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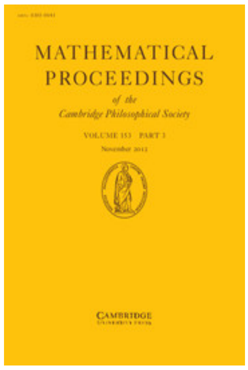

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Enumerating projectively equivalent bundles

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1. *Introduction*

This paper asks: given a vector bundle ξ and a line bundle λ over the same base space, are $\lambda \otimes \xi$ and ξ equivalent? We concentrate on real bundles ξ. Although the question is sensible in its own right, we explain in Section 2 our immediate motivation for studying it. In Section 3 we make some general comments about the question, the most significant being that under certain restrictions the answer depends on the stable class of ξ rather than on ξ itself (Proposition 3·4).

The rest of the paper tackles an interesting special case. To state the main result, let $P(\mathbb{R}^{n+1})$ denote n-dimensional real projective space, H the Hopf line bundle over it, and a_{n+1} the order of the reduced Grothendieck group $\widetilde{KO}^0(P(\mathbb{R}^{n+1}))$.

Theorem 1·1. *Let ξ be a real n-plane bundle over $P(\mathbb{R}^{n+1})$. Then $H \otimes \xi$ and ξ are bundle equivalent if and only if n is even and ξ is stably equivalent either to $\frac{1}{2}nH$ or to $\frac{1}{2}(n + a_{n+1})H$.*

The layout of the proof is described at the end of Section 4.

When ξ is the tangent bundle τ of $P(\mathbb{R}^{n+1})$ Theorem 1·1 implies that $\tau \otimes H$ and τ are equivalent if and only if n is 2 or 6; this is similar to the fact that the almost complex spheres are precisely S^2 and S^6 (see Example 5·3).

There are some connections between our results and those of [**1**], and we mention one of these in Section 6. To conclude this introduction, we draw attention to three points. First, our proof of Theorem 1·1 involves twisted K-theory; this has been studied in [**5**, **10**] and applied in [**1**, **5**], but otherwise has perhaps been underexploited. Proposition 5·4 is a desuspension result for such twisted structures, and Theorem 8·1 computes some twisted K-groups. Secondly, to clarify our use of spin^c structures in Section 10 we describe a result (Proposition 9·1) which is essentially contained in [**13**]. Finally, Proposition 10·4 is of some general interest for establishing equivalence of bundles.

2. *Motivation*

We now describe how the question in Section 1 arose. Throughout the paper, an equality between vector bundles means a bundle equivalence. We denote the product real n-plane bundle (over any base space) by n or by $\mathbb{R}^n$, according to context. In particular $n\xi = n \otimes \xi = \mathbb{R}^n \otimes \xi$.

Let G be a compact topological group, P a principal G-bundle over a space X. A *bundle automorphism* of P means a fibrewise self-map of P which respects the action of G. The set of all bundle automorphisms of P, under composition, forms a group $\mathscr{G}(P)$ called the gauge group of the bundle. Clearly if P and P' are equivalent principal G-bundles then $\mathscr{G}(P)$ and $\mathscr{G}(P')$ are isomorphic. However, there are broader circumstances under which these groups may be related. For example in [**17**] Morgan and Piccinini define, for fixed G and X, a group they call 'the local gauge group' which depends on a choice of open cover for X. They then show that under suitable hypotheses $\mathscr{G}(P)$ and $\mathscr{G}(P')$ are conjugate in this local gauge group if and only if P and P' are fundamentally equivalent in the sense of [**14**]. We may use *projective equivalence* as an alternative name for fundamental equivalence, since it means equivalence of the associated bundles with fibre G and group the inner automorphisms of G, and this group is the same as the projective group, the quotient of G by its centre.

Fundamental equivalence has been further studied in [**16**]. It is observed there that, when G is the orthogonal group $O(n)$ or the unitary group $U(n)$, two principal G-bundles are fundamentally equivalent if and only if their associated (real or complex) vector bundles ξ and ξ' satisfy $\xi' = \lambda \otimes \xi$ for some line bundle λ. This suggests using the tensor action by the group of line bundles to enumerate the principal bundles in a given fundamental class (or the vector bundles in a given projective class). However, as noted in [**16**], one comes up against the problem that the action may have 'isotropy': for a vector bundle ξ over a base space X, we shall refer to the subgroup of (isomorphism classes of) line bundles λ over X such that $\lambda \otimes \xi = \xi$ as 'the isotropy group of ξ'. To enumerate the real vector bundles in the projective class of ξ we must divide the order of $H^1(X; \mathbb{Z}/2)$ by the order of this isotropy group. Hence the question in Section 1.

One could more generally try to enumerate the principal G-bundles in a given projective class, say for G a compact Lie group (the case $G = \mathrm{Spin}\,(n)$ might be interesting) or even for G a compact topological group; in [**18**] the Steenrod–Milgram classifying space construction is reworked in the context of weak Hausdorff k-spaces, and an application of this will be to see how far one can get with a general compact group. However, in this paper G will almost always be $O(n)$, although in Section 3 it is briefly $U(n)$.

3. *General remarks*

Here are some remarks which are useful pointers to where the real problems lie.

Remark 3·1. If ξ is a real n-plane bundle with n odd, then the isotropy group of ξ is trivial.

This is a special case of Proposition 11 in [**16**]; if $\lambda \otimes \xi = \xi$ then

$$w_1(\xi) = w_1(\lambda \otimes \xi) = w_1(\xi) + nw_1(\lambda) = w_1(\xi) + w_1(\lambda),$$

so $w_1(\lambda) = 0$ and λ is trivial. Alternatively we may consider determinant bundles: if $\lambda \otimes \xi = \xi$, then $\det\, \xi = \det\,(\lambda \otimes \xi) = \lambda^{\otimes n}\,.\,\det\, \xi = \lambda\,.\,\det\, \xi$ as line bundles, so λ is trivial.

In looking for isotropy groups of real n-plane bundles, we may therefore restrict to even n.

Remark 3·2. The isotropy group of any complex n-plane bundle over a suspension ΣA is the same as that of the trivial n-plane bundle over ΣA. (In terms of classifying maps, tensor product by line bundles is given by the action of the homotopy set $[X, BS^1]$ on $[X, BU(n)]$ induced by Bm, where S^1 is the centre of $U(n)$ and m: $S^1 \times U(n) \to U(n)$ is the multiplication homomorphism. If $X = \Sigma A$ then

$$H^2(\Sigma A, \mathbb{Z}) = [\Sigma A, BS^1] \to [\Sigma A, BU(n)] \to [\Sigma A, BPU(n)]$$

is an exact sequence of groups and homomorphisms, and the isotropy group for any $\xi \in [\Sigma A, BU(n)]$ is just the image of $H^2(\Sigma A, \mathbb{Z})$ in $[\Sigma A, BU(n)]$.)

The corresponding statement for real bundles is true, but has no content, at least when A is connected, since then there are no non-trivial real line bundles over ΣA.

Remark 3·3. A real or complex vector bundle is projectively trivial if and only if it is a sum of a number of copies of a fixed line bundle.

The final observation in this section will be useful later in the paper; it says that, at least over a manifold, if ξ is 'just unstable' then its isotropy group depends only on its stable class.

PROPOSITION 3·4. *Suppose that ξ and η are stably equivalent real n-plane bundles over a closed connected n-manifold X, and suppose that $\lambda \otimes \xi = \xi$ for some line bundle λ. Then $\lambda \otimes \eta = \eta$.*

Proof. By Remark 3·1 it is sufficient to consider even n. It is easy to see that $\lambda \otimes \eta$ and η are stably equivalent. Let $\mathbb{Z}(\xi)$ denote the integer local coefficient system defined by ξ.

We now distinguish two cases: (i) $w_1\xi \neq w_1 X$, (ii) $w_1\xi = w_1 X$.

In case (i), $H^n(X; \mathbb{Z}(\xi)) \approx \mathbb{Z}/2$, and by standard obstruction theory (see [**20**]) there are at most two elements in the set S of (equivalence classes of) n-plane bundles over X in the stable class of ξ. If $|S| = 1$ there is nothing to prove. If $|S| = 2$ we note that $\lambda \otimes \lambda$ is trivial so tensoring with λ gives an involution on S; the result follows in this case since a permutation of a 2-element set has either two fixed points or none.

Remark 3·5. One can decide as follows whether $|S|$ is 1 or 2 in the above (see [**15**] for oriented equivalence and [**19**] for unoriented equivalence). For any g in $KO^{-1}(X)$ let α_g be a stable bundle over $S^1 \times X$ corresponding to g, and let $\sigma w_{i+1}(\alpha_g)$ be the element of $H^i(X;\, \mathbb{Z}/2)$ corresponding under suspension to the Stiefel-Whitney class w_{i+1} of α_g. Then $|S| = 2$ if and only if for every g in $KO^{-1}(X)$,

$$\sum_{i=0}^{n} \sigma w_{i+1}(\alpha_g) w_{n-i}(\xi)[X] = 0,$$

where $[X]$ is the fundamental class of X in $H_n(X;\, \mathbb{Z}/2)$. For the particular ξ discussed in Section 10, when $X = P(\mathbb{R}^{n+1})$ with $n \equiv 0 \bmod 4$, we can use this formula to check that $|S| = 2$ whenever $n > 8$. Hence equivalence of $\lambda \otimes \xi$ with ξ in these cases is a non-trivial matter.

The proof of Proposition 3·4 will be complete when we deal with case (ii); this follows from the next result, which is stronger. □

Proposition 3·6. *Suppose that η is a real n-plane bundle over a closed connected n-manifold X with $w_1(\eta) = w_1(X)$ and that $\lambda \otimes \eta$ and η are stably equivalent for some line bundle λ over X. Then $\lambda \otimes \eta = \eta$.*

Proof of Proposition 3·6. First, for any n-plane bundle ζ over X, let $e(\zeta)$ denote the Euler class of ζ in $H^n(X;\, \mathbb{Z}(\zeta))$, the obstruction to existence of a non-zero cross-section of ζ as described in [**20**]. Note that this kind of Euler class is well-defined without any orientations being involved; when ζ is orientable we may define an Euler class in $H^n(X;\, \mathbb{Z})$ by choosing an orientation for ζ, i.e. an isomorphism $\mathbb{Z}(\zeta) \approx \mathbb{Z}$. To prove Proposition 3·6 we shall use the following result from standard obstruction theory:

Proposition 3·7. *Suppose that X is a closed connected n-manifold and that ζ, ζ' are stably equivalent n-plane bundles over X with $w_1\zeta = w_1X$. For some $N \geqslant 1$, let $f : \zeta \oplus N \to \zeta' \oplus N$ be a stable equivalence, and identify $H^n(X;\, \mathbb{Z}(\zeta))$ with $H^n(X;\, \mathbb{Z}(\zeta'))$ using the isomorphism of $\mathbb{Z}(\zeta) \approx \mathbb{Z}(\zeta')$ determined by f. Then f desuspends to an equivalence of ζ with ζ' if and only if $e(\zeta) = e(\zeta')$.*

Since we can compose any such f with a self-equivalence of $\zeta \oplus N$ which changes the sign of one co-ordinate in N, a corollary of Proposition 3·7 is that with the same hypotheses, and identifying $H^n(X;\, \mathbb{Z}(\zeta))$ with $H^n(X;\, \mathbb{Z}(\zeta'))$ using *some* stable equivalence of ζ with ζ', we have $\zeta = \zeta'$ if and only if $e(\zeta) = \pm e(\zeta')$.

In our case, there is a canonical isomorphism of $\mathbb{Z}(\lambda \otimes \eta)$ with $\mathbb{Z}(\eta)$; for n is even so there is a canonical class of isomorphism of the fibres η_x and $(\lambda \otimes \eta)_x$ for any x in X. We may therefore identify $H^*(X;\mathbb{Z}(\eta))$ with $H^*(X;\, \mathbb{Z}(\lambda \otimes \eta))$. We shall prove that $e(\lambda \otimes \eta) = e(\eta)$.

There is a general formula (see [**6**]) which shows that $e(\lambda \otimes \eta)$ and $e(\eta)$ differ at most by 2-torsion. However, a transfer argument suffices here: let $\pi\colon \tilde{X} \to X$ be the double cover associated with the line bundle λ. Then π induces a homomorphism π^*: $H^*(X;\, \mathbb{Z}(\eta)) \to H^*(\tilde{X};\, \mathbb{Z}(\pi^*(\eta)))$ and there is a cohomology transfer homomorphism $t\colon H^*(\tilde{X};\mathbb{Z}(\pi^*(\eta))) \to H^*(X;\, \mathbb{Z}(\eta))$. Moreover, $t \circ \pi^* = 2$. A similar formula holds for $\lambda \otimes \eta$. As above, we may identify $H^*(\tilde{X};\, \mathbb{Z}(\pi^*(\eta)))$ with $H^*(\tilde{X};\mathbb{Z}(\pi^*(\lambda \otimes \eta)))$. But $\pi^*(\lambda \otimes \eta) = \pi^*(\eta)$ since $\pi^*(\lambda) = 1$. This gives $\pi^*(e(\lambda \otimes \eta)) = \pi^*(e(\eta))$, so $2e(\lambda \otimes \eta) = 2e(\eta)$. Now $e(\lambda \otimes \eta) = e(\eta)$ as required, since $H^n(X;\mathbb{Z}(\eta)) \approx \mathbb{Z}$.

4. *The stable situation for* $P(\mathbb{R}^{n+1})$

We shall sometimes use the same notation for a bundle and its stable class. Since $\widetilde{KO}^0(P(\mathbb{R}^{n+1}))$ is cyclic of order a_{n+1} generated by $H - 1$, any real vector bundle over $P(\mathbb{R}^{n+1})$ is stably equivalent to sH for some integer s, and we may assume that $0 \leqslant s < a_{n+1}$. The next proposition shows in particular that the stable constraint on ξ in Theorem 1·1 is necessary.

Proposition 4·1. *Let ξ be a real r-plane bundle over $P(\mathbb{R}^{n+1})$ which is stably equivalent to sH, with $0 \leqslant s < a_{n+1}$. Then $H \otimes \xi$ and ξ are stably equivalent if and only if either $s = \frac{1}{2}r$ or $s = \frac{1}{2}(r + a_{n+1})$.*

Proof. By Remark 3·1 we may assume that r is even. Recall that $a_{n+1}H = a_{n+1}$ over $P(\mathbb{R}^{n+1})$. Then $\xi \oplus a_{n+1} = sH \oplus (a_{n+1} + r - s)$, while

$$H \otimes \xi \oplus a_{n+1} = H \otimes (\xi \oplus a_{n+1}) = H \otimes (sH \oplus (a_{n+1} + r - s)) = s \oplus (a_{n+1} + r - s)H.$$

Hence $\xi \oplus a_{n+1} = H \otimes \xi \oplus a_{n+1}$ if and only if $s \equiv a_{n+1} + r - s \mod a_{n+1}$, from which the result follows. □

In particular, for an n-plane bundle ξ over $P(\mathbb{R}^{n+1})$, if $H \otimes \xi = \xi$ then ξ must be stably equivalent either to $\frac{1}{2}n$ or to $\frac{1}{2}(n + a_{n+1})$.

We may now summarise the rest of the proof of Theorem 1·1.

For $n \equiv 2 \mod 4$, it is easily checked that Propositions 4·1 and 3·6 together prove the theorem. These cases are covered again in what follows.

Suppose that for $s = \frac{1}{2}n$ or $\frac{1}{2}(n + a_{n+1})$, we can show that *there exists* an n-plane bundle ξ over $P(\mathbb{R}^{n+1})$ stably equivalent to sH and with $H \otimes \xi = \xi$. Then by Proposition 3·4, the same will hold for any n-plane bundle stably equivalent to ξ, and Theorem 1·1 will be proved. When $s = \frac{1}{2}n$ it is of course trivial to prove such an existence result – we simply take $\xi = \frac{1}{2}nH \oplus \frac{1}{2}n$. The crux of the proof is to establish existence when $s = \frac{1}{2}(n + a_{n+1})$.

We first give a unified proof of this for $n \equiv 2, 4$ or $6 \mod 8$, although the cases $n \equiv 2$ or $6 \mod 8$ are already covered. We then deal with $n \equiv 0 \mod 4$, noting on the way a particularly simple proof for $n \equiv 4 \mod 8$ (Remark 10·3). Thus there are several overlaps between the cases; we believe that the methods used are all of sufficient interest to merit inclusion. In order of increasing difficulty the cases may be listed: $n \equiv 4 \mod 8$, $n \equiv 2 \mod 4$, $n \equiv 0 \mod 8$.

The layout of the rest of the paper is as follows. In Section 5 we review twisted complex structures, which are used in Sections 6, 7 and 8. Section 6 contains a naïve treatment of the cases $n \equiv 2, 4$ or $6 \mod 8$ using Clifford modules but no explicit K-theory. In Sections 7 and 8 we calculate the relevant twisted K-theory groups and reprove the results of Section 6 as a corollary. In Section 9 we give a result about spin structures, with a corollary for spin^c structures which clarifies a point in Section 10, where we deal with the case $n \equiv 0 \mod 4$.

5. *Twisted complex structures*

A twisted complex structure on a real vector bundle ξ over a space X is like a complex structure, except that the pure imaginary scalars live in a real euclidean line bundle λ over X, instead of in a constant 'imaginary axis i$\mathbb{R}$'. Thus in place of $\mathbb{C}$ we define $\mathbb{C}_\lambda$ to be the bundle of fields which has underlying real bundle $1 \oplus \lambda$ and whose fibrewise multiplication is determined by setting $v^2 = -1$ for any v in λ with $||v|| = 1$. Like ordinary complex structures, and like the analogous twisted symplectic structures in [**5**], twisted complex structures have a useful desuspension property (Proposition 5·4 below).

Although a complex structure on ξ involves a scalar action of $\mathbb{C}$ on the fibres of ξ, the usual definition of complex structure concentrates on the action of the pure imaginary scalars: we define a complex structure on ξ to be a fibrewise linear map

$J: \xi \to \xi$ such that $J^2 = -1$. The analogous definition of twisted complex structures shows immediately why we are interested in them here. Let ξ be a real vector bundle, λ a real euclidean line bundle over the same base, and let ξ_b, λ_b denote the fibres over a point b.

Definition 5·1. A λ-twisted complex structure on ξ is a fibrewise linear map J: $\lambda \otimes \xi \to \xi$ such that '$J^2 = -1$'; more precisely, for any b and any u in ξ_b, v in λ_b with $||v|| = 1$, we require that $J(v \otimes J(v \otimes u)) = -u$.

Thus if ξ has a λ-twisted complex structure, then $\lambda \otimes \xi$ and ξ are equivalent in a special way.

When λ is trivial a λ-twisted complex structure on ξ is just a complex structure on ξ. Since any line bundle λ is locally trivial, locally a twisted complex structure is the same as a complex structure. Given a λ-twisted complex structure J on ξ, we get a corresponding fibrewise scalar action of $\mathbb{C}_\lambda$ on ξ, and in particular this gives each fibre of ξ a complex structure. For if we choose a unit vector v in λ_b, we may denote a point in $(1 \oplus \lambda)_b$ by $x \oplus yv$ where $x, y \in \mathbb{R}$, and define the scalar action by $(x \oplus yv) . u = xu + yJ(v \otimes u)$. This is well-defined since if we use $-v$ in place of v the recipe gives

$$(x \oplus (-y)(-v)) . u = xu + (-y)J((-v) \otimes u) = xu + yJ(v \otimes u).$$

We call this scalar action a $\mathbb{C}_\lambda$-structure and ξ equipped with a $\mathbb{C}_\lambda$-structure is called a $\mathbb{C}_\lambda$-bundle.

Example 5·2. $1 \oplus \lambda$ admits a natural $\mathbb{C}_\lambda$-structure.

Example 5·3. Let τ be the tangent bundle of $P(\mathbb{R}^{n+1})$, with $n = 2$ or 6. Here is an explicit check that τ admits a $\mathbb{C}_H$-structure. First note that for these values of n, there is a 'vector product' on $\mathbb{R}^{n+1}$. If we identify $\mathbb{R}^3$ ($\mathbb{R}^7$) with the purely imaginary quaternions (Cayley numbers) we may construct such a vector product by taking $c \times d$ to be the imaginary part of the product cd.

Let us represent the total space of τ as the quotient of

$$\{(c, d) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : ||c|| = 1, \langle c, d\rangle = 0\}$$

by the equivalence relation $(-c, -d) \sim (c, d)$, and the total space of H as the quotient of

$$\{(c, y) \in \mathbb{R}^{n+1} \times \mathbb{R} : ||c|| = 1\}$$

by $(-c, -y) \sim (c, y)$. Then the total space of $H \otimes \tau$ is the quotient of

$$\{(c, d) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : ||c|| = 1, \langle c, d\rangle = 0\}$$

by $(-c, d) \sim (c, d)$. Let $J(c, d) = (c, c \times d)$, where $c \times d$ is the vector product. It is easily checked that this gives a well-defined J: $H \otimes \tau \to \tau$ with '$J^2 = -1$'. Since τ is stably equivalent to $(n + 1)H$, and $\frac{1}{2}(n + a_{n+1}) = n + 1$ when $n = 2$ or 6, this gives a proof (to be superseded) of two positive cases in Theorem 1·1.

The Whitney sum of $\mathbb{C}_\lambda$-bundles for fixed λ is again a $\mathbb{C}_\lambda$-bundle, and over a finite-dimensional base space we can 'stabilise' $\mathbb{C}_\lambda$-bundles by adding multiples of $\mathbb{C}_\lambda$, in the same sense that complex bundles are stabilised by adding multiples of the trivial bundle $\mathbb{C}$. In particular the following 'twisted desuspension' theorem holds.

PROPOSITION 5·4. *Let λ be a real line bundle over a $2m$-dimensional CW-complex X, and let ζ be a $\mathbb{C}_\lambda$-bundle of 'complex' dimension $m+N$ over X, with $N \geqslant 0$. Then there is a (unique) $\mathbb{C}_\lambda$-bundle η of complex dimension m over X such that ζ and $\eta \oplus N\mathbb{C}_\lambda$ are isomorphic as $\mathbb{C}_\lambda$-bundles.*

Proof. This is essentially the same as the proof of the analogue for complex bundles.

In detail, to desuspend one step, suppose ζ is a $\mathbb{C}_\lambda$-bundle over X whose (complex) fibre dimension r satisfies $r > m$. Then the real bundle underlying ζ has a nowhere-zero cross-section s, which is unique up to homotopy. We may define an explicit $\mathbb{C}_\lambda$-monomorphism $f\colon \mathbb{C}_\lambda \to \zeta$ as follows: over $b \in X$ any point of $\mathbb{C}_\lambda$ may be represented by $x \oplus v$ where $x \in \mathbb{R}$ and $v \in \lambda_b$. We define $f(x\oplus v) = xs(b) + J(v \otimes s(b))$, where J is the λ-twisted complex structure on ζ. We may now split off a copy of $\mathbb{C}_\lambda$ from ζ, using a '$\mathbb{C}_\lambda$-invariant' metric on ζ – more precisely, a euclidean metric $\langle\, ,\, \rangle$ on ζ such that for any $v_1, v_2 \in \lambda_b$ and $u_1, u_2 \in \zeta_b$ we have

$$\langle J(v_1 \otimes u_1), J(v_2 \otimes u_2)\rangle = \langle v_1, v_2\rangle\langle u_1, u_2\rangle$$

where we use the previously chosen metric on λ. Such a metric on ζ can be constructed as usual with the help of a partition of unity on X. Then we may check that $\zeta = f(\mathbb{C}_\lambda) \oplus \zeta'$, where ζ' is the orthogonal complement of $f(\mathbb{C}_\lambda)$ in ζ with respect to this metric. $\square$

Remark 5·5. Just as in the ordinary case, λ-twisted complex structures are related to non-degenerate skew-symmetric fibrewise maps $\xi \otimes \xi \to \lambda$ (we may pass from one to the other by making a choice of euclidean metric on ξ). This shows up the similarity with the twisted symplectic structures in [**5**].

6. *A proof of Theorem* 1·1 *when* $n \equiv 2$, 4 *or* 6 mod 8

Suppose we can show that the trivial bundle a_{n+1} over $P(\mathbb{R}^{n+1})$ admits a $\mathbb{C}_H$-structure. By Proposition 5·4 then a_{n+1} is isomorphic as a $\mathbb{C}_H$-bundle to $\eta \oplus \frac{1}{2}(a_{n+1} - n)\mathbb{C}_H$ for some $\mathbb{C}_H$-bundle η of complex dimension $\frac{1}{2}n$. Let ξ be the real n-plane bundle underlying η. An easy calculation shows that ξ is stably equivalent to $\frac{1}{2}(n + a_{n+1})H$; and $H \otimes \xi = \xi$ since ξ admits a $\mathbb{C}_H$-structure.

We aim to show that for $n \equiv 2$, 4 or 6 mod 8, the trivial bundle a_{n+1} admits a $\mathbb{C}_H$-structure. At this point we could simply quote from proposition 7·1 of [**1**]. However, we shall give a self-contained argument using Clifford algebra bundles and modules. As in [**3**], all Clifford algebras will be taken with respect to a negative definite quadratic form, and for a euclidean bundle, the quadratic form on each fibre is the negative of the form given by the metric. Note that for the values of n involved, $a_{n+1} = a_{n+2}$.

For clarity in this proof, we temporarily distinguish some trivial bundles from their fibres. Viewing H as in Example 5·3, we see that it is contained fibrewise linearly in $P(\mathbb{R}^{n+1}) \times \mathbb{R}^{n+1}$ – the class of the pair (c, y) is included at the point $([c], yc)$, where $[c]$ is the point in $P(\mathbb{R}^{n+1})$ represented by c. Hence $1 \oplus H$ is similarly contained in $P(\mathbb{R}^{n+1}) \times \mathbb{R}^{n+2}$, so there is an inclusion of the Clifford algebra bundle $C(1 \oplus H)$ in $P(\mathbb{R}^{n+1}) \times C(\mathbb{R}^{n+2})$.

Now $\mathbb{R}^{a_{n+1}} \oplus \mathbb{R}^{a_{n+1}}$ is a simple $\mathbb{Z}/2$-graded module for the graded Clifford algebra $C(\mathbb{R}^{n+2})$, since $a_{n+1} = a_{n+2}$; so the same is true at the trivial bundle level. Hence the fibrewise inclusion of $C(1 \oplus H)$ in $P(\mathbb{R}^{n+1}) \times C(\mathbb{R}^{n+2})$ makes $P(\mathbb{R}^{n+1}) \times (\mathbb{R}^{a_{n+1}} \oplus \mathbb{R}^{a_{n+1}})$ a $\mathbb{Z}/2$-graded module-bundle for $C(1 \oplus H)$. Now let e denote a unit vector in $\mathbb{R}$ and also the corresponding vector in any fibre of the above trivial bundle 1. If v is any vector in $P(\mathbb{R}^{n+1}) \times \mathbb{R}^{n+1}$, then e and v anti-commute as elements in the Clifford algebra bundle $P(\mathbb{R}^{n+1}) \times C(\mathbb{R}^{n+2})$; in particular this holds when v comes from H, say in the fibre over $b \in P(\mathbb{R}^{n+1})$. Now each of e and v interchanges the graded components of the Clifford module $\{b\} \times (\mathbb{R}^{a_{n+1}} \oplus \mathbb{R}^{a_{n+1}})$, so the Clifford product ev preserves these components. If v is a unit vector, ev also acts with square -1. Thus the formula $J(v \otimes u) = evu$ (for any $v \in H_b$, $u \in \{b\} \times \mathbb{R}^{a_{n+1}}$) defines an H-twisted complex structure on $P(\mathbb{R}^{n+1}) \times \mathbb{R}^{a_{n+1}}$, making it a $\mathbb{C}_H$-bundle as required.

When $n = 2$ or 6 this is related to Example 5·3: for example when $n = 2$, we have $a_3 = 4 = 4H = \tau \oplus 1 \oplus H = \tau \oplus \mathbb{C}_H$, so the $\mathbb{C}_H$-structure we constructed for $\tau(P(\mathbb{R}^3))$ in Example 5·3 gives a_3 a $\mathbb{C}_H$-structure too.

7. *Twisted K-groups*

We continue with the notation of Section 6, and recall that $\mathbb{C}_\lambda$-bundles can be stabilized by adding multiples of $\mathbb{C}_\lambda$. In fact, as in the symplectic case described on pp. 135–136 of [**5**], we can form the Grothendieck group $K^0_\lambda(X)$ associated with stable $\mathbb{C}_\lambda$-bundles. Chapter 9 of [**7**] describes a general setting for topological Hermitian K-theory.

In this section we make brief remarks about twisted K-groups, and then describe a general pattern for calculating them, used in the next section. This discussion is not essential for proving Theorem 1·1, but it throws extra light on our methods and we believe it may have other applications.

We shall use notation and results from [**7**] and [**8**]. In particular L denotes the trivial bundle $\mathbb{R}$ with $\mathbb{Z}/2$ acting as ± 1, and if ζ is a vector bundle then $L \otimes \zeta$ denotes the $\mathbb{Z}/2$-vector bundle consisting of ζ with the antipodal action of $\mathbb{Z}/2$ on fibres. For any euclidean vector bundle ζ we denote by $S\zeta$ and $D\zeta$ the associated sphere-bundle and unit disc-bundle.

We shall employ $\mathbb{Z}/2$-equivariant KO-theory with coefficients. A reference for this is [**8**]. We recall that if ζ is a real $\mathbb{Z}/2$-vector bundle over X then one may define $KO^*_{\mathbb{Z}/2}(X;\zeta)$ as $\widetilde{KO}^*_{\mathbb{Z}/2}(X^\zeta)$, where the $\mathbb{Z}/2$-space X^ζ is the Thom complex of ζ. Similarly if Y is a subcomplex of X we may define

$$KO^*_{\mathbb{Z}/2}(X, Y; \zeta) = KO^*_{\mathbb{Z}/2}(X^\zeta, Y^{\zeta|Y}).$$

We shall re-interpret K_λ-groups as $\mathbb{Z}/2$-equivariant KO-groups, a technique used in [**4**] and attributed there to G.B.Segal. Note that we are considering trivial $\mathbb{Z}/2$ action on X. Vector bundles over $\mathbb{C}_\lambda$ correspond to $\mathbb{Z}/2$-graded module-bundles over the Clifford algebra bundle $C(1 \oplus \lambda)$. Hence, as a special case of Theorem 6.1 of [**7**] (see also the references to Karoubi and Segal cited there):

Proposition 7·1. *There is an isomorphism* $K^0_\lambda(X) \approx KO^0_{\mathbb{Z}/2}(X; L \oplus L \otimes \lambda)$.

This gives one way to define $K^i_\lambda(X)$ for any i: set $K^i_\lambda(X) = KO^i_{\mathbb{Z}/2}(X; L \oplus L \otimes \lambda)$.

We shall need a generalization of the standard exact sequence relating real and complex K-theory:

$$\to K^*(X) \xrightarrow{r} KO^*(X) \xrightarrow{\cdot\eta} KO^{*-1}(X) \to \qquad (7{\cdot}2)$$

(see, for example [**2**]).

PROPOSITION 7·3. *There is an exact sequence*:

$$\to K^*_\lambda(X) \xrightarrow{r} KO^*(X) \xrightarrow{\cdot\eta_\lambda} KO^*(X;\,\lambda) \to .$$

When λ is trivial this reduces to (7·2). The proof of Proposition 7·3 is given by the next lemma; the diagram in it will be referred to as the *main diagram*; its point is that the right-hand sequence is known to be exact, while the left-hand one contains our target group $K^0_\lambda(X)$ together with groups likely to be known in applications.

LEMMA 7·4. *There exists a commutative diagram as follows, with exact vertical sequences*:

$$\begin{tikzcd}
KO^{-1}(X) \arrow[r, "\cdot\eta_L", "\approx"'] \arrow[d, "\cdot\eta_\lambda"', "q"] & KO^{-1}_{\mathbb{Z}/2}(X;\,L) \arrow[d, "p^*"] \\
KO^{-1}(X;\,\lambda) \arrow[r, "\theta", "\approx"'] \arrow[d] & KO^{-1}_{\mathbb{Z}/2}(S(L\otimes\lambda);\,L) \arrow[d] \\
K^0_\lambda(X) \arrow[r, "\approx"'] \arrow[d, "r"] & KO^0_{\mathbb{Z}/2}(X;\,L\oplus L\otimes\lambda) \arrow[d] \\
KO^0(X) \arrow[r, "\cdot\eta_L", "\approx"'] \arrow[d, "\cdot\eta_\lambda"', "s"] & KO^0_{\mathbb{Z}/2}(X;\,L) \arrow[d, "p^*"] \\
KO^0(X;\,\lambda) \arrow[r, "\theta", "\approx"'] & KO^0_{\mathbb{Z}/2}(S(L\otimes\lambda);\,L).
\end{tikzcd}$$

Proof. The right-hand vertical sequence is the Gysin sequence of the sphere-bundle $S(L\otimes\lambda)$ over X in $\mathbb{Z}/2$-equivariant KO-theory with coefficients in L, which is the $KO_{\mathbb{Z}/2}(\ \ ;\,L)$ sequence of the pair $(D(H\otimes\lambda), S(H\otimes\lambda))$. In this Gysin sequence, we have replaced $KO^*_{\mathbb{Z}/2}(D(L\otimes\lambda), S(L\otimes\lambda);\,L)$ by $KO^*_{\mathbb{Z}/2}(X;\,L\oplus L\otimes\lambda)$ using (1·3) of [**9**], which holds equally well $\mathbb{Z}/2$-equivariantly.

The middle isomorphism is as in Proposition 7·1. We now explain the isomorphisms labelled $.\eta_L$. The paragraph on p. 124 of [**8**] containing (2·3) holds equally well in the real case: recall that the real representation ring $RO\,(\mathbb{Z}/2)$ is the same as $KO^0_{\mathbb{Z}/2}(*) \approx \mathbb{Z}\oplus\mathbb{Z}t$ generated as a ring by the class $[L]$, which we call t, and $t^2 = 1$. The exact sequence of the pair $X \times (D(L), S(L))$ can be written as in (2·3) of [**8**]:

$$0 \to KO^*_{\mathbb{Z}/2}(X;\,L) \to KO^*_{\mathbb{Z}/2}(X) \to KO^*(X) \to 0,$$

which is split exact. In particular when X is replaced by a point, $KO^0_{\mathbb{Z}/2}(*;\,L)$ is infinite cyclic generated by a unique class η_L mapping to $1-t$ in $KO^0_{\mathbb{Z}/2}(*)$. Reverting to our general X, multiplication by η_L gives an isomorphism of $KO^*(X)$ with $KO^*_{\mathbb{Z}/2}(X;\,L)$ as required.

Next we explain the top and bottom squares. As $\mathbb{Z}/2$ acts freely on $S(L\otimes\lambda)$ with X as quotient, $KO^*(X)$ is naturally isomorphic with $KO^*_{\mathbb{Z}/2}(S(L\otimes\lambda))$. Similarly, $\mathbb{Z}/2$ acts freely on $S(L\otimes\lambda)\times(D(L), S(L))$ with quotient $(D(\lambda), S(\lambda))$ over X, giving the natural isomorphisms labelled θ in the main diagram. To see what q and s are, and why these squares commute, it is convenient to consider the commutative diagram:

$$\begin{tikzcd} KO^*(X)\otimes KO^0_{\mathbb{Z}/2}(*;\,L) \arrow[r, "\approx"] \arrow[d, "1\otimes c^*"] & KO^*(X;\,L) \arrow[d, "="] \\ KO^*(X)\otimes KO^0_{\mathbb{Z}/2}(X;\,L) \arrow[r] \arrow[d, "1\otimes p^*"] & KO^*(X;\,L) \arrow[d, "p^*"] \\ KO^*(X)\otimes KO^0_{\mathbb{Z}/2}(S(L\otimes\lambda);\,L) \arrow[r] & KO^*_{\mathbb{Z}/2}(S(L\otimes\lambda);\,L). \end{tikzcd}$$

Here the top isomorphism is given by (external) tensor product and the other horizontal homomorphisms by (internal) tensor product; to make sense of the bottom one we note that $KO^*(X)\approx KO^*_{\mathbb{Z}/2}(S(L\otimes\lambda)\times D(L))$ while $KO^0_{\mathbb{Z}/2}(S(L\otimes\lambda);\,L)\approx KO^0_{\mathbb{Z}/2}(S(L\otimes\lambda)\times(D(L),S(L)))$. We define η_λ to be the image of η_L under the homomorphism induced by the (constant) map $c\circ p$. By commutativity of the diagram, for any α in $KO^*(X)$ we have $p^*(\alpha\,.\,\eta_L)=\alpha\,.\,\eta_\lambda$. Thinking of η_λ as an element of $KO^0(X;\,L)$ via the isomorphism θ, we may define q and s in the main diagram to be multiplication by η_λ, and we see that the top and bottom squares there commute.

By considering the nature of the Karoubi–Segal isomorphism, we see that the third square in the main diagram commutes, where r takes underlying real bundles (here r may be considered as restriction from $C(1\oplus H)$-bundles to $C(1)$-bundles). $\square$

Remark 7·5. Another description of the twisted K-groups uses Atiyah's Real K-theory, [**2**]. Let $\mathrm{i}\lambda$ denote the real vector bundle λ with the involution -1. Then it is easy to identify $\mathbb{C}_\lambda$-bundles over X with Real vector bundles over the double cover $S(\mathrm{i}\lambda)$. We have an equivalence of cohomology theories:

$$K^*_\lambda(X)=KR^*(S(\mathrm{i}\lambda)).$$

This may be used to give another derivation of the exact sequence relating K_λ- and KO-theory as the KR-exact sequence of the pair $(D(\mathrm{i}\lambda),S(\mathrm{i}\lambda))$.

Remark 7·6. There is also a twisted version of Wood's description of the exact sequence (7·2) in terms of the Hopf fibration over the Riemann sphere. Let P_λ denote the $\mathbb{C}_\lambda$-projective space construction on $\mathbb{C}_\lambda$-bundles and H_λ the corresponding Hopf $\mathbb{C}_\lambda$-line bundle. Then there is an equivalence:

$$K^*_\lambda(X)=KO^*(P_\lambda(\mathbb{C}_\lambda\oplus\mathbb{C}_\lambda);\,\mathbb{C}_\lambda-H_\lambda),$$

and the exact sequence of Proposition 7·3 may be identified with the fibrewise cofibre sequence of a fibrewise Hopf map

$$\eta_\lambda\colon(\lambda\oplus\mathbb{C}_\lambda)^+_X\to(\mathbb{C}_\lambda)^+_X,$$

where ${}^+_X$ denotes fibrewise one-point compactification.

Remark 7·7. Although we do not use them here, one can define (by any of the standard methods) twisted Chern classes $c_j(\zeta)$ in $H^{2j}(X;\,\mathbb{Z}(\lambda)^{\otimes j})$ for a $\mathbb{C}_\lambda$-bundle ζ over X, and a corresponding Chern character

$$K^*_\lambda(X)\ \xrightarrow{ch}\ \bigoplus_j H^{*+4j}(X;\,\mathbb{Q})\oplus\bigoplus_j H^{*+4j+2}(X;\,\mathbb{Q}(\lambda)).$$

In fact, taken with the Chern–Pontrjagin characters defined on $KO^{*-1}(X;\,\lambda)$ and on $KO^*(X)$, this is compatible with the exact sequence in Proposition 7·3.

Such twisted Chern classes are (as the referee observes) well known in algebraic geometry (see, for example, [**12**]).

8. *Calculations for projective spaces*

In this section we use the methods of Section 7 to describe $K^0_H(P(\mathbb{R}^{n+1}))$. To state the result, let $h = [\mathbb{C}_H]$ denote the class of $\mathbb{C}_H$ in this group. We write the name of a generator alongside each cyclic group.

THEOREM 8·1. *The groups $K^0_H(P(\mathbb{R}^{n+1}))$ are as follows:*

$$K^0_H(P(\mathbb{R}^{n+1})) = \begin{cases} \mathbb{Z}h & \text{for } n \equiv 0,\ 1,\ 3,\ 7 \mod 8, \\ \mathbb{Z}h \oplus \mathbb{Z}e_n & \text{for } n \equiv 2,\ 6 \mod 8, \\ \mathbb{Z}h \oplus \mathbb{Z}/2\, e_n & \text{for } n \equiv 4,\ 5 \mod 8 \end{cases}$$

where e_n restricts to $\frac{1}{2}a_{n+1}(H-1)$ in $KO^0(P(\mathbb{R}^{n+1}))$, and e_{8k+4}, e_{8k+5} are the restrictions of e_{8k+6} to subspaces.

Together with Proposition 5·4 this reproves the positive part of Theorem 1·1 for $n \equiv 2$, 4, or 6 mod 8. For $e_n + \frac{1}{2}nh$ gives a stable $\mathbb{C}_H$-bundle whose underlying real bundle is easily seen to be stably equivalent to $\frac{1}{2}(n + a_{n+1})H$. It also shows that the methods of Section 6 cannot be used to prove Theorem 1·1 when n is divisible by 8.

The class e_n in Theorem 8·1 can be related to the class constructed explicitly in Section 6: the latter is $e_n + \frac{1}{2}a_{n+1}h$. We omit the proof.

Theorem 8·1 goes slightly further than Proposition (7·1) of [**1**], which concentrates on the image of $K^0_H(P(\mathbb{R}^{n+1}))$ in $KO^0(P(\mathbb{R}^{n+1}))$.

Proof of Theorem 8·1. We take $X = P(\mathbb{R}^{n+1})$ and $\lambda = H$ in the main diagram of Section 7. Then of the groups in the left-hand sequence, $KO^*(P(\mathbb{R}^{n+1}))$ is well known. So too is $KO^*(P(\mathbb{R}^{n+1});\, H)$, for it is the same as $\widetilde{KO}^*(P(\mathbb{R}^{n+1})^H)$ and $P(\mathbb{R}^{n+1})^H$ is homotopy equivalent to $P(\mathbb{R}^{n+2})$.

Our next goal is to show that η_H generates $KO^0(P(\mathbb{R}^{n+1});\, H)$, and s maps $i \oplus j(H-1)$ in $KO^0(P(\mathbb{R}^{n+1}))$ to $(i-2j)\eta_H$ in $KO^0(P(\mathbb{R}^{n+1});\, H)$ for any integers i and j. To see the latter it is enough to show that $[H]\,.\,\eta_H = -\eta_H$.

It is convenient at this stage to observe that η_H may be considered as an element of yet another group; for there is an obvious $\mathbb{Z}/2$-equivariant homeomorphism ϕ making the following diagram commute:

$$\begin{array}{ccc} S(L\otimes H) & \xrightarrow{\phi} & S(L\otimes \mathbb{R}^{n+1}) \\ \Big\downarrow p & & \Big\downarrow p_1 \\ P(\mathbb{R}^{n+1}) & = & P(\mathbb{R}^{n+1}), \end{array}$$

where p_1 is the usual projection. Thus $\eta_H \in KO^0_{\mathbb{Z}/2}(S(L\otimes H);\, L)$ may be considered as an element of $KO^0_{\mathbb{Z}/2}(S(L\otimes\mathbb{R}^{n+1});\, L)$ via the isomorphism induced by ϕ. Equivalently it is the image of η_L under

$$KO^0_{\mathbb{Z}/2}(*;\, L) \approx KO^0_{\mathbb{Z}/2}(D(L\otimes \mathbb{R}^{n+1});\, L) \to KO^0_{\mathbb{Z}/2}(S(L\otimes \mathbb{R}^{n+1});\, L),$$

where the second map is induced by restriction. Also, $[L]$ maps to $[H]$ under the similar map $KO^0_{\mathbb{Z}/2}(*) \to KO^0_{\mathbb{Z}/2}(S(L\otimes\mathbb{R}^{n+1})) \approx KO^0(P(\mathbb{R}^{n+1}))$. Hence to prove $[H]\,.\,\eta_H = -\eta_H$, it is enough to prove $[L]\,.\,\eta_L = -\eta_L$. But η_L has image $t-1 = [L]-1$

under the monomorphism $KO^0_{\mathbb{Z}/2}(*;\, L) \to KO^0_{\mathbb{Z}/2}(*)$. It is therefore enough to show $[L].([L]-1) = -([L]-1)$, which is true since $[L]^2 = 1$.

To see that η_H generates $KO^0(P(\mathbb{R}^{n+1});\, H) \approx KO^0_{\mathbb{Z}/2}(S(L\otimes\mathbb{R}^{n+1});\, L)$, we use surjectivity of the restriction homomorphism:

$$KO^0_{\mathbb{Z}/2}(*;\, L) \approx KO^0_{\mathbb{Z}/2}(D(L\otimes\mathbb{R}^{n+1});\, L) \to KO^0_{\mathbb{Z}/2}(S(L\otimes\mathbb{R}^{n+1});\, L).$$

This holds since the next group in the $KO^0_{\mathbb{Z}/2}(\quad;\, L)$ exact sequence of the pair $(D(L\otimes\mathbb{R}^{n+1}), S(L\otimes\mathbb{R}^{n+1}))$ is

$$KO^1_{\mathbb{Z}/2}((D(L\otimes\mathbb{R}^{n+1}), S(L\otimes\mathbb{R}^{n+1}));\, L) \approx KO^1_{\mathbb{Z}/2}(*;\, L\oplus(n+1)L),$$

and it follows from table 3·1 in [**8**] that this group is zero, since $KO^1(*)$, $K^1(*)$ and $KSp^1(*)$ are zero.

Next we show that the homomorphism q is surjective unless $n \equiv 2 \mod 4$, in which case its cokernel is $\mathbb{Z}$. Since

$$\widetilde{KO}^{-1}(P(\mathbb{R}^{n+1})) = \begin{cases} \mathbb{Z}/2\,\sigma_n & \text{for } n \not\equiv 2 \mod 4,\\ \mathbb{Z}\oplus\mathbb{Z}/2\,\sigma_n & \text{for } n \equiv 2 \mod 4,\end{cases}$$

it is enough to prove that q maps onto $\mathbb{Z}/2\,\sigma_n$ for all n, so since σ_m is the restriction of σ_n for $n \geqslant m \geqslant 1$, it is enough to prove that q maps $KO^{-1}(P(\mathbb{R}^2))$ onto $\widetilde{KO}^{-1}(P(\mathbb{R}^3))$. But this follows from exactness in the main diagram for $n = 1$, since $K^0_H(P(\mathbb{R}^2)) = K^0_H(S^1) = \mathbb{Z}h$.

We are finally ready for the algebraic calculation, based on the main diagram, which will prove Theorem 8·1. It is easy to see that

$$\operatorname{Ker} s = \begin{cases} \mathbb{Z}(1+H) & \text{for } n \equiv 0,1,3,7 \mod 8,\\ \mathbb{Z}(1+H)\oplus\mathbb{Z}/2\ \tfrac{1}{2}a_{n+1}(H-1) & \text{for } n \equiv 2,4,5,6 \mod 8.\end{cases}$$

When $n \not\equiv 2 \mod 4$ this together with exactness in the main diagram completes the calculation, since $\operatorname{Coker} q = 0$. When $n \equiv 2 \mod 4$, we have $\operatorname{Coker} q = \mathbb{Z}$ and there is a short exact sequence

$$0 \to \mathbb{Z} \to K^0_H(P(\mathbb{R}^{n+1})) \to \mathbb{Z}\oplus\mathbb{Z}/2 \to 0.$$

To resolve the extension problem and complete the proof of Theorem 8·1, we use the K_H sequence of the pair $(P(\mathbb{R}^{n+1}), P(\mathbb{R}^n))$. Since the pull-back of H by the attaching map $S^n \to P(\mathbb{R}^{n+1})$ for $P(\mathbb{R}^{n+2})$ is trivial, over S^n the twisted and ordinary K-theories coincide, so the sequence may be written:

$$\tilde{K}^0(S^{n+1}) \to K^0_H(P(\mathbb{R}^{n+2})) \to K^0_H(P(\mathbb{R}^{n+1})) \to \tilde{K}^0(S^n).$$

When $n \equiv 2 \mod 8$, this gives an exact sequence

$$\mathbb{Z} \approx \tilde{K}^0(S^n) \to K^0_H(P(\mathbb{R}^{n+1})) \to K^0_H(P(\mathbb{R}^n)) \approx \mathbb{Z} \to 0,$$

and the extension problem is resolved as in the statement of Theorem 8·1. When $n \equiv 6 \mod 8$, the exact sequence

$$0 \to \mathbb{Z} \approx K^0_H(P(\mathbb{R}^{n+2})) \to K^0_H(P(\mathbb{R}^{n+1})) \to \tilde{K}^0(S^n) \approx \mathbb{Z}$$

resolves the extension problem, while if we take $e_n \in K^0_H(P(\mathbb{R}^{n+1}))$ in this case to map to a generator of $\mathbb{Z}$ in the above sequence, then the commutative diagram

$$\begin{array}{ccc} K^0_H(P(\mathbb{R}^{n+1})) & \longrightarrow & K^0_H(P(\mathbb{R}^{n})) \\ \big\downarrow {\scriptstyle r_n} & & {\scriptstyle r_{n-1}} \big\downarrow {\scriptstyle \approx} \\ \mathbb{Z}\oplus\mathbb{Z}/2 \approx \operatorname{Ker} s_n & \xrightarrow{\approx} & \operatorname{Ker} s_{n-1} \approx \mathbb{Z}\oplus\mathbb{Z}/2 \end{array}$$

shows that the restriction of e_n to $K^0_H(P(\mathbb{R}^n))$ may be taken as e_{n-1}. Similarly its restriction to $K^0_H(P(\mathbb{R}^{n-1}))$ serves as e_{n-2}. This completes the calculation. □

9. *Stable spin structures*

In this section we outline a proof of the following proposition, which is essentially contained in [**13**], and give a corollary for spin^c structures (9·3).

Proposition 9·1. *Suppose that ζ, ζ' are oriented m-dimensional real vector bundles over a finite CW-complex X of dimension $n < m$. Suppose that they are equivalent* (*as oriented bundles*) *and that each has a spin structure. Then there is an* (*oriented*) *equivalence between them which preserves the spin structures.*

This is closely related to a result in [**11**] (also in [**13**]), explained below:

Proposition 9·2. *Any two spin structures on a stable bundle are bundle equivalent.*

To explain these results we recall two equivalent definitions of *spin structure*.

Let p: Spin $(m) \to SO(m)$ denote the usual 2-fold covering map, and for a principal Spin(m)-bundle α with projection π: $P \to X$ let $V(\alpha)$ denote the associated vector bundle over X; thus $V(\alpha)$ has total space $P \times_{\mathrm{Spin}\,(m)} \mathbb{R}^m$, where Spin (m) acts on $\mathbb{R}^m$ via p. In our first definition, a spin structure on an oriented m-plane bundle ζ is a pair $(\alpha,\ f)$, where α is a principal Spin (m)-bundle and f: $V(\alpha) \to \zeta$ is an equivalence of oriented vector bundles. Two such structures $(\alpha,\ f)$ and $(\beta,\ g)$ on the same ζ are said to be *equivalent* if there is an isomorphism θ: $\alpha \to \beta$ of principal Spin (m)-bundles such that the following diagram commutes:

$$\begin{array}{ccc} V(\alpha) & \xrightarrow{f} & \zeta \\ \big\downarrow {\scriptstyle V(\theta)} & & \big\downarrow {\scriptstyle =} \\ V(\beta) & \xrightarrow{g} & \zeta\,. \end{array}$$

On the other hand, the spin structures are said to be *bundle equivalent* if there just exists an isomorphism θ: $\alpha \to \beta$ of principal Spin (m)-bundles. If (α, f) is a spin structure on ζ and k: $\zeta \to \zeta'$ is an oriented bundle equivalence, clearly $(\alpha, k \circ f)$ is a spin structure on ζ'. Finally, when (α, f) and (β, g) are spin structures on oriented vector bundles ζ and ζ', we say that an oriented equivalence k: $\zeta \to \zeta'$ *preserves the spin structures* if $(\alpha, k \circ f)$ and (β, g) are equivalent.

Our second definition is in terms of classifying maps. Let us choose a fixed classifying map $\phi : X \to BSO(m)$ for ζ, and let $Bp : B\,\mathrm{Spin}\,(m) \to BSO(m)$ be the map associated with the homomorphism p. Then a spin structure on ζ is a lift of ϕ to a map ψ: $X \to B\,\mathrm{Spin}\,(m)$ (so $Bp \circ \psi = \phi$). Two such structures are *equivalent* if they are homotopic through lifts of ϕ, and *bundle equivalent* if they are homotopic (not necessarily through lifts of ϕ). Here our maps and homotopies may be free or basepoint-preserving at will, since the target spaces are simply-connected.

We use both these definitions, omitting the proof that they agree. (A non-canonical one-one correspondence between the equivalence classes of spin structures in the two definitions is obtained by choosing a classifying morphism of ζ to the universal bundle over $BSO(m)$. This is covered in [**13**].)

The second approach gives a short proof of Proposition 9·2. For the covering $\mathbb{Z}/2 \to \mathrm{Spin}\,(m) \to SO\,(m)$ gives rise to a fibre sequence

$$SO(m) \xrightarrow{\sigma w_2} P(\mathbb{R}^\infty) = B\mathbb{Z}/2 \longrightarrow B\,\mathrm{Spin}\,(m) \xrightarrow{Bp} BSO(m)$$

and hence to an exact sequence (of groups, since dim $X < m$)

$$[X,\, SO\,(m)] \to [X,\, B\mathbb{Z}/2] \to [X,\, B\,\mathrm{Spin}\,(m)] \to [X,\, BSO(m)].$$

Now the inclusion i: $P(\mathbb{R}^m) \to P(\mathbb{R}^\infty)$ induces a surjection $[X,\, P(\mathbb{R}^m)] \twoheadrightarrow [X,\, P(\mathbb{R}^\infty]$ since dim $X < m$, and the standard map $c:\ P(\mathbb{R}^m) \to SO\,(m)$ satisfies $\sigma w_2 \circ c = i$. Hence $[X, SO\,(m)] \to [X, P(\mathbb{R}^\infty)]$ is surjective, and it follows that Bp_* : $[X,\, B\,\mathrm{Spin}\,(m)] \to [X,\, BSO(m)]$ is injective, proving Proposition 9·2.

We may now prove Proposition 9·1. Let ζ, ζ' be as in the statement of the proposition. Suppose that (α, f) and (β, g) are spin structures on ζ and ζ' in the sense of the first definition, and that h: $\zeta \to \zeta'$ is an equivalence of oriented bundles. Then $(\alpha, h\circ f)$ is also a spin structure on ζ', and by Proposition 9·2 there is an isomorphism θ: $\alpha \to \beta$ of principal Spin (m)-bundles. At this stage we have a diagram (in general non-commutative):

$$\begin{array}{ccc} V(\alpha) & \xrightarrow{f} & \zeta \\ \big\downarrow{\scriptstyle V(\theta)} & & \big\downarrow{\scriptstyle h} \\ V(\beta) & \xrightarrow{g} & \zeta'. \end{array}$$

Now $k = g \circ V(\theta) \circ f^{-1}$ is an oriented equivalence from ζ to ζ' preserving spin structures as required.

Corresponding to the two definitions of spin structure there are analogous definitions of spin^c structure. The analogues of Propositions 9·1 and 9·2 do not in general hold for spin^c structures: a simple counterexample is given by taking $X = S^2$. However, the following corollary of Proposition 9·1 will suffice for our needs in Section 10. In the statement, $\mathbb{T}$ denotes the circle group.

Corollary 9·3. *Proposition* 9·1 *holds with spin replaced by spin*^{c}, *provided we add the following hypothesis*: *the map* $[X, B\mathbb{Z}/2] \to [X, B\mathbb{T}]$ *induced by the inclusion homomorphism* $\mathbb{Z}/2 \hookrightarrow \mathbb{T}$ *is onto.*

Proof. This follows from the proof of Proposition 9·1, together with inspection of the commutative diagram:

$$\begin{array}{ccccccc} [X,\, SO(m)] & \to & [X,\, B\mathbb{Z}/2] & \to & [X,\, B\,\mathrm{Spin}\,(m)] & \to & [X,\, BSO(m)] \\ \big\downarrow{\scriptstyle =} & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle =} \\ [X,\, SO(m)] & \to & [X,\, B\mathbb{T}] & \to & [X,\, B\,\mathrm{Spin}^c(m)] & \to & [X,\, BSO(m)], \end{array}$$

where the lower sequence arises from the exact sequence of homomorphisms

$$1 \to \mathbb{T} = U(1) \to \mathrm{Spin}^c(m) \to SO(m) \to 1. \quad \square$$

Another way of expressing the additional hypothesis is: every complex line bundle over X is the complexification of some real line bundle.

10. *The case* $n \equiv 0 \mod 4$

In this section we complete the proof of Theorem 1·1 by establishing:

PROPOSITION 10·1. *Suppose* $n \equiv 0 \mod 4$. *Then there exists a real vector bundle* ξ *of dimension* n *over* $P(\mathbb{R}^{n+1})$ *such that* $\xi = H \otimes \xi$ *and* ξ *is stably equivalent to* $\frac{1}{2}(n + a_{n+1})H$.

Proof. The construction of a suitable ξ can be carried out as follows. Let $n = 4k$ for some integer k. First for any non-negative integers r, s with $r + s \geqslant 2k$, consider the vector bundle $\zeta = \mathbb{R}^{2r} \oplus H \otimes \mathbb{R}^{2s}$ over $P(\mathbb{R}^{4k+1})$. Then ζ is the real bundle underlying the complex bundle $\mathbb{C}^r \oplus H \otimes \mathbb{C}^s$. The latter desuspends, uniquely, to a complex bundle η of complex dimension $2k$; thus

$$\eta \oplus \mathbb{C}^{r+s-2k} = \mathbb{C}^r \oplus H \otimes \mathbb{C}^s; \tag{10·2}$$

for the obstructions to desuspension lie in $H^*(P(\mathbb{R}^{4k+1}); \pi_{*-1}(U(r+s)/U(2k)))$, and $\pi_i(U(r+s)/U(2k)) = 0$ for $i \leqslant 4k+2$.

We now specialise to the case $s = k + 2^{2k-2+e}$, where e is 0 for k even, 1 for k odd. Then the real bundle ξ underlying η is stably equivalent to $\frac{1}{2}(n + a_{n+1})H$.

Hence ξ and $H \otimes \xi$ are stably equivalent by Proposition 4·1.

Remark 10·3. For k odd, it is easy to check that η and $H \otimes \eta$ are stably equivalent complex bundles. Hence, as above, they are isomorphic. This gives at once an easy verification of Proposition 10·1 in the case that $n \equiv 4 \mod 8$.

To prove $\xi = H \otimes \xi$ for any n divisible by 4 we use the following lemma; the meaning of the Euler classes appearing in it will be explained shortly.

PROPOSITION 10·4. *Let* X *be a connected closed manifold of even dimension* $2m$, *such that* w_1X *is non-zero and* w_2X *is reduction of an integral class in* $H^2(X; \mathbb{Z})$. *Let* ξ, ξ' *be* $2m$*-dimensional spin*^{c} *bundles over* X, *and suppose that there exists a stable isomorphism* $f\colon \xi \oplus \mathbb{R}^N \to \xi' \oplus \mathbb{R}^N$ (*where* $N \geqslant 1$) *under which the spin*^{c} *structures correspond. Then* f *desuspends to an isomorphism of* ξ *to* ξ' *if and only if the Euler classes of* $L \otimes \xi$ *and* $L \otimes \xi'$ *in* $K^0_{\mathbb{Z}/2}(X)$ *are equal.*

Let us assume this proposition for the moment. Then it remains to check that it applies to the ξ and $H \otimes \xi$ of Proposition 10·1.

The complex structures on η, $H \otimes \eta$ define spin^c structures on ξ, $H \otimes \xi$. Since ξ and $H \otimes \xi$ are stably equivalent, it follows that with the orientations given by their spin^c structures they are stably oriented equivalent. By Corollary 9·3 there is a stable oriented equivalence between them which preserves their spin^c structures.

We now show that the Euler classes of $L \otimes \xi$ and $L \otimes H \otimes \xi$ in $K^0_{\mathbb{Z}/2}(P(\mathbb{R}^{4k+1}))$ are equal; this is more technical.

Recall from [**9**] how such Euler classes are defined. For an arbitrary real vector bundle ξ over X the K-theory Euler class $\gamma(\xi)$ is defined (as the Hurewicz image of the stable cohomotopy Euler class) in $K^0(X; -\xi)$. When ξ has even dimension $2m$ and is equipped with a spin^c structure, we may use the associated Bott class $u \in K^0(X; \xi)$ to define a K-theory Euler class $e(\xi) = u \,.\, \gamma(\xi) \in K^0(X)$, depending, of course, on the choice of spin^c structure. By construction, it is the restriction to the zero-section X in ξ of the Bott class u. This is similar to the situation in ordinary cohomology already touched on in Section 3; the Euler class we dealt with there in $H^n(X; \mathbb{Z}(\zeta))$ is analogous to the Euler class in $K^0(X; -\xi)$ here, while the Euler class in $K^0(X)$ here corresponds to the Euler class in $H^n(X;\ \mathbb{Z})$, which depends on a choice of orientation for ζ.

Recall that a spin^c structure on ξ consists of a principal $\mathrm{Spin}^c(2m)$-bundle P and an isomorphism

$$P \times_{\mathrm{Spin}^c(2m)} \mathbb{R}^{2m} \to \xi.$$

Let S^+ and S^- denote the standard irreducible complex $\mathrm{Spin}^c(2m)$-modules of dimension 2^m, and write $S^+(\xi)$ and $S^-(\xi)$ for the associated vector bundles over X. Then the K-theory Euler class $e(\xi)$ is the difference $[S^+(\xi)] - [S^-(\xi)]$ in $K^0(X)$. The Bott class u can be described as follows. The pair $(S^+(\xi), S^-(\xi))$ has the structure of a graded module over the complex Clifford algebra bundle $C(\xi)$, and u is represented, using K-theory with compact supports, by Clifford multiplication:

$$v\colon S^+(\xi) \to S^-(\xi)$$

over $v \in \xi$ (see [**3**]).

We need a $\mathbb{Z}/2$-equivariant Bott class for $L \otimes \xi$ in order to define an Euler class $e(L \otimes \xi) \in K^0_{\mathbb{Z}/2}(X)$. This can be written explicitly. The map above is compatible with the involution -1 on ξ and on $S^-(\xi)$ (and $+1$ on $S^+(\xi)$) and gives

$$S^+(\xi) \to L \otimes S^-(\xi)$$

over $L \otimes \xi$, defining a Bott class in $K^0_{\mathbb{Z}/2}(X;\ L \otimes \xi)$. The associated Euler class is

$$e(L \otimes \xi) = [S^+(\xi)] - [S^-(\xi)]\,.\,t \in K^0(X) \otimes (\mathbb{Z} \oplus \mathbb{Z}t),$$

where $t = [L]$ as before.

(Notice that the same construction defines a spin^c structure and Euler class for $\lambda \otimes \xi$, for any real line bundle λ.)

In the calculations which follow, ξ will be the real bundle underlying a complex vector bundle η. We take the natural spin^c structure determined by the complex structure with $S^+(\xi)$ and $S^-(\xi)$ the sums of the even and odd complex exterior powers $\Lambda^j \eta$, respectively.

The condition in Proposition 10.4 can be made quite explicit: $S^+(\xi) = S^+(\xi')$ and $S^-(\xi) = S^-(\xi')$. (We are in the stable range.)

Remark 10·5. The Euler class is unchanged by an orientation-preserving equivalence $\xi \to \xi$. Changing the spin^c structure by a class in $H^2(X;\mathbb{Z})$ multiplies the Euler class by the corresponding complex line bundle. Changing the orientation of ξ interchanges S^+ and S^-.

As in the real case treated in Section 8 we have

$$K^0_{\mathbb{Z}/2}(P(\mathbb{R}^{4k+1})) \approx K^0(P(\mathbb{R}^{4k+1})) \otimes K^0_{\mathbb{Z}/2}(*) \approx (\mathbb{Z} \oplus \mathbb{Z}/2^{2k}\,x) \otimes (\mathbb{Z} \oplus \mathbb{Z}t),$$

where x is the class of $H - 1$.

Lemma 10·6. *With the above notation, the $K_{\mathbb{Z}/2}$-Euler class of the complex bundle $L \otimes \xi$ is*

$$2^{2k-1}(1-t) - 2^{2k-2}(\epsilon_+ - \epsilon_- t)x \in K^0_{\mathbb{Z}/2}(P(\mathbb{R}^{4k+1})),$$

where

$$\epsilon_- = \sum_{1\leqslant i<2k} (-1)^i \binom{s}{i} \bmod 4, \quad \epsilon_+ = \epsilon_- + 2\binom{s}{2k} \bmod 4.$$

Proof. We shall lift from $\mathbb{Z}/2$ to $\mathbb{T}$-equivariant K-theory. The motive for doing so is as follows: $K^0_{\mathbb{Z}/2}(*) \approx \mathbb{Z}\oplus\mathbb{Z}t$ has divisors of zero; for example $t^2 = 1$, so $(t-1)(t+1) = 0$. On the other hand $K^0_{\mathbb{T}}(*) \approx \mathbb{Z}[z, z^{-1}]$, where z is the class of the standard 1-dimensional complex representation E of $\mathbb{T}$, and this has no divisors of zero. We shall therefore calculate in $K^0_{\mathbb{T}}(P(\mathbb{R}^{4k+1})) \approx K^0(P(\mathbb{R}^{4k+1})) \otimes K^0_{\mathbb{T}}(*)$ and at the end substitute t for z to get the answer in $K^0_{\mathbb{Z}/2}(P(\mathbb{R}^{4k+1}))$; thus initially we deal with Euler classes in $\mathbb{T}$-equivariant K-theory.

We compute the Euler class of $E \otimes_{\mathbb{C}} \eta$ (thinking of E as $\mathbb{C}$ with $\mathbb{T}$ acting by left multiplication). It is an element $A(z) + B(z)x$, say, of $K^0_{\mathbb{T}}(P(\mathbb{R}^{4k+1}))$ which is $(\mathbb{Z} \oplus \mathbb{Z}/2^{2k}\, x) \otimes \mathbb{Z}[z, z^{-1}]$. By (10·2) above and multiplicativity of Euler classes, we have

$$e(E \otimes_{\mathbb{C}} \eta).e((r + s - 2k)E) = e(rE \oplus sE \otimes H).$$

Now as in (2.2) of [**8**] we have $e(E) = 1 - z$ and similarly

$$e(E \otimes H) = 1 - (1 + x)z.$$

Hence

$$(A(z) + B(z)x)(1 - z)^{r+s-2k} = (1 - z)^r \,.\, (1 - (1 + x)z)^s.$$

It follows that the Euler class we seek is

$$(1 - z)^{2k} \sum_{i\geqslant 0} \binom{s}{i} (-z/(1 - z))^i x^i,$$

which, after a short manipulation using $(1 + x)^2 = 1$ and $2^{2k}x = 0$, becomes

$$(1 - z)^{2k} - \sum_{1\leqslant i\leqslant 2k} \binom{s}{i} z^i (1 - z)^{2k-i} 2^{i-1} x.$$

Now as described earlier, we replace z by t. Noting that $t^2 = 1$, we get that the Euler class of $L \otimes \xi$ in $K^0_{\mathbb{Z}/2}(P(\mathbb{R}^{4k+1}))$ is as stated in the lemma. □

Now if we began with $H \otimes \xi$ in place of ξ, we would get a $K_{\mathbb{T}}$-Euler class $A_1(z) + B_1(z)x$ where

$$(A_1(z) + B_1(z)x)(1 - (1 + x)z)^{r+s-2k} = (1 - (1 + x)z)^r \,.\, (1 - z)^s.$$

and calculating as above we would get $K_{\mathbb{Z}/2}$-Euler class as in Lemma 10·6 except with t replaced by $(1 + x)t$. The difference between the Euler classes of $L \otimes \xi$ and $L \otimes H \otimes \xi$ is therefore

$$2^{2k-1}xt + 2^{2k-2}\epsilon_- xtx = 2^{2k-1}xt(1 - \epsilon_-),$$

and this is zero provided ϵ_- is odd, since $2^{2k}x = 0$. But working mod 2 we get, for $i < 2k$,

$$\binom{s}{i} = \binom{k + 2^{2k-2+e}}{i} = \binom{k}{i} \bmod 2$$

provided $k \geqslant 1$, so

$$\epsilon_- = \sum_{1\leqslant i<2k} \binom{k}{i} = \sum_{1\leqslant i\leqslant k} \binom{k}{i} = 2^k - 1 \mod 2$$

and ϵ_- is odd as required.

Thus the Euler classes of $L \otimes \xi$ and $L \otimes H \otimes \xi$ in $K^0_{\mathbb{Z}/2}(P(\mathbb{R}^{4k+1}))$ are equal, and the proof of Proposition 10·1 is complete once we prove Proposition 10·4. □

Proof of Proposition 10·4. We begin by explaining how the relevant obstruction theory fits into the framework described in Section 1 of [**8**]. There is an obvious fibrewise inclusion i_0 of the trivial bundle $\mathbb{R}^N$ over X into $\xi \oplus \mathbb{R}^N$, and a similar inclusion i' into $\xi' \oplus \mathbb{R}^N$. The composition $i_1 = f^{-1} \circ i'$ gives another inclusion of $\mathbb{R}^N$ into $\xi \oplus \mathbb{R}^N$. To desuspend f to an isomorphism of ξ with ξ' we are interested in extending the inclusion of $\mathbb{R}^N$ into $\xi \oplus \mathbb{R}^N$ over $X \times \dot{I}$ given by i_0 and i_1 to an inclusion over $X \times I$. Thus as in [**8**] the obstruction is a relative Euler class in stable cohomotopy

$$\begin{aligned}\omega^0((X\times I,\, X\times \dot{I})\times P(\mathbb{R}^N);\, -H\otimes(\xi\oplus\mathbb{R}^N))&\\ \approx \omega^{-1}(X\times P(\mathbb{R}^N);\, -H\otimes(\xi\oplus\mathbb{R}^N)) &\approx \mathbb{Z}/2.\end{aligned}$$

In fact this obstruction group maps isomorphically all the way down to ordinary cohomology (with coefficients in $\mathbb{Z}(\xi) \approx \mathbb{Z}$), but in order to detect the obstruction as a difference of Euler classes we again concentrate on $K_{\mathbb{Z}/2}$-theory. Note that since f preserves spin^c structures, f^* maps the $K_{\mathbb{Z}/2}$-Euler class of $L \otimes \xi'$ to that of $L \otimes \xi$.

As on p. 119 of [**8**] we lift to $\mathbb{Z}/2$-equivariant theory, and then as in Section 3 of [**8**] pass to $K_{\mathbb{Z}/2}$-theory, to get an obstruction in $K^{-1}_{\mathbb{Z}/2}(X \times S(NL);\, -L \otimes (\xi \oplus N))$. We shall look at the image under the coboundary map

$$\begin{aligned}\delta\colon\ & K^{-1}_{\mathbb{Z}/2}(X\times S(NL);\, -L\otimes(\xi\oplus N))\\ &\to K^0_{\mathbb{Z}/2}(X\times(D(NL),S(NL));\, -L\otimes(\xi\oplus N)) \approx K^0_{\mathbb{Z}/2}(X;\, -L\otimes\xi),\end{aligned}$$

where the isomorphism follows as before from (1.3) of [**9**]. As in Section 1 of [**8**], the image will be the difference of the Euler classes of i_0 and i_1. We just need to check that this detects the obstruction, by showing that δ is injective.

In order to do this, we consider relative groups of (X, Y), where Y is the complement of an open $2m$-disc in X. We have the following commutative diagram:

$$\begin{array}{ccc}
K^{-1}_{\mathbb{Z}/2}((X,Y)\times S(NL);-(2m+N)L) & \xrightarrow{\delta} & K^0_{\mathbb{Z}/2}((X,Y);-2mL)\\
\big\downarrow\approx & & \big\downarrow\approx\\
K^{-1}_{\mathbb{Z}/2}((X,Y)\times S(NL);-L\otimes(\xi\oplus N)) & \xrightarrow{\delta} & K^0_{\mathbb{Z}/2}((X,Y);-L\otimes\xi)\\
\big\downarrow & & \big\downarrow\\
K^{-1}_{\mathbb{Z}/2}(X\times S(NL);-L\otimes(\xi\oplus N)) & \xrightarrow{\delta} & K^0_{\mathbb{Z}/2}(X;-L\otimes\xi)\\
\big\downarrow\approx & & \big\downarrow\approx\\
K^{-1}_{\mathbb{Z}/2}(X\times S(NL);-(2m+N)L) & \xrightarrow{\delta} & K^0_{\mathbb{Z}/2}(X;-2mL).
\end{array}$$

Here all the indicated isomorphisms arise from Bott periodicity, since ξ is a spin^c bundle (as is $2mL$).

Since $K^{-1}_{\mathbb{Z}/2}(X \times S(NL); -L \otimes (\xi \oplus N)) \approx \mathbb{Z}/2$, to show that third δ is injective it is sufficient to show that the composition across the top and down the right-hand side is non-zero.

The right-hand vertical is

$$\begin{array}{c} K^0_{\mathbb{Z}/2}((X,Y); -2mL) \approx \tilde{K}^0(S^{2m}) \otimes K^0_{\mathbb{Z}/2}(*; -2mL) \approx \tilde{K}^0(S^{2m}) \otimes (\mathbb{Z}1 \oplus \mathbb{Z}t) \\ \big\downarrow \\ K^0_{\mathbb{Z}/2}(X; -2mL) \approx K^0(X) \otimes K^0_{\mathbb{Z}/2}(*; -2mL) \approx K^0(X) \otimes (\mathbb{Z}1 \oplus \mathbb{Z}t). \end{array}$$

We shall show below that the image of the map $\tilde{K}^0(S^{2m}) \to K^0(X)$ has order 2.

It is therefore sufficient to show that $\tilde{K}^0(S^{2m}) \otimes (1 + t)$ is the image of the top δ in the diagram. One simple way to check this is as follows: since

$$\begin{aligned} &K^{-1}_{\mathbb{Z}/2}((X,Y) \times S(NL); -(2m+N)L) \\ &\qquad \approx K^{-1}_{\mathbb{Z}/2}(S^{2m} \times S(NL); -(2m+N)L) \approx K^{-1}_{\mathbb{Z}/2}(S(NL); -NL), \end{aligned}$$

we may regard the top δ in the diagram as a map

$$K^{-1}_{\mathbb{Z}/2}(S(NL); -NL) \to K^0_{\mathbb{Z}/2}((X,Y); -2mL) \approx K^0_{\mathbb{Z}/2}(*) \approx \mathbb{Z} \oplus \mathbb{Z}t.$$

Now we can reduce to the case when $N = 1$ by the commutative diagram:

$$\begin{tikzcd} K^{-1}_{\mathbb{Z}/2}(S(L); -L) \arrow[r, "\delta"] \arrow[d, "\approx"] & K^0_{\mathbb{Z}/2}(D(L), S(L); -L) \arrow[d, "\approx"] \\ K^{-1}_{\mathbb{Z}/2}((S(NL), S((N-1)L)); -NL) \arrow[r] \arrow[d] & K^0_{\mathbb{Z}/2}(*) \arrow[d, "\approx"] \\ K^{-1}_{\mathbb{Z}/2}(S(NL); -NL) \arrow[r, "\delta"] & K^0_{\mathbb{Z}/2}(*). \end{tikzcd}$$

The next homomorphism in the top sequence here is

$$K^0_{\mathbb{Z}/2}(D(L), S(L); -L) \to K^0_{\mathbb{Z}/2}(D(L); -L) \approx K^0_{\mathbb{Z}/2}(*),$$

and we may read off the fact that $1 + t$ is in $\delta(K^{-1}_{\mathbb{Z}/2}(S(L); -L))$ from table 3·1 of [**8**].

To complete the proof we must look at the map $\tilde{K}^0(S^{2m}) \to K^0(X)$. Using periodicity and duality, and letting τ denote the tangent bundle of X, we can rewrite this as the map

$$\tilde{K}_0(S^0) \to \tilde{K}_0(X^{2m-\tau}) \tag{10·7}$$

in K-homology induced by the inclusion of the bottom cell in the Thom complex of the stable normal bundle. The condition that $w_2 X$ should lift to an integral class is equivalent to the existence of a spin^c structure on $\tau \oplus \lambda$, where λ is the determinant line bundle of τ. (It is also the condition that τ admit a pin^c structure; but we do not need this interpretation.) The Bott isomorphism given by a choice of spin^c structure identifies (10·7) with the homomorphism

$$\tilde{K}_0(S^0) \to \tilde{K}_0(X^{\lambda-1})$$

induced again by the inclusion of the bottom cell of the Thom complex. Let ℓ: $X \to P(\mathbb{R}^\infty)$ be the classifying map of λ. Composition with

$$\ell_*\colon \tilde{K}_0(X^{\lambda-1}) \to \tilde{K}_0(P(\mathbb{R}^\infty)^{H-1})$$

maps $\mathbb{Z} = \tilde{K}_0(S^0)$ non-trivially to $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z} = \tilde{K}_0(P(\mathbb{R}^\infty)^{H-1})$. This verifies that the K-theory Euler classes detect the $\mathbb{Z}/2$ obstruction to desuspension. □

Remark 10·8. The hypotheses of Proposition 10·4 can be weakened to cover the case that ξ is orientable but not necessarily spin^c if we replace the condition that w_2X lift to an integral class by the integrality of $w_2(\tau - \xi)$. The K-theory Euler class must then be considered in $K^0_{\mathbb{Z}/2}(X;\, -L \otimes \xi)$.

Remark 10·9. We have seen in Section 7 that the case $n \equiv 0 \bmod 8$ cannot be dealt with using twisted $\mathbb{C}_\lambda$-structures. As noted in Remark 5·5, such structures correspond to skew-symmetric non-degenerate bilinear forms with values in λ. One might look instead at symmetric forms: $\xi \otimes \xi \to \lambda$ over X. These correspond, up to homotopy, to maps $T\colon \lambda \otimes \xi \to \xi$ with 'square' $+1$. Such pairs (ξ, T) are in 1–1 correspondence with real vector bundles over the double cover $S(\lambda)$: the pull-back to the double cover has an honest involution and we take the $+1$-eigenspace. (Cf. Remark 7·5.)

In our projective space example: $X = P(\mathbb{R}^{4k+1})$, with k even, if our $4k$-dimensional bundle admitted a non-singular symmetric form with values in H, then it would correspond to a stably non-trivial $2k$-bundle over the $4k$-sphere. But there is no such bundle, because $\pi_{4k-1}(O(2k))$ is finite.

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