates $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2}\right)$. Thus, we have eight tetrahedra inside $P$ with center at $O$, each of these eight tetrahedra also has volume equal to $A$. This gives the desired result. □

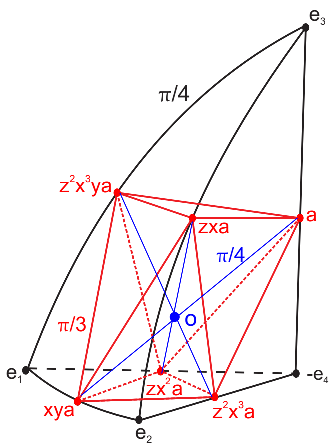



FIGURE 6

3.2. **Cellular decomposition.** If $X = S^n/\pi$, where $\pi$ is some finite group acting freely by isometries, $\mathcal{F}$ is the fundamental domain of the action of $\pi$, and $q : S^n \to X$ is the natural quotient map, then, $q$ is a covering map, $S^n = \tilde{X}$ is the universal covering space of $X$, $q|_{\mathcal{F}} : \mathcal{F} \to X$ is surjective, and it is a relative homeomorphism for the pair $(\mathcal{F}, \partial\mathcal{F}) \to (X, X^{(n-1)})$, where $X^{(n-1)}$ is the $(n-1)$-skeleton of $X$ (recall $X$ is a compact manifold). A $\pi$-equivariant cell decomposition $\tilde{K}$ of $S^n$ determines a cell decomposition $\tilde{L}$ of $\mathcal{F}$, where $\tilde{L}$ is a subcomplex of $\tilde{K}$, $\tilde{K} = \pi\tilde{L}$, and $q(\tilde{L})$ is a cell decomposition $L$ of $X$. Moreover, at least one lift of each cell of $L$ will lie in $\tilde{L}$. Therefore, we can choose for each cell $c \in L$ a single cell $\tilde{c} \in \tilde{L}$, that will be called a *representative lift* of $c$. Since all the other cells of $\tilde{L}$ are in the $\pi$-orbit of some cell $\tilde{c}$ (i.e. are in the $\pi$-orbit determined by the action of some element of the group), the cell complex of $\tilde{L}$ can be described using the cells $\tilde{c}$ and some actions of $\pi$ over $\widetilde{c}$. This set of cells will be called a *minimal set* of representative lifts, and will give a *minimal cell decomposition* of $\mathcal{F}$, that we denote by $\widetilde{Z}$. It is clear that $q(\tilde{Z}) = L$. Taking all the complete orbits of the cells in $\tilde{Z}$, we obtain the cell complex $\tilde{K} = \pi\tilde{Z}$, that is a $\pi$-equivariant cellular decomposition of $S^n$.

We pass now to determining a $P_{24}$-equivariant cell decomposition of $S^3$. For this we define a cell decomposition of $S^3$ as follows: fix a point of $S^3$ and identify this point with $a$. Write $S^3$ as the join of two circles and consider the decomposition introduced in Section 3. This gives a cell decomposition $\widetilde{L}$ of the fundamental domain $\mathcal{F}_{24,3}$. Then, 0-cells of the cell decomposition $\widetilde{K}$ of $S^3$ are the elements of the orbit $P_{24}a$. The 1-cells, 2-cells and 3-cells of $S^3$ are the 1-cells, 2-cells and 3-cells, respectively, of the fundamental domain translated by the actions of the elements of $P_{24}$. It is clear that $\widetilde{K} = P_{24}\widetilde{L}$, i.e. that this is an equivariant cell decomposition. This gives the following lemma.

**Lemma 3.6.** *A $P_{24}$ equivariant cell decomposition of $S^3$ is $\widetilde{K} = P_{24}\widetilde{L}$, where $\widetilde{L}$ is the cell decomposition of $\mathcal{F}_{24,3}$ given in Section 3. The quotient $L = \widetilde{L}/\alpha(P_{24})$ gives a cell decomposition of $\mathscr{P}_3$.*

Now we will describe the minimal set of lifts $\widetilde{Z}$ of the cells of $L$ in $S^3$, and their boundaries. In order to describe this complex, after fixing one 0-cell $a$, we denote the other 0-cells by action of the elements of the group. Higher dimensional cells are obtained by joining lower dimensional ones.

**Proposition 3.2.** *A minimal set $\tilde{Z}$ of representative lifts $\tilde{c}_{q,s}$ (where the first index denotes the dimension) of the cells of the cellular decomposition $L$ of $\mathscr{P}_3$ defined in Lemma 3.6 is:*

$$\widetilde{c}_{0,1} = a = \left(0, 0, \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\right),$$

$$\begin{array}{ll} \widetilde{c}_{1,1} = a * z^2x^3ya, & \widetilde{c}_{1,2} = a * zxa, \\ \widetilde{c}_{1,3} = a * z^2x^3a, & \widetilde{c}_{1,4} = a * zx^2a, \end{array}$$

$$\begin{array}{ll} \widetilde{c}_{2,1} = a * zxa * z^2x^3a, & \widetilde{c}_{2,2} = a * zx^2a * z^2x^3ya, \\ \widetilde{c}_{2,3} = a * zx^2a * z^2x^3a, & \widetilde{c}_{2,4} = a * zxa * z^2x^3ya, \end{array}$$

$$\begin{aligned} \widetilde{c}_{3,1} =& a * xya * zx^2a * z^2x^3a \cup a * zxa * xya * z^2x^3a \cup a * zx^2a * z^2x^3ya * xya \\ & \cup a * zxa * z^2x^3ya * xya, \end{aligned}$$

*with boundaries:*

$$\begin{aligned}
&\partial_1(\widetilde{c}_{1,1}) = (z^2x^3y - 1)\widetilde{c}_{0,1}, \\
&\partial_1(\widetilde{c}_{1,2}) = (zx - 1)\widetilde{c}_{0,1}, \\
&\partial_1(\widetilde{c}_{1,3}) = (z^2x^3 - 1)\widetilde{c}_{0,1}, \\
&\partial_1(\widetilde{c}_{1,4}) = (zx^2 - 1)\widetilde{c}_{0,1}, \\
&\partial_2(\widetilde{c}_{2,1}) = -z^2x^3\,\widetilde{c}_{1,1} + \widetilde{c}_{1,2} - \widetilde{c}_{1,3}, \\
&\partial_2(\widetilde{c}_{2,2}) = -\widetilde{c}_{1,1} - z^2x^3y\,\widetilde{c}_{1,3} + \widetilde{c}_{1,4}, \\
&\partial_2(\widetilde{c}_{2,3}) = -zx^2\,\widetilde{c}_{1,2} + \widetilde{c}_{1,3} - \widetilde{c}_{1,4}, \\
&\partial_2(\widetilde{c}_{2,4}) = \widetilde{c}_{1,1} - \widetilde{c}_{1,2} - zx\widetilde{c}_{1,4}, \\
&\partial_3(\widetilde{c}_{3,1}) = (1 - z^2x^3y)\widetilde{c}_{2,1} + (1 - z^2x^3)\widetilde{c}_{2,2} + (1 - zx)\widetilde{c}_{2,3} + (1 - zx^2)\widetilde{c}_{2,4}.
\end{aligned}$$

*Proof.* The proof is based on an explicit description of the cell decomposition $\tilde{L}$ of the fundamental domain. First, since $\mathscr{P}_3$ is connected, one 0-cell $c_{0,1}$ is sufficient in $L$ and we fix the lift $\tilde{c}_{0,1}$ of $c_{0,1}$ by identifying it with the point $a = \left(0, 0, \dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$. Next, since the quotient space is a manifold and the fundamental domain a 3-dimensional disc, it is clear that we can take just one top cell, namely we can lift the top cell of $L$ and this will be exactly $\tilde{L}$, with boundary glued on the 2-skeleton of $\tilde{L}$, that is precisely its boundary. The 0-cells are vertices of $\mathfrak{F}_{24,3}$, that is, the points:

$$c_{0,1} = a, \quad c_{0,2} = zxa, \quad c_{0,3} = z^2x^3a, \quad c_{0,4} = z^2x^3ya, \quad c_{0,5} = zx^2a, \quad c_{0,6} = xya.$$

The 1-cells are arcs:

$$\begin{array}{lll}
c_{1,1} = a * z^2x^3a, & c_{1,2} = a * zxa, & c_{1,3} = a * zx^2a, \\
c_{1,4} = z^2x^3a * zxa, & c_{1,5} = zxa * xya, & c_{1,6} = z^2x^3a * xya, \\
c_{1,7} = z^2x^3ya * xya, & c_{1,8} = zxa * z^2x^3ya, & c_{1,9} = a * z^2x^3ya, \\
c_{1,10} = z^2x^3ya * z^2xa & c_{1,11} = zx^2a * xya, & c_{1,12} = zx^2a * z^2x^3a.
\end{array}$$

The 2-cells are the sets:

$$
\begin{array}{ccc}
c_{2,1} = a * zxa * z^2x^3a, & c_{2,2} = zxa * xya * z^2x^3a, & c_{2,3} = zxa * z^2x^3ya * xya, \\
c_{2,4} = z^2x^3ya * xya * zx^2a, & c_{2,5} = z^2x^3ya * zx^2a * a, & c_{2,6} = a * zx^2a * z^2x^3a, \\
c_{2,7} = a * zxa * z^2x^3ya, & c_{2,8} = zx^2a * z^2x^3a * xya. &
\end{array}
$$

We verify that the union of the eight 2-cells previously presented coincides with the boundary of the fundamental domain, $\partial\mathfrak{F}_{24,3}$. Similarly, the boundary of each 2-cell is contained in the union of 1-cells, and the boundary of each 1-cell is contained in the union of 0-cells. This shows that this set of cells provides a cell decomposition of the fundamental domain. We verify that the following is a set of linearly independent 1-cells that generates all the other 1-cells:

$$
\begin{array}{cc}
\widetilde{c}_{1,1} = c_{1,9} = a * z^2x^3ya, & \widetilde{c}_{1,2} = c_{1,2} = a * zxa, \\
\widetilde{c}_{1,3} = c_{1,1} = a * z^2x^3a, & \widetilde{c}_{1,4} = c_{1,3} = a * zx^2a,
\end{array}
$$

for

$$
\begin{array}{cccc}
c_{1,4} = z^2x^3\, c_{1,9}, & c_{1,5} = zx\, c_{1,1}, & c_{1,6} = z^2x^3\, c_{1,3}, & c_{1,7} = z^2x^3y\, c_{1,2}, \\
c_{1,8} = zx\, c_{1,3}, & c_{1,10} = z^2x^3y\, c_{1,1}, & c_{1,11} = zx^2\, c_{1,9}, & c_{1,12} = zx^2\, c_{1,2}.
\end{array}
$$

Next, we give representative lifts in $\tilde{Z}$ of the 2-cells. The method is similar to the previous one, we identify the orbits of 2-cells in $\widetilde{L}$, we get:

$$
c_{2,2} = z^2x^3c_{2,5}, \quad c_{2,3} = zxc_{2,6}, \quad c_{2,4} = z^2x^3yc_{2,1}, \quad c_{2,8} = zx^2c_{2,7}.
$$

Thus, a set of minimal lifts of 2-cells is:

$$
\begin{array}{cc}
\widetilde{c}_{2,1} = c_{2,1} = a * zxa * z^2x^3a, & \widetilde{c}_{2,2} = c_{2,5} = z^2x^3ya * zx^2a * a, \\
\widetilde{c}_{2,3} = c_{2,6} = a * zx^2a * z^2x^3a, & \widetilde{c}_{2,4} = c_{2,7} = a * zxa * z^2x^3ya.
\end{array}
$$

The unique 3-cell of $\widetilde{Z}$ is $\widetilde{c}_{3,1}$ and it coincides with the fundamental domain. Figure 7 shows the fundamental domain with all the cells in $\widetilde{Z}$.

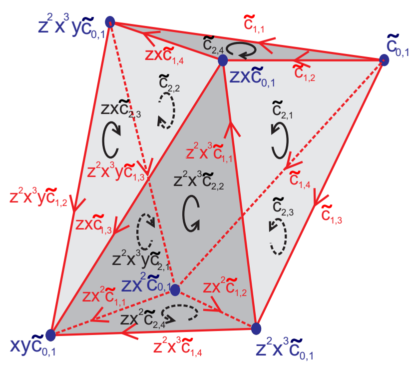



Figure 7

Finally, we calculate the boundaries geometrically and we write the coefficients in $\mathbb{Z}P_{24}$. $\square$

## 4. The higher dimensional case

The fundamental domain and the cellular decomposition for the higher dimensional cases will follow by the tridimensional case and some general results. Applying [3, Lemma 2.3.1], the fundamental domain of the action of $P_{24}$ on $S^{4n-1}$, via the representation $\alpha$, follows immediately from the definition of the fundamental domain $\mathcal{F}_{24,3}$ given in Proposition 3.1.

**Proposition 4.1.** *A fundamental domain for the action of the group $P_{24}$ on $S^{4n-1}$, $n \geq 1$, via the representation $\alpha$ is*

$$\mathcal{F}_{24,4n-1} = \Sigma_1 * \Sigma_2 * \cdots * \Sigma_{2(n-1)} * \mathcal{F}_{24,3},$$

*with $\mathcal{F}_{24,3}$ inside $\Sigma_{2n-1} * \Sigma_{2n}$.*

In a similar way, the equivariant cell decomposition of $S^{4n-1}$ follows from that of $S^3$ given in Propositions 3.2. It remains to consider the boundary of the cell $\tilde{c}_{4q,1}$, $q > 0$. But this follows easily considering that $\tilde{c}_{4q,1} = S^{4q-1} * \tilde{c}_{4q-1,1}$, and hence its boundary is given by the collection of all the cells of $S^{4q-1}$, i.e. all the orbits of $\pi$. All the details can be found in the chain complex explicitly given in the next section.

### 4.1. The chain complex.

We now describe the chain complex of $\mathbb{Z}P_{24}$ modules for the tetrahedral spherical space form $\mathscr{P}_{4n-1}$.

Following standard notation in algebraic topology, we will denote by $C(\tilde{K};\mathbb{Z}\pi)$ the $\mathbb{Z}\pi$-chain complex of the universal covering space of a finite complex $K$ with the action of the fundamental group acting by covering transformations. This is a complex of free finitely generated modules over $\mathbb{Z}\pi$.

**Theorem 4.1.** *The chain complex $C(\mathscr{P}_{4n-1};\mathbb{Z}P_{24})$ of the universal covering space of the tetrahedral spherical space forms $\mathscr{P}_{4n-1}$ with the action of the fundamental group acting by covering transformations is the following complex of free finitely generated $\mathbb{Z}P_{24}$ modules:*

$$0 \longrightarrow C_{4n-1} \longrightarrow \cdots \xrightarrow{\partial_{4q-1}} C_{4q-2} \xrightarrow{\partial_{4q-2}} C_{4q-3} \xrightarrow{\partial_{4q-3}} \cdots \longrightarrow C_0 \longrightarrow 0,$$

*where*

$$\begin{aligned}
C_{4q-4} =& \mathbb{Z}P_{24}[\mathbf{c}_{4q-4}],\\
C_{4q-3} =& \mathbb{Z}P_{24}[\mathbf{c}_{4q-3,1}, \mathbf{c}_{4q-3,2}, \mathbf{c}_{4q-3,3}, \mathbf{c}_{4q-3,4}],\\
C_{4q-2} =& \mathbb{Z}P_{24}[\mathbf{c}_{4q-2,1}, \mathbf{c}_{4q-2,2}, \mathbf{c}_{4q-2,3}, \mathbf{c}_{4q-2,4}],\\
C_{4q-1} =& \mathbb{Z}P_{24}[\mathbf{c}_{4q-1}],
\end{aligned}$$

*with boundary ($\partial_0 = 0$):*

$$\begin{aligned}
\partial_{4q-4,1}(\mathbf{c}_{4q-4}) =& (1 + x + x^2 + x^3)(1 + y)(1 + z + z^2)\mathbf{c}_{(4q-4)-1},\\
\partial_{4q-3}(\mathbf{c}_{4q-3,1}) =& (z^2x^3y - 1)\mathbf{c}_{4q-4},\\
\partial_{4q-3}(\mathbf{c}_{4q-3,2}) =& (zx - 1)\mathbf{c}_{4q-4},\\
\partial_{4q-3}(\mathbf{c}_{4q-3,3}) =& (z^2x^3 - 1)\mathbf{c}_{4q-4},\\
\partial_{4q-3}(\mathbf{c}_{4q-3,4}) =& (zx^2 - 1)\mathbf{c}_{4q-4},
\end{aligned}$$

$$\begin{aligned}
\partial_{4q-2}(\mathbf{c}_{4q-2,1}) =& - z^2x^3\,\mathbf{c}_{4q-3,1} + \mathbf{c}_{4q-3,2} - \mathbf{c}_{4q-3,3},\\
\partial_{4q-2}(\mathbf{c}_{4q-2,2}) =& - \mathbf{c}_{4q-3,1} - z^2x^3y\,\mathbf{c}_{4q-3,3} + \mathbf{c}_{4q-3,4},\\
\partial_{4q-2}(\mathbf{c}_{4q-2,3}) =& - zx^2\mathbf{c}_{4q-3,2} + \mathbf{c}_{4q-3,3} - \mathbf{c}_{4q-3,4},\\
\partial_{4q-2}(\mathbf{c}_{4q-2,4}) =& \mathbf{c}_{4q-3,1} - \mathbf{c}_{4q-3,2} - zx\mathbf{c}_{4q-3,4},\\
\partial_{4q-1}(\mathbf{c}_{4q-1}) =& (1-z^2x^3y)\mathbf{c}_{4q-2,1} + (1-z^2x^3)\mathbf{c}_{4q-2,2} + (1-zx)\mathbf{c}_{4q-2,3} +\\
&+(1-zx^2)\mathbf{c}_{4q-2,4}.
\end{aligned}$$

*This complex is exact in all middle dimensions, namely* $\mathrm{Im}(\partial_{q+1}) = \mathrm{Ker}(\partial_q)$, *for* $0 < q < 4n-1$.

*Proof.* The complex emerges directly from the equivariant cellular decomposition of $S^3$ described in Proposition 3.2. Exactness follows by the fact that the complex is composed of the cells of the cellular decomposition of $S^{4n-1}$, and the homology of $S^{4n-1}$ is zero, in middle levels. $\square$

The chain complex $C(\mathscr{P}_{4n-1};\mathbb{Z}P_{24})$ of the universal covering space $(S^{4n-1})$ is exact in all middle dimensions, as we saw in Theorem 4.1. Taking the augmentation map and letting $n \to \infty$, we construct a 4-periodic resolution of $\mathbb{Z}$ over $\mathbb{Z}P_{24}$, and we have the following result.

**Corollary 4.1.** *The complex (the boundaries are given in Theorem 4.1)*

$$\cdots \longrightarrow C_{4q-3} \xrightarrow{\partial_{4q-3}} C_{4q-4} \xrightarrow{\partial_{4q-4}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

*is a* 4*-periodic resolution of* $\mathbb{Z}$ *over* $\mathbb{Z}P_{24}$.

4.2. **Remarks.** With integer coefficients, the complex in Theorem 4.1 is the complex of groups

$$\begin{aligned}
C_{4q-4} =& \mathbb{Z}[\mathbf{c}_{4q-4}],\\
C_{4q-3} =& \mathbb{Z}[\mathbf{c}_{4q-3,1}, \mathbf{c}_{4q-3,2}, \mathbf{c}_{4q-3,3}, \mathbf{c}_{4q-3,4}],\\
C_{4q-2} =& \mathbb{Z}[\mathbf{c}_{4q-2,1}, \mathbf{c}_{4q-2,2}, \mathbf{c}_{4q-2,3}, \mathbf{c}_{4q-2,4}],\\
C_{4q-1} =& \mathbb{Z}[\mathbf{c}_{4q-1}],
\end{aligned}$$

with boundary ($\partial_0 = 0$):

$$\begin{aligned}
\partial_{4q-4}(\mathbf{c}_{4q-4}) &= 24\mathbf{c}_{(4q-4)-1},\\
\partial_{4q-3} &= 0,\\
\partial_{4q-2}(\mathbf{c}_{4q-2,1}) &= -\mathbf{c}_{4q-3,1} + \mathbf{c}_{4q-3,2} - \mathbf{c}_{4q-3,3},\\
\partial_{4q-2}(\mathbf{c}_{4q-2,2}) &= -\mathbf{c}_{4q-3,1} - \mathbf{c}_{4q-3,3} + \mathbf{c}_{4q-3,4},\\
\partial_{4q-2}(\mathbf{c}_{4q-2,3}) &= -\mathbf{c}_{4q-3,2} + \mathbf{c}_{4q-3,3} - \mathbf{c}_{4q-3,4},\\
\partial_{4q-2}(\mathbf{c}_{4q-2,4}) &= \mathbf{c}_{4q-3,1} - \mathbf{c}_{4q-3,2} - zx\mathbf{c}_{4q-3,4},\\
\partial_{4q-1}(\mathbf{c}_{4q-1}) &= 0.
\end{aligned}$$

This gives $H_0(\mathscr{P}_{4n-1};\mathbb{Z}) = \mathbb{Z}$, $H_{4q-4,q>1}(\mathscr{P}_{4n-1}) = 0$, $H_{4q-2}(\mathscr{P}_{4n-1}) = 0$, $H_{4q-1}(\mathscr{P}_{4n-1};\mathbb{Z}) = \mathbb{Z}/24\mathbb{Z}$ (since the determinant of the matrix of $\partial_{4q-2}$ is different from 0). In order to compute $H_{4q-3}(\mathscr{P}_{4n-1})$, we change the basis of $C_{4q-3}$ and

$C_{4q-2}$ to the new bases

$$\mathbf{c}_{4q-3,4}, \mathbf{c}_{4q-3,1} + \mathbf{c}_{4q-3,4}, -\mathbf{c}_{4q-3,2} + \mathbf{c}_{4q-3,3} - \mathbf{c}_{4q-3,4}, \mathbf{c}_{4q-3,3} + \mathbf{c}_{4q-3,4},$$

and

$$-2\mathbf{c}_{4q-2,1} + \mathbf{c}_{4q-2,2} - \mathbf{c}_{4q-2,3} - \mathbf{c}_{4q-2,4}, -\mathbf{c}_{4q-2,1} - \mathbf{c}_{4q-2,3}, \mathbf{c}_{4q-2,3}, -\mathbf{c}_{4q-2,1} - \mathbf{c}_{4q-2,4}.$$

In these new bases the matrix of $\partial_{4q-2}$ is diagonal with eigenvalues $3, 1, 1, 1$, and therefore $H_{4q-3}(\mathscr{P}_{4n-1}) = \mathbb{Z}/3\mathbb{Z}$.

The cohomology follows by the universal coefficient theorem:

$$\begin{aligned}
H^0(P_{24}; \mathbb{Z}) &= \mathbb{Z}, \\
H^{4q-4, q>1}(P_{24}; \mathbb{Z}) &= \mathbb{Z}/24\mathbb{Z}, \\
H^{4q-3}(P_{24}; \mathbb{Z}) &= 0, \\
H^{4q-2}(P_{24}; \mathbb{Z}) &= \mathbb{Z}/3\mathbb{Z}, \\
H^{4q-1}(P_{24}; \mathbb{Z}) &= 0.
\end{aligned}$$

This agrees with [7] Theorem 4.4. Furthermore, we now show explicitly that the complex described in Theorem 4.1 is in fact chain equivalent to the one given in Proposition 4.1 of [7]. For, first recall that in [7] the following presentation for the group $P_{24}$ is used

$$P'_{24} = \langle S, T; STS = T^2, TST = S^2 \rangle.$$

An isomorphism $\varphi : P_{24} \to P'_{24}$ is the following (we thank the referee for suggesting this isomorphism)

$$\varphi(x) = TS^2T, \qquad \varphi(y) = TS, \qquad \varphi(z) = S^4,$$

with inverse

$$\psi(T) = z^2x^3y, \qquad \psi(S) = zx^2.$$

Consistently, we denote by $C'$ the resolution of $\mathbb{Z}$ over $P'_{24}$ described in Proposition 4.1 of [7], that we recall here

$$\begin{aligned}
C'_0 &= \mathbb{Z}P'_{24}[a], & \partial'_0(a) &= 0; \\
C'_1 &= \mathbb{Z}P'_{24}[b, b'], & \partial'_1(b) &= (S-1)a, \\
& & \partial'_1(b') &= (T-1)a; \\
C'_2 &= \mathbb{Z}P'_{24}[c, c'], & \partial'_2(c) &= (T-S-1)b + (1+TS)b', \\
& & \partial'_2(c') &= (1+ST)b + (S-T-1)b'; \\
C'_3 &= \mathbb{Z}P'_{24}[d], & \partial'_3(d) &= (S-1)c + (T-1)c'; \\
C'_4 &= \mathbb{Z}P'_{24}[a_4], & \partial'_4(a_4) &= Nd.
\end{aligned}$$

We now present an explicit proof of the equivalence of $C$ and $C'$. Using the isomorphism $\varphi$, we identify the element of $P_{24}$ with their image in $P'_{24}$. We have the chain maps

$$\varphi_q : C_q \to C'_q$$

$$\begin{aligned}
\varphi_0(c_0) &= a, & & \\
\varphi_1(c_{1,1}) &= b', & \varphi_1(c_{1,2}) &= S^{-1}T^{-1}((1-T)b-b'),\\
\varphi_1(c_{1,3}) &= T^{-1}(b-b'), & \varphi_1(c_{1,4}) &= b,\\
\varphi_2(c_{2,1}) &= -S^{-1}T^{-1}c-T^{-1}c', & \varphi_2(c_{2,2}) &= 0,\\
\varphi_2(c_{2,3}) &= 0, & \varphi_2(c_{2,4}) &= S^{-1}T^{-1}c,\\
\varphi_3(c_3) &= T^{-1}d,
\end{aligned}$$

and

$$\psi_q : C'_q \to C_q$$

$$\begin{aligned}
\psi_0(a) &= c_0, & &\\
\psi_1(b) &= c_{1,4}, & \psi_1(b') &= c_{1,1},\\
\psi_2(c) &= -c_{2,2}-Tc_{2,3}+TSc_{2,4}, & \psi_2(c') &= -Tc_{2,1}+c_{2,2}-Tc_{2,4},\\
\psi_3(d) &= Tc_3,\\
\psi_4(a_4) &= Tc_4.
\end{aligned}$$

It is easy to verify that $\varphi\psi$ is the identity of $C'$. On the other side, $\psi\varphi$ is chain equivalent to the identity of $C$ by the chain homotopy $D_q : C_q \to C_{q+1}$,

$$D_1(c_{1,2}) = S^{-1}T^{-1}c_{2,1}+S^{-1}c_{2,3}, \qquad D_1(c_{1,3}) = T^{-1}c_{2,2},$$

and $D_q$ is zero otherwise.

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UFU, Universidade Federal de Uberlândia, Uberlândia, Brazil.
*E-mail address*: ligialf@famat.ufu.br

UFU, Universidade Federal de Uberlândia, Uberlândia, Brazil.
*E-mail address*: anapaula@famat.ufu.br

ICMC, Universidade de São Paulo, São Carlos, Brazil.
*E-mail address*: ozimneto@icmc.usp.br

Università del Salento, Lecce, Italy.
*E-mail address*: mauro.spreafico@unisalento.it