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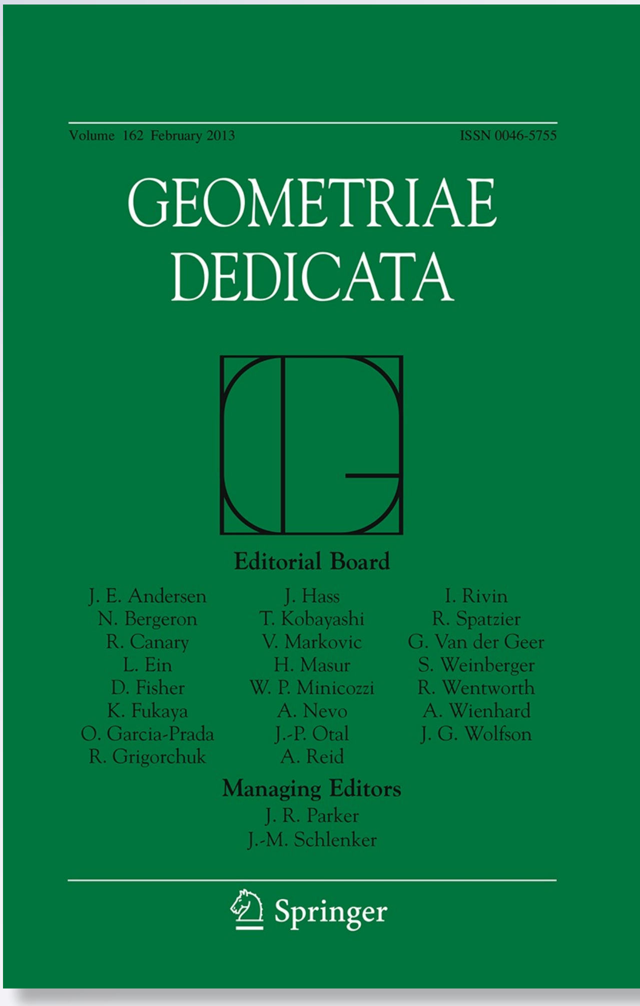

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ORIGINAL PAPER

Cellular decomposition of quaternionic spherical space forms

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Abstract We obtain an explicit cellular decomposition of the quaternionic spherical space forms, manifolds of positive constant curvature that are factors of an odd sphere by a free orthogonal action of a generalized quaternionic group. The cellular structure gives and explicit description of the associated cellular chain complex of modules over the integral group ring of the fundamental group. As an application we compute the Whitehead torsion of these spaces for any representation of the fundamental group.

Keywords Quaternionic groups · Spherical space forms · Cellular decomposition

Mathematics Subject Classification 57N60 · 57Q10

1 Introduction

Explicit cellular decompositions of spaces are fundamental devices in algebraic topology. In general, however, beside the trivial classical cases of spheres and projective spaces, such cellular decompositions are not explicitly known. A first non trivial case, where cellular decomposition and chain complex are explicitly known, is that of the lens spaces. Lens spaces are the simplest example of spherical space forms, that are the manifolds obtained as quotients of a sphere by a free orthogonal action of some finite group acting through some orthogonal representation. Lens spaces correspond to spherical space forms with cyclic fundamental group, and were originally investigated precisely in the context of solving the so

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called *spherical space forms problem* (see for example [5] and [8] for an overview on the spherical space forms problem), namely the classification of the spherical space forms. In order to study the topological type of the lens spaces, Franz [4], Reidemeister [7] and De Rham [3] introduced an invariant, called Franz–Reidemeister torsion, that was consequently generalized by Whitehead to the Whitehead torsion [6]. Producing an explicit cellular decomposition and the associated chain complex of lens spaces, they were able to give the simple homotopy classification of these spaces, and from that the topological classification follows [2,6]. Afterwards, instead of proceeding along this line of investigation, the spherical space forms problem was actually solved by means of the results of several authors in group theory and representation theory. We address the interested readers to the book of Wolf [8]. Going back to the original line of investigation, the purpose of this work is to present an explicit cellular decomposition of the spherical space forms with fundamental group a generalized quaternionic group, that we call *quaternionic spherical space forms*. The method used to obtain such a decomposition is geometric, and generalizes in some sense the one used for the lens spaces. Once we have the cellular decomposition, we easily obtain the explicit description of the associated cellular chain complex of modules over the integral group ring of the fundamental group. With this complex, we can compute all homology and cohomology groups with respect to any representation, namely with any twist of the coefficients. We can also compute the Whitehead torsion, that as for the lens spaces gives the simple homotopy classification of the quaternionic spherical space forms. We present these calculations as an application of our construction. We conclude observing that the approach described in this work is developed with the aim of investigating the other non abelian cases, that are being studied in some works in progress.

2 Quaternionic spherical space forms: preliminaries and notations

Let t be a positive integer. We denote by $\mathbf{Q}_{4t}$ the *group of generalized quaternions* of order $4t$. This is the group with two generators and presentation $\mathbf{Q}_{4t} = \langle x, y : x^t = y^2, yxy^{-1} = x^{-1}\rangle$. It is easy to see that the elements of $\mathbf{Q}_{4t}$ can be listed as follows: $\mathbf{Q}_{4t} = \{x^0, x, x^2, \ldots, x^{2t-1}, x^0y, xy, \ldots, x^{2t-1}y\}$, namely that $\mathbf{Q}_{4t} = C_{2t} \cup C_{2t}y$, where $C_{2t} = \langle x : x^{2t} = 1\rangle$ is a cyclic subgroup of index 2. We introduce two particular elements: $N_x = 1 + x + x^2 + \cdots + x^{t-1}$ and $L_x = 1 + x + x^2 + \cdots + x^{2t-1}$, that will be useful in the following.

Let S^{4n-1} be the unit sphere in the real $4n$-space, with the standard orientation. Identify S^{4n-1} with the unit sphere in $\mathbb{C}^{2n}$, $\{(z_1, z_2, \ldots, z_{2n}) \in \mathbb{C}^{2n} \mid \sum_{j=1}^{2n} |z_j|^2 = 1\}$. Let $\theta = \frac{\pi}{t}$, $\zeta = \mathrm{e}^{i\theta}$ a primitive root of unity, $q_1, q_2, \ldots, q_n$ integers (not necessarily distinct) with $(q_l, 2t) = 1$, and $r_1, r_2, \ldots, r_n$ their multiplicative inverse modulo $2t$. The unitary representation:

$$\alpha_{q_j} : \mathbf{Q}_{4t} \to U(2, \mathbb{C}),$$
$$\alpha_{q_j} : \begin{cases} x \mapsto \begin{pmatrix} \zeta^{q_j} & 0 \\ 0 & \zeta^{-q_j} \end{pmatrix}, \\ y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{cases}$$

defines a unitary representation

$$\alpha = \alpha_{q_1,\ldots,q_n} = \alpha_{q_1} \oplus \cdots \oplus \alpha_{q_n} : \mathbf{Q}_{4t} \to U(2n, \mathbb{C}), \tag{1}$$

whose explicit action is

$$\begin{aligned}&\alpha(x,(z_1,z_2,\ldots,z_{2n}))=(\zeta^{q_1}z_1,\zeta^{-q_1}z_2,\zeta^{q_2}z_3,\zeta^{-q_2}z_4,\ldots,\zeta^{q_n}z_{2n-1},\zeta^{-q_n}z_{2n}),\\&\alpha(y,(z_1,z_2,\ldots,z_{2n}))=(z_2,-z_1,\ldots,z_{2n},-z_{2n-1}).\end{aligned}\tag{2}$$

The quotient space of the action α, $Q(4t;q_1,\ldots,q_n)=S^{4n-1}/\alpha_{q_1,\ldots,q_n}(\mathbf{Q}_{4t})$, is called *quaternionic spherical space form*. We will also use the simplified notation $Q(4t)$, when no ambiguity arises.

We conclude this section with some remarks that will be useful in the applications in Sect. 6. If we identify S^{4n-1} with the unit sphere in quaternionic n-space, $S^{4n-1}=\{(w_1,w_2,\ldots,w_n)\in\mathbb{H}^n\mid\sum_{j=1}^n|w_j|^2=1\}$, viewing $\mathbb{H}$ as a left $\mathbb{H}$-module, the inclusion of $\mathbf{Q}_{4t}$ into $\mathbb{C}\oplus j\mathbb{C}=\mathbb{H}$, induced by the identification

$$\alpha_q:\begin{cases}x\mapsto\zeta^q,\\y\mapsto j,\end{cases}$$

gives the left $\mathbf{Q}_{4t}$ action on S^3. On S^{4n-1} this reads as the linear transformation of $\mathbb{H}$-modules defined by the representation

$$\begin{aligned}\alpha_{q_1,\ldots,q_n}&:\mathbf{Q}_{4t}\to Gl(n,\mathbb{H}),\\\alpha_{q_1,\ldots,q_n}=\alpha_{q_1}\oplus\cdots\oplus\alpha_{q_n}&:\begin{cases}x\mapsto\zeta^{q_1}\oplus\cdots\oplus\zeta^{q_n},\\y\mapsto j\oplus\cdots\oplus j.\end{cases}\end{aligned}\tag{3}$$

Whenever the representation is not relevant, we will use the notation gX to denote the action of the group element g on the subset X.

Lemma 1 *All the possible free actions of the generalized quaternionic groups on a sphere are of the form given in Eq.* (1) *(or in Eq.* (3)*).*

Lemma 2 *Any quaternionic spherical space form is homeomorphic to one of the type* $Q(4t;q_1,\ldots,q_n)$, *with* $1\le q_1\le q_2\le\cdots\le q_n<t$.

Lemma 1 follows by direct verification using the classification of Wolf [8, Theorem 6.1.11, (7.4.1), 7.2.18, and 5.5.6]. The proof of Lemma 2 follows the same line as the proofs of the similar results for lens spaces in [6, Section12].

3 Curved join

Given two unitary complex numbers $z_1,z_2\in\mathbb{C}$, consider the ordered pair (z_1,z_2) in $\mathbb{C}\times\mathbb{C}=\mathbb{R}^4$. Since there exists only one plane in $\mathbb{R}^4$ through the origin, z_1 and z_2, we can take the oriented arc from z_1 to z_2 in the unitary circle on this plane and with length $\pi/2$. We denote this arc by z_1*z_2 and its end points by $z_1*\emptyset$ and $\emptyset*z_2$. If the two points, say w_1,w_2, lay on the same circle, we use the notation $[w_1,w_2]$ for the oriented arc from w_1 to w_2. For any two subsets Z_1 and Z_2, with $Z_1\times Z_2\subset S^1\times S^1\subset\mathbb{C}\times\mathbb{C}$, we define their *curved joint* by

$$Z_1*Z_2=\{z_1*z_2\mid z_1\in Z_1,z_2\in Z_2\}.\tag{4}$$

For example: $S^1*S^1=S^3$. This process generalizes as follows. Identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$, and, given the standard orthonormal basis $\{e_1,\ldots,e_{2m}\}$ of $\mathbb{R}^{2m}$, for each $r\neq s$, denote by $\Pi_{r,s}$ the plane generated by $\{e_r,e_s\}$. Suppose $\Pi_{r_1,s_1}\cap\Pi_{r_2,s_2}=\{0\}$. Let Z_1 and Z_2 be subsets of the unit circles of Π_{r_1,s_1} and Π_{r_2,s_2}, respectively. Then the curved joint Z_1*Z_2 is well defined by Eq. (4). In particular, let Σ_l be the unit circle laying in the lth complex hyperplane of $\mathbb{C}^{2n}$. Then, we have an homeomorphism of the iterated curved joint $\Sigma_1 * \cdots * \Sigma_{2n}$ with S^{4n-1}.

To deal with the general case it is useful to identify the basis vector with their final points, and to use the e_j to denote the points in S^{4n-1}. Then, for example the canonical base of $\mathbb{C}^2$ in S^3 is

$$e_1 = 1 * \emptyset, \quad e_2 = i * \emptyset, \quad e_3 = \emptyset * 1, \quad e_4 = \emptyset * i.$$

We conclude by recalling the following formula for the boundary:

$$\partial(X * Y) = \partial(X) * Y + (-1)^{\dim X+1} X * \partial(Y). \tag{5}$$

4 Cellular decomposition of quaternionic spherical space forms

In this section we give a fundamental domain for the action α of $\mathbf{Q}_{4t}$ over S^{4n-1}, and we describe a cellular decomposition of the space $Q(4t)$. For we first consider the simpler three dimensional case, and then we generalize the construction to higher dimensions. Recall that the notation gK, where $g \in \mathbf{Q}_{4t}$, and K is a subset of S^{4n-1}, means that g acts by α on all the points of K. It is easy to verify that $\alpha_p(\mathbf{Q}_{4t}) = \alpha_q(\mathbf{Q}_{4t})$, for any p, q. Whence the fundamental domain does not depend on the representation.

Lemma 3 *The fundamental domain for the action of $\mathbf{Q}_{4t}$ on S^3 is the set $\mathcal{F}_{4t,3} = [\zeta, 1] * [1, -1] = [\zeta e_1, e_1] * [e_3, -e_3]$ (recall $\zeta = \mathrm{e}^{\frac{\pi i}{t}}$).*

Proof The proof is by direct verification. Since the fundamental domain does not depend on the representation, fix $q = 1$. First we observe that the union of the translated $g\mathcal{F}_{4t,3}$, for $g \in \mathbf{Q}_{4t}$ covers all S^3, and second that for each $g \neq h \in \mathbf{Q}_{4t}$, $g\mathcal{F}_{4t,3}$ and $h\mathcal{F}_{4t,3}$ intersect only along the boundaries. We give some details for the case $t = 2$ and $q = 1$. Then, by definition in Eq. (3), $\alpha_1 : \mathbf{Q}_{4t} \to U(2, \mathbb{C})$ is

$$\alpha_1(x) = \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix} \quad \text{and} \quad \alpha(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In other words, x determines a counter clockwise rotation of θ in the plane $\Pi_{1,2}$ and a clockwise rotation of θ in the plane $\Pi_{3,4}$; y first switches these two planes, and then rotates $\Pi_{3,4}$ by π. Observing that the canonical base of $\mathbb{C}^2$ is $e_1 = 1 * \emptyset$, $e_2 = \zeta e_1 = \zeta * \emptyset$, $e_3 = \emptyset * 1$, $e_4 = \zeta e_3 = \emptyset * \zeta$, we find that the action α_1 is

$$\begin{aligned} xe_1 &= e_2, \quad xe_2 = -e_1, \quad xe_3 = -e_4, \quad xe_4 = e_3, \\ ye_1 &= -e_3, \quad ye_2 = -e_4, \quad ye_3 = e_1, \quad ye_4 = e_2, \\ xye_1 &= e_4, \quad xye_2 = -e_3, \quad xye_3 = e_2, \quad xye_4 = -e_1; \end{aligned}$$

this is shown in Fig. 1, with the fundamental domain highlighted. □

Next we give a $\mathbf{Q}_{4t}$-cellular decomposition of S^3. This is obtained as follows. First, we give in the next lemma the cellular chain complex of the quotient space $\mathcal{F}_{4t,3}/\mathbf{Q}_{4t} = S^3/\mathbf{Q}_{4t}$ as a complex of free finitely generated $\mathbb{Z}\mathbf{Q}_{4t}$-modules. By lifting the cells corresponding to the modules generators, we obtain a cell complex whose realization is a subset of $\mathcal{F}_{4t,3}$. Eventually, by translating the last complex with the group action, we obtain the desired $\mathbf{Q}_{4t}$-cellular decomposition of S^3.

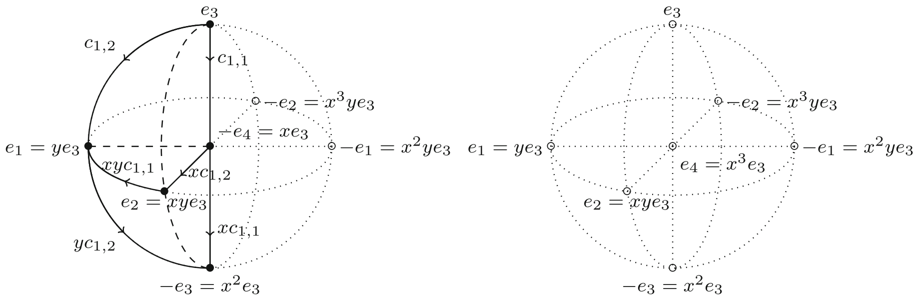

Fig. 1 Fundamental domain $\mathcal{F}_{8,3}$ and action of $\mathbf{Q}_8$ on the 3-dimensional sphere S^3 by α_1

Lemma 4 *A cellular chain complex of $\mathcal{F}_{4t,3}/\mathbf{Q}_{4t}$ is the complex of $\mathbb{Z}\mathbf{Q}_{4t}$-modules with one generator in dimension 0, and 3, two generators in dimensions 1 and 2, whose lifts are the following cells of $\mathcal{F}_{4t,3}$ (the first index denotes the dimension):*

$$\begin{aligned}
c_0 &= \emptyset * 1 = e_3,\\
c_{1,1} &= \emptyset * [1, \bar{\zeta}] = [e_3, \bar{\zeta} e_3], \quad c_{1,2} = -(1 * 1) = e_3 * e_1,\\
c_{2,1} &= 1 * [1, -1] = e_1 * [e_3, -e_3], \quad c_{2,2} = -(\zeta * [1, \bar{\zeta}] \cup [\zeta, 1] * 1),\\
c_3 &= [\zeta, 1] * [1, -1] = [\zeta e_1, e_1] * [e_3, -e_3].
\end{aligned}$$

The boundaries of these cells over the group ring $\mathbb{Z}\mathbf{Q}_{4t}$ are:

$$\partial_3(c_3) = (1 - x^{r_1})c_{2,1} + (1 - x^{r_1}y)c_{2,2},$$

$$\begin{aligned}
\partial_2(c_{2,1}) &= N_{x^{r_1}} c_{1,1} - (1 + y)c_{1,2},\\
\partial_2(c_{2,2}) &= -(1 + x^{r_1}y)c_{1,1} + (1 - x^{r_1})c_{1,2},
\end{aligned}$$

$$\begin{aligned}
\partial_1(c_{1,1}) &= (x^{r_1} - 1)c_0,\\
\partial_1(c_{1,2}) &= (y - 1)c_0.
\end{aligned}$$

Proof We give details for the case $t = 2$. The one 3-cell c_3 coincides with the whole fundamental domain. By taking the intersection of the translated of the fundamental domain, it is easy to see that we end up with the following set of cells. The 0-cells are the vertices of $\mathcal{F}_{8,3}$, i.e. the points:

$$\begin{aligned}
&c_{0,1} = \emptyset * 1 = e_3, \quad c_{0,2} = \emptyset * \bar{\zeta} = \bar{\zeta} e_3, \quad c_{0,3} = \emptyset * -1 = -e_3,\\
&c_{0,4} = 1 * \emptyset = e_1, \quad c_{0,5} = \zeta * \emptyset = \zeta e_1;
\end{aligned}$$

the 1-cells are the arcs:

$$\begin{aligned}
c_{1,1} &= \emptyset * [1, \bar{\zeta}] = [e_3, \bar{\zeta} e_3], \quad c_{1,2} = -(1 * 1) = e_3 * e_1,\\
c_{1,3} &= \emptyset * [\bar{\zeta}, -1] = [\bar{\zeta} e_3, -e_3], \quad c_{1,4} = 1 * -1 = e_1 * -e_3,\\
c_{1,5} &= [\zeta, 1] * \emptyset = [\zeta e_1, e_1], \quad c_{1,6} = -(\zeta * \bar{\zeta}) = \bar{\zeta} e_3 * \zeta e_1;
\end{aligned}$$

the 2-cells are the sets:

$$\begin{aligned}
c_{2,1} &= 1 * [1, -1] = e_1 * [e_3, -e_3],\\
c_{2,2} &= [\zeta, 1] * 1 \cup \zeta * [1, \bar{\zeta}] = e_3 * \left(\bar{\zeta} e_3 * \zeta e_1 \cup [\zeta e_1, e_1]\right),\\
c_{2,3} &= [\zeta, 1] * (-1) \cup \zeta * [\bar{\zeta}, -1] = -e_3 * \left(\bar{\zeta} e_3 * \zeta e_1 \cup [\zeta e_1, e_1]\right).
\end{aligned}$$

We verify that the union of the four 2-cells $c_{2,j}$ coincides with the boundary of the fundamental domain, $\partial\mathcal{F}_{8,3}$, that the boundary of each 2-cell is contained in the union of the 1-cells, and that the boundary of each 1-cell is contained in the union of the 0-cells. This shows that this set of cells provides a cellular decomposition of the fundamental domain. Next, using the group action, we show that all cells are given by $\mathbb{Z}\mathbf{Q}_8$ linear combinations of the minimal set of cells given in the statement of the Lemma. Note that this minimal set is naturally suggested once we fix a starting point, namely one 0-cell: the two 1-cells are then constructed by considering the action of the two group's generators on the chosen 0-cell. Note also that in order to preserve generality with respect to the representation, we need to use the correct power of the generators, and this depends on the representation. Fixing $c_{0,1}$ as the one 0-cell, it is easy to see that the other 0-cells are:

$$c_{0,2} = x^{r_1} c_{0,1}, \quad c_{0,3} = x^{2r_1} c_{0,1}, \quad c_{0,4} = y c_{0,1}, \quad c_{0,5} = x^{r_1} y c_{0,1},$$

for the 1-cells, we have:

$$c_{1,3} = x^{r_1} c_{1,1}, \quad c_{1,4} = y c_{1,2}, \quad c_{1,5} = x^{r_1} y c_{1,1}, \quad c_{1,6} = x^{r_1} c_{1,2},$$

and for the 2-cells: $c_{2,3} = -x^{r_1} c_{2,1} - x^{r_1} y c_{2,2}$.

To conclude the proof we compute the boundaries. Using Eq. (5) and the previous formulas, we obtain

$$\begin{aligned}
\partial(c_3) &= 1 * [1,-1] - \zeta * [1,-1] + [\zeta,1] * (-1) - [\zeta,1] * 1\\
&= c_{2,1} - (\zeta * [1,\bar{\zeta}] + \zeta * [\bar{\zeta},-1]) + [\zeta,1] * (-1) - [\zeta,1] * 1\\
&= c_{2,1} - (\zeta * [1,\bar{\zeta}] + [\zeta,1] * 1) + ([\zeta,1] * (-1) - \zeta * [\bar{\zeta},-1])\\
&= c_{2,1} + c_{2,2} + c_{2,3}\\
&= (1 - x^{r_1}) c_{2,1} + (1 - x^{r_1} y) c_{2,2}.
\end{aligned}$$

For the 2-cells we have

$$\partial(c_{2,1}) = \emptyset * [1,-1] - 1 * (-1) + 1 * 1 = N_{x^{r_1}} c_{1,1} - (1+y) c_{1,2},$$

$$\partial(c_{2,2}) = -(1 + x^{r_1} y) c_{1,1} + (1 - x^{r_1}) c_{1,2},$$

$$\partial(c_{2,3}) = (x^{r_1} y - x^{r_1}) c_{1,1} + (x^{r_1} + y) c_{1,2}.$$

The boundaries of the 1-cells are easily identified.

In the general case of the group $\mathbf{Q}_{4t}$, the same argument works. One starts with a larger number of cells, see Fig. 2, but using the geometry one is able to prove that the cells necessary to generate the $\mathbb{Z}\mathbf{Q}_{4t}$ chain modules always reduces to the cells described in the statement of the lemma. The calculation of the boundary is then straightforward. □

This construction generalizes to the case of $\mathbf{Q}_{4t}$ acting on S^{4n-1}. We state the results with a sketch of the proofs, that are generalizations of the previous ones.

Lemma 5 *The fundamental domain of the action $\alpha_{q_1,\dots,q_n}$ of $\mathbf{Q}_{4t}$ on S^{4n-1} is*

$$\mathcal{F}_{4t,4n-1} = \Sigma_1 * \cdots * \Sigma_{2(n-1)} * [\zeta e_{4n-3}, e_{4n-3}] * [e_{4n-1}, -e_{4n-1}].$$

Proof The action α of $\mathbf{Q}_{4t}$ is relevant only on the four top dimensions. Here $[\zeta e_{4n-3}, e_{4n-3}] * [e_{4n-1}, -e_{4n-1}]$ is a 'copy' of $\mathcal{F}_{4t,3} = [\zeta, 1] * [1, -1]$ (that is, can be described with the same cellular decomposition with dimensions set up accordingly) settled in the last two complex planes, and

$$\Sigma_1 * \cdots * \Sigma_{2(n-1)} \simeq S^{4(n-1)-1}$$

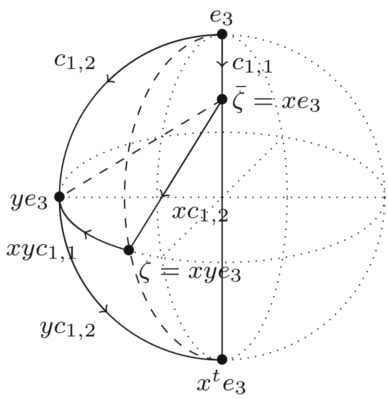

Fig. 2 Fundamental domain $\mathcal{F}_{4t,3}$

is α-invariant, namely it is sent onto itself by the action α. It follows that $\mathcal{F}_{4t,4n-1}$ above covers the whole sphere, for $[\zeta e_{4n-3}, e_{4n-3}] * [e_{4n-1}, -e_{4n-1}]$ covers Σ_{2n}. □

Lemma 6 *A cellular chain complex of $\mathcal{F}_{4t,4n-1}/\mathbf{Q}_{4t}$ is the complex of $\mathbb{Z}\mathbf{Q}_{4t}$-modules with one generator in dimensions $4k-1$ and $4k-4$, two generators in dimensions $4k-2$ and $4k-3$. The lifts of the cells representing these generators are described in the course of the proof, and have the following boundaries (except $\partial_0 = 0$):*

$$\partial_{4k-1}(c_{4k-1}) = (1 - x^{r_k})c_{4k-2,1} + (1 - x^{r_k}y)c_{4k-2,2},$$

$$\begin{aligned}\partial_{4k-2}(c_{4k-2,1}) &= N_{x^{r_k}} c_{4k-3,1} - (1+y)c_{4k-3,2},\\ \partial_{4k-2}(c_{4k-2,2}) &= -(1 + x^{r_k}y)c_{4k-3,1} + (1 - x^{r_k})c_{4k-3,2},\end{aligned}$$

$$\begin{aligned}\partial_{4k-3}(c_{4k-3,1}) &= (x^{r_k} - 1)c_{4k-4},\\ \partial_{4k-3}(c_{4k-3,2}) &= (y-1)c_{4k-4},\end{aligned}$$

$$\partial_{4k-4}(c_{4k-4}) = L_{x^{r_k}}(1+y)c_{4(k-1)-1}.$$

Proof Generalizing the proof of Lemma 4, we obtain the following reduced set of cells

$$\begin{aligned}c_{4k-4} &= \Sigma^{2(k-1)} * \emptyset * 1,\\ c_{4k-3,1} &= \Sigma^{2(k-1)} * \emptyset * [1, \bar{\zeta}], \quad c_{4k-3,2} = -(\Sigma^{2(k-1)} * 1 * 1),\\ c_{4k-2,1} &= \Sigma^{2(k-1)} * 1 * [1, -1], \quad c_{4k-2,2} = \Sigma^{2(k-1)} * ([\zeta, 1] * 1 \cup \zeta * [1, \bar{\zeta}]),\\ c_{4k-1} &= \Sigma^{2(k-1)} * [\zeta, 1] * [1, -1].\end{aligned}$$

The boundary operators follow, up to the one in dimension $4k-4 \neq 0$. For this one, note that

$$\begin{aligned}\partial_{4k-4}(c_{4k-4}) &= \Sigma^{2(k-1)}\\ &= \Sigma^{2(k-2)} * (1 + x^{r_{k-1}} + x^{2r_{k-1}} + \cdots + x^{(2t-1)r_{k-1}})(1+y)c_{4(k-1)-1}\\ &= L_{x^{r_k}}(1+y)c_{4(k-1)-1}.\end{aligned}$$ □

Corollary 1 *The cellular complex $K(4t)$ described in Lemma 6 is a finite connected cellular decomposition of dimension $4n-1$ of the space $Q(4t)$.*

5 The chain complex

We are now able to give the relevant chain complex associated to $K(4t)$. Recall this is defined as follows. Let (K, L) be a pair of connected finite cell complexes of dimension m, and $(\tilde{K}, \tilde{L})$ its universal covering complex pair, and identify the fundamental group of K with the group of the covering transformations of $\tilde{K}$. Note that covering transformations are cellular. Let $C((\tilde{K}, \tilde{L}); \mathbb{Z})$ be the chain complex of $(\tilde{K}, \tilde{L})$ with integer coefficients. The action of the group of covering transformations makes each chain group $C_q((\tilde{K}, \tilde{L}); \mathbb{Z})$ into a module over the group ring $\mathbb{Z}\pi_1(K)$, and each of these modules is $\mathbb{Z}\pi_1(K)$-free (see [6, p. 377]) and finitely generated by the natural choice of the q-cells of $K - L$. Since K is finite it follows that $C((\tilde{K}, \tilde{L}); \mathbb{Z})$ is free and finitely generated over $\mathbb{Z}\pi_1(K)$. We obtain a complex of free finitely generated modules over $\mathbb{Z}\pi_1(K)$ that we denote by $C((\tilde{K}, \tilde{L}); \mathbb{Z}\pi_1(K))$.

Using the cellular decomposition and the boundary of the cells described in the previous section, we obtain the following result in the case under investigation.

Lemma 7 *The chain complex $C(\tilde{K}(4t; q_1, \ldots, q_n); \mathbb{Z}\mathbf{Q}_{4t})$, of free finitely generated $\mathbb{Z}\mathbf{Q}_{4t}$ modules, is $(1 \leq k \leq n)$:*

$$
\begin{aligned}
C_{4k-1}(C(\tilde{K}(4t; q_1, \ldots, q_n); \mathbb{Z}\mathbf{Q}_{4t}) &= \mathbb{Z}\mathbf{Q}_{4t}[c_{4k-1}],\\
C_{4k-2}(C(\tilde{K}(4t; q_1, \ldots, q_n); \mathbb{Z}\mathbf{Q}_{4t}) &= \mathbb{Z}\mathbf{Q}_{4t}[c_{4k-2,1}, c_{4k-2,2}],\\
C_{4k-3}(C(\tilde{K}(4t; q_1, \ldots, q_n); \mathbb{Z}\mathbf{Q}_{4t}) &= \mathbb{Z}\mathbf{Q}_{4t}[c_{4k-3,1}, c_{4k-3,2}],\\
C_{4k-4}(C(\tilde{K}(4t; q_1, \ldots, q_n); \mathbb{Z}\mathbf{Q}_{4t}) &= \mathbb{Z}\mathbf{Q}_{4t}[c_{4k-4}],
\end{aligned}
$$

with boundary operators given by (except $\partial_0 = 0$)

$$
\partial_{4k-1} = \begin{pmatrix} 1 - x^{r_k} & 1 - x^{r_k} y \end{pmatrix}, \quad \partial_{4k-2} = \begin{pmatrix} N_{x^{r_k}} & -1 - y \\ -1 - x^{r_k} y & 1 - x^{r_k} \end{pmatrix},
$$

$$
\partial_{4k-3} = \begin{pmatrix} x^{r_k} - 1 \\ y - 1 \end{pmatrix}, \qquad \partial_{4k-4} = \begin{pmatrix} L_x(1 + y) \end{pmatrix}.
$$

We observe here that $L_x(1 + y) = L_{x^{r_k}}(1 + y)$ for $L_x = L_{x^{r_k}}$ since that $(r_k, 2t) = 1$.

Proof The complex follows from the cellular decomposition given in Sect. 4. We first need to show that the two basis chains in degrees $4k - 2$ and $4k - 3$ are independent, namely that $c_{4k-j,1} \neq \alpha c_{4k-j,2}$, and viceversa, for all $\alpha \in \mathbb{Z}\mathbf{Q}_{4t}$, and $j = 2, 3$. This follows from geometry, just looking at the concrete cells, as defined in the proof of Lemma 6. It is quite straightforward for the case $j = 3$, and needs some little more care for $j = 2$. By construction, the complex $C(\tilde{K}(4t; q_1, \ldots, q_n); \mathbb{Z}\mathbf{Q}_{4t})$ is exact; however, we present here an explicit verification of semi exactness for S^7, that extends easily to the general case. For we compute

$$
\begin{aligned}
\partial_6 \partial_7 &= \big((1 - x^{r_2}) N_{x^{r_2}} - (1 - x^{r_2} y)(1 + x^{r_2} y) \\
&\qquad -(1 - x^{r_2})(1 + y) + (1 - x^{r_2} y)(1 - x^{r_2}) \big) \\
&= \big(1 - x^{t r_2} - 1 + (x^{r_2} y)^2 \\
&\qquad -1 + x^{r_2} - y + x^{r_2} y - x^{r_2} + x^{r_2} y x^{r_2} + 1 - x^{r_2} y \big) = \begin{pmatrix} 0 & 0 \end{pmatrix},
\end{aligned}
$$

because $x^{t r_2} = y^2 = (x^{r_2} y)^2$ and $x^{r_2} y x^{r_2} = y$. Note that this works also for the composite $\partial_2 \partial_3$, replacing r_2 by r_1. Next, it is straightforward to verify that $\partial_5 \partial_6 = \partial_1 \partial_2$ and $\partial_4 \partial_5 = \partial_0 \partial_1$ are trivial, while

$$\partial_3\partial_4 = \left(L_{x^{r_2}}(1+y)(1-x^{r_2}) \; L_{x^{r_2}}(1+y)(1-x^{r_2}y) \right) = \left(0 \; 0 \right),$$

due to the fact that

$$\begin{aligned}
&L_{x^{r_2}}(1+y)(1-x^{r_2}y)\\
&= (1+x^{r_2}+\cdots+x^{r_2(2t-2)}+x^{r_2(2t-1)}) + \underbrace{(y+x^{r_2}y+\cdots+x^{r_2(2t-1)}y)}_{a}\\
&-\underbrace{(x^{r_2}y+x^2y+\cdots+x^{r_2(2t-1)}y+x^{r_22t}y)}_{a} - (yx^{r_2}y+(x^{r_2}y)^2+\cdots+x^{r_2(2t-2)}(x^{r_2}y)^2)\\
&= (1+x^{r_2}+\cdots+x^{r_2(2t-2)}+x^{r_2(2t-1)}) - (yx^{r_2}y+(x^{r_2}y)^2+\cdots+x^{r_2(2t-2)}(x^{r_2}y)^2)\\
&= (1+x^{r_2}+\cdots+x^{r_2(2t-2)}+x^{r_2(2t-1)}) - (x^{r_2(t-1)}+x^{r_2t}+\cdots+x^{r_2(t-2)}) = 0,\\
&L_{x^{r_2}}(1-x)\\
&= (1+x^{r_2}+\cdots+x^{r_2(2t-2)}+x^{r_2(2t-1)}) + (y+x^{r_2}y+\cdots+x^{r_2(2t-1)}y)\\
&-(x^{r_2}+x^{2r_2}+\cdots+x^{r_2(2t-1)}+x^{r_22t}) - (yx^{r_2}+x^{r_2}yx^{r_2}+\cdots+x^{r_2(2t-1)}yx^{r_2}) = 0,
\end{aligned}$$

because $yx^{r_2} = x^{-r_2}y = x^{r_2(2t-1)}y$. □

Since $K(4t)$ is a finite cell decomposition of the space $Q(4t)$, the chain complex $C(\tilde{K}(4t); \mathbb{Z}\mathbf{Q}_{4t})$ described in Lemma 7 is the desired complex associated to $Q(4t)$.

6 Applications: Whitehead torsion and simple homotopy type

The aim of this section is to present the calculation of the Whitehead torsion and to prove a classification theorem for quaternionic spherical space forms, as anticipated in the introduction. For the reader's benefit, we briefly recall the main definitions involved. This section is essentially based on [1,2,6].

6.1 Non commutative determinants

Let $\mathbb{K}$ be a division ring (not necessarily abelian). We denote by $Gl(n,\mathbb{K})$ the group of the n-square invertible matrices with coefficients in $\mathbb{K}$, and with product $AB = (\sum_j a_{i,j}b_{j,k})$, where $A = (a_{j,k})$ denotes the matrix with element $a_{j,k}$ in the line j, column k. We denote by $E(n,\mathbb{K})$ the normal subgroup of $Gl(n,\mathbb{K})$ generated by the *elementary matrices*, namely the matrices that coincide with the identity matrix except for one off diagonal element. Then, $E(n,\mathbb{K})$ coincides with the commutator subgroup $[Gl(n,\mathbb{K}), Gl(n,\mathbb{K})]$, except when $n = 2$ and $\mathbb{K} = \mathbb{Z}/2$ or $\mathbb{Z}/3$. Note that for a ring R this is no longer true, as we can see from the proof of Lemma 1.1 of [6], the identification holds only in the limit group $Gl(R)$, see next section. Let denote by $D(x)$ the matrix in $Gl(n,\mathbb{K})$ that coincides with the identity matrix except for the element $d_{n,n} = x$. Then, we have the following decomposition, that in general is not unique [1, 4.1].

Lemma 8 *For each $A \in Gl(n,\mathbb{K})$, there exist $0 \neq x \in \mathbb{K}$ and $B \in E(n,\mathbb{K})$ such that $A = D(x)B$.*

Let $\mathbb{K}^\times$ denote the group of the units of $\mathbb{K}$, that coincides with $\mathbb{K} - \{0\}$. Let $\mathsf{ab}(\mathbb{K}^\times) = \frac{\mathbb{K}^\times}{[\mathbb{K}^\times,\mathbb{K}^\times]}$ denotes the commutative factor group (we denote by $\mathsf{ab}(a)$ the class of a in $\mathsf{ab}(\mathbb{K}^\times)$). Denoting by $M(n,\mathbb{K})$ the ring of the square n-matrices over $\mathbb{K}$, the *Dieudonné determinant* is the homomorphism $\mathrm{Det} : M(n,\mathbb{K}) \to \mathsf{ab}(\mathbb{K}^\times) \cup \{0\}$, defined as follows: $\mathrm{Det}(A) = 0$, if A is not invertible, $\mathrm{Det}(A) = \mathsf{ab}(x)$, if $A = D(x)B$, for some $x \in \mathbb{K}$, $B \in E(n, \mathbb{K})$. This definition is well posed, and we have the following result [1, 4.3] (note in particular that $D(x)$ is an element of $E(n, \mathbb{K})$ if and only if $\mathsf{ab}(x) = \mathsf{ab}(1)$ is the unit of $\mathsf{ab}(\mathbb{K}^\times)$).

Lemma 9 *The mapping* $\mathrm{Det} : Gl(n, \mathbb{K}) \to \mathsf{ab}(\mathbb{K}^\times)$ *is an epimorphism of groups, with kernel* $E(n, \mathbb{K})$. *The quotient group* $Gl(n, \mathbb{K})/E(n, \mathbb{K})$ *is isomorphic to* $\mathsf{ab}(\mathbb{K}^\times)$.

Because of Lemma 9, we use also the notation $Sl(n, \mathbb{K}) = E(n, \mathbb{K})$. The Dieudonné determinant satisfies the following properties:

$$\mathrm{Det}(AB) = \mathrm{Det}(A)\mathrm{Det}(B) = \mathrm{Det}(BA), \quad \mathrm{Det}(B) = \mathsf{ab}(1), \ \forall B \in E(n, \mathbb{K}).$$

The following calculation rule will also be useful:

$$\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \mathsf{ab}(ad - aca^{-1}b), & a \neq 0, \\ \mathsf{ab}(cb), & a = 0. \end{cases} \tag{6}$$

Next we specialize to quaternions, so let $\mathbb{K} = \mathbb{H}$. We have the following facts:

(I) $[\mathbb{H}^\times, \mathbb{H}^\times]$ is isomorphic to the set of unit quaternions.

(II) The homomorphism

$$|\,| : \mathsf{ab}(\mathbb{H}^\times) = \frac{\mathbb{H}^\times}{[\mathbb{H}^\times, \mathbb{H}^\times]} \to \mathbb{R}^+, \quad |\,| : \mathsf{ab}(x) \mapsto |x|,$$

is an isomorphism.

(III) We have the following commutative diagram, where the two arrows in the last line are isomorphisms,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Sl(n, \mathbb{H}) & \xrightarrow{\ i\ } & Gl(n, \mathbb{H}) & \xrightarrow{\mathrm{Det}} & \mathsf{ab}(\mathbb{H}^\times) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & K(n, \mathbb{H}) := Gl(n, \mathbb{H})/E(n, \mathbb{H}) & \xrightarrow{\mathrm{Det}} & \mathsf{ab}(\mathbb{H}^\times) & \xrightarrow{|\,|} & \mathbb{R}^+. \end{array}$$

(IV) With respect to the injective algebra homomorphisms $\rho_{\mathbb{C}}$ and $\rho_{\mathbb{R}}$, we have:

$$|\mathrm{Det}(A)|^2 = \det(\rho_{\mathbb{C}}(A)), \quad |\mathrm{Det}(A)|^4 = \det(\rho_{\mathbb{C}}\rho_{\mathbb{R}}(A)), \quad |\det(A)|^2 = \det(\rho_{\mathbb{R}}(A)).$$

6.2 Whitehead torsion

Let R be an associative ring with unit. Let $Gl(R)$ the direct limit of the canonical inclusions $Gl(1, R) \subset Gl(2, R) \subset \cdots$. The subgroup $E(R) < Gl(R)$ generated by all elementary matrices is equal to the commutator subgroup of $Gl(R)$, and the abelian quotient group $K_1(R) = Gl(R)/E(R)$ is called the *Whitehead group* of R. Let $[(-1)] \in K_1(R)$ denotes the element of order 2 corresponding to the unit $(-1) \in Gl(1, R) \subset Gl(R)$. Then the quotient $\tilde{K}_1(R) = K_1(R)/\{[(1)], [(-1)]\}$ is called the *reduced Whitehead group* of R. The class $[A]$ in $K_1(R)$ or $\tilde{K}_1(R)$ is called the *torsion* of the matrix A, and the projection is denoted by $\tau : Gl(R) \to K_1(R)$ or $\tau : Gl(R) \to \tilde{K}_1(R)$.

Remark 1 If $R = \mathbb{K}$ is a division ring, the Whitehead group coincides with the direct limit of the groups $K(n, \mathbb{K})$, the homomorphism (Dieudonné determinant) $\mathrm{Det} : K_1(\mathbb{K}) \to$

$\mathsf{ab}(\mathbb{K}^\times)$, Det : $[A] \mapsto \mathrm{Det}(A)$ is an isomorphism, and we have the split short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Sl(\mathbb{K}) & \longrightarrow & Gl(\mathbb{K}) & \xrightarrow{\mathrm{Det}} & \mathsf{ab}(\mathbb{K}^\times) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \tilde{K}_1(\mathbb{K}) & \xrightarrow{\mathrm{Det}} & \mathsf{ab}(\mathbb{K}^\times)/\{\pm 1\} & & \end{array}$$

where $Sl(\mathbb{K}) = E(\mathbb{K})$, the arrow in the last line is an isomorphism, and the inverse homomorphism is induced by the inclusion of $\mathbb{K}^\times = Gl(1, \mathbb{K}) \subset Gl(\mathbb{K})$.

From Remark 1 and fact (II), it follows that $\tilde{K}_1(\mathbb{H}) = K_1(\mathbb{H})$ is isomorphic to the multiplicative group of the positive reals $\mathbb{R}^+$, and the isomorphism is the Dieudonné determinant: $|\mathrm{Det}(\,)| : \tilde{K}_1(\mathbb{H}) \to \mathbb{R}^+$, $|\mathrm{Det}(\,)| : [A] \mapsto |\mathrm{Det}(A)|$. Note also that if $G \le \mathbb{H}^\times$, then $\mathbb{R}G \le \mathbb{H}^\times$, and we have a ring homomorphism $\mathbb{R}G \to \mathbb{H}$ that induces an abelian group homomorphism $K_1(\mathbb{R}G) \to K_1(\mathbb{H})$. Let $\rho : G \to O(4n, \mathbb{R})$ be the restriction of the representation $\rho_{\mathbb{C}}\rho_{\mathbb{R}}$, then, we have that, for all $[A] \in \tilde{K}_1(\mathbb{R}G) \le \tilde{K}_1(\mathbb{H})$, $|\mathrm{Det}([A])|^4 = |\det(\rho(A))|$.

Assume that the ring R has *invariant basis number*, namely that: if M is any finitely generated free module over R, then any two bases of M have the same cardinality. Note that this assumption is satisfied if $R = \mathbb{Z}G$ is the group ring of some group G [2, 9.2], and if R is a division ring [2, 9.1]. All our modules are finitely generated left R-modules. Let M be a (finite dimensional) free R-module. Let $x = \{x_1, \ldots, x_m\}$ and $y = \{y_1, \ldots, y_m\}$ be two bases for M. We denote by (y/x) the non singular m-square matrix A over R defined by the change of bases, namely $(y/x) = A$, where $y_j = A_{j,k}x_k$, and we denote by $[y/x]$ the class of A in $\tilde{K}_1(R)$. Let

$$C : C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

be a chain complex of finite length m of (finite dimensional) free R-modules. Denote by $Z_q = \ker(\partial_q : C_q \to C_{q-1})$, $B_q = \mathrm{Im}(\partial_{q+1} : C_{q+1} \to C_q)$, and $H_q(C) = Z_q/B_q$ the homology groups of C. In order to define the torsion of C we assume either that B_q is a free R-module for each q, or that stably free bases for all modules are used, see [6, Section 4], or [2, Section 13] for details. Also note that if R is a division ring (in particular a field), any module over R is free, and that each submodule of a free finitely generated module over a PID is free and finitely generated. Assume that H_q is a free R-module for each q. For each q, fix a basis c_q for C_q, and a set of independent elements h_q in Z_q that project on a fixed basis $\bar{h}_q$. Let b_q be a set of elements of C_q such that $\partial_q(b_q)$ is a basis for B_{q-1}. Then the set of elements $\{\partial_{q+1}(b_{q+1}), h_q, b_q\}$ is a basis for C_q. We define the *Whitehead torsion* of C with respect to c and h to be the class

$$\tau_{\mathrm{W}}(C; h) = \sum_{q=0}^{m} (-1)^q [\partial_{q+1}(b_{q+1}), h_q, b_q/c_q], \tag{7}$$

in $\tilde{K}_1(R)$. The torsion $\tau_{\mathrm{W}}(C; h)$ does not depend on the choice of the graded basis $b = \{b_q\}$ [6, p. 365]. Also note that the torsion is functorial with respect to the change or ring. If M is a free right R'-module and $\varphi : R \to R'$ is a ring homomorphism, then $\tau_{\mathrm{W}}(M \otimes_\varphi C; h') \in \tilde{K}_1(R')$ is defined and we have [6, p. 385]

$$\tau_{\mathrm{W}}(M \otimes_\varphi C; h') = \varphi_* \tau_{\mathrm{W}}(C; h). \tag{8}$$

Let π be a group and $\rho : \pi \to \mathrm{Aut}_R(M)$ be a representation into the group of the automorphisms of some free right module M over some ring with unit R. Then, ρ extends to a ring homomorphism $\rho : \mathbb{Z}\pi \to \mathbb{Z}\mathrm{Aut}_R(M)$, and (together with the inclusion of $\mathbb{Z}$ into R) endows M with a left $\mathbb{Z}\pi$-module structure. Let C be a finite complex of free finitely generated $\mathbb{Z}\pi$-modules, and consider the complex $C_\rho = M \otimes_\rho C_q$ of free finitely generated R-modules. Assuming that the Whitehead torsion $\tau_{\mathrm{W}}(C_\rho; h)$ of C_ρ is defined, then it is an element of $\tilde{K}_1(\mathbb{Z}\mathrm{Aut}_R(M))$, because the relevant ring is $\mathbb{Z}\mathrm{Aut}_R(M)$. By Eq. (8), $\tau_{\mathrm{W}}(C_\rho; h) = \rho_* \tau_{\mathrm{W}}(C; h')$.

Now, suppose $C_\rho = C((\tilde{K}, \tilde{L}); \mathbb{Z}\pi_1(K))$ to be the complex of free finitely generated modules over $\mathbb{Z}\pi_1(K)$ described at the beginning of Sect. 5. Assuming that the homology modules are free, any fixed choice of a basis c_q for $C_q((\tilde{K}, \tilde{L}); \mathbb{Z}\pi_1(K))$ and of cycles h_q representing a basis of $H_q(C((\tilde{K}, \tilde{L}); \mathbb{Z}\pi_1(K)))$ allows to define the torsion $\tau_{\mathrm{W}}(C((\tilde{K}, \tilde{L}); \mathbb{Z}\pi_1(K)); h)$ in $\tilde{K}_1(\mathbb{Z}\pi_1(K))$. In the present case the geometry determines a preferred base c_q. There is still some ambiguity due to the arbitrariness of the choice of the representative cells in the covering space $\tilde{K}$, projecting over the cells representing the fixed bases of C_q. However, different choices of representative cells in the covering give different torsions in $\tilde{K}_1(\mathbb{Z}\pi_1(K))$ that project to the same class if we take the quotient by the π_1 action. More precisely, to avoid this ambiguity it is sufficient to factor by the image of $\pi_1(K)$ inside $\mathrm{Aut}_R(M)$ obtained by applying the representation ρ. Namely, using the sequence of group homomorphisms:

$$\pi_1(K) \xrightarrow{\rho} \mathrm{Aut}_R(M) \longrightarrow Gl(\mathbb{Z}\mathrm{Aut}_R(M)) \xrightarrow{\tau} \tilde{K}_1(\mathbb{Z}\mathrm{Aut}_R(M)),$$

and the natural projection

$$q : \tilde{K}_1(\mathbb{Z}\mathrm{Aut}_R(M)) \to \tilde{K}_1(\mathbb{Z}\mathrm{Aut}_R(M))/\tau\rho(\pi_1(K)).$$

This suggests the following definition.

Definition 1 Let R be a ring with unit, and M be a free right R-module. Let (K, L) a pair of connected finite cellular complexes, and $\rho : \pi_1(K) \to \mathrm{Aut}_R(M)$ a representation of the fundamental group of K. Assume the complex $C((K, L); M_\rho)$ is free with free homology. Then, the R torsion of (K, L) with respect to the representation ρ and the graded basis h is the class $\tau_{\mathrm{R}}((K, L); \rho, h) = q(\tau_{\mathrm{W}}(C((K, L); M_\rho); h))$, in $\tilde{K}_1(\mathbb{Z}\mathrm{Aut}_R(M))/\tau\rho(\pi_1(K))$.

6.3 Whitehead torsion of quaternionic spherical space forms

Let $\zeta = e^{i\theta} \neq 1$ be a $2t$-th root of the unity, and a, b, c integers with $(a, 2t) = (c, 2t) = 1$, $(b, 4) = 1$. Let $\rho_{a,b,c} : \mathbf{Q}_{4t} \to \mathbb{H}^\times$ be the inclusion of $\mathbf{Q}_{4t}$ into the division ring $\mathbb{H}$ of the quaternions defined by

$$\rho_{a,b,c}(x) = \zeta^a, \quad \rho_{a,b,c}(y) = \zeta^c j^b. \tag{9}$$

The representation induces a ring homomorphism $\rho_{a,b,c} : \mathbb{Z}\mathbf{Q}_{4t} \to \mathbb{H}$, and twisting the chain complex $C(\tilde{K}(4t); \mathbb{Z}\mathbf{Q}_{4t})$ by $\rho_{a,b,c}$, we obtain the chain complex $C(K(4t); \mathbb{H}_{\rho_{a,b,c}}) = \mathbb{H} \otimes_{\rho_{a,b,c}} C(\tilde{K}(4t); \mathbb{Z}\mathbf{Q}_{4t})$, as in Sect. 6.2:

$$\begin{aligned} &\cdots \longrightarrow \mathbb{H}[c_{4k-1}] \xrightarrow{\partial_{4k-1}} \mathbb{H}[c_{4k-2,1}, c_{4k-2,2}] \xrightarrow{\partial_{4k-2}} \\ &\xrightarrow{\partial_{4k-2}} \mathbb{H}[c_{4k-3,1}, c_{4k-3,2}] \xrightarrow{\partial_{4k-3}} \mathbb{H}[c_{4k-4}] \xrightarrow{\partial_{4k-4}} \cdots \end{aligned} \tag{10}$$

with boundary

$$\partial_{4k-1} = \left(1-\zeta^{ar_k} \;\; 1-\zeta^{ar_k+c}j^b\right), \; \partial_{4k-2} = \begin{pmatrix} N_{\zeta^{ar_k}} & -1-\zeta^c j^b \\ -1-\zeta^{ar_k+c}j^b & 1-\zeta^{ar_k} \end{pmatrix},$$

$$\partial_{4k-3} = \begin{pmatrix} \zeta^{ar_k}-1 \\ \zeta^c j^b - 1 \end{pmatrix}, \qquad \partial_{4k-4} = 0.$$

This is a periodic complex of free finitely generated $\mathbb{H}$-modules. Since $\mathbb{H}$ is a division ring, boundaries and cycles are free, and therefore the torsion $\tau_{\mathrm{R}}(Q(4t); \rho_{a,b,c}, h)$ is well defined. Since $K_1(\mathbb{H}) = \mathbb{R}^+$ and $\tau\rho_{a,b,c}(x) = |\mathrm{Det}(x)| = 1$, for all $x \in \mathbf{Q}_{4t}$ (see Sect. 6.1), it follows that $\tau_{\mathrm{R}}(Q(4t); \rho_{a,b,c}, h)$ is a class in the factor group

$$K_1(\mathbb{H})/\tau\rho_{a,b,c}(\pi_1(K(4t))) \cong \mathbb{R}^+,$$

namely a positive real number, where the identification is obtained by taking the module of the Dieudonné determinant. For simplicity, we will identify in the following this group with the positive reals, and we will write τ_{R} for the image $|\mathrm{Det}\tau_{\mathrm{R}}|$.

We are now in the position of calculating the torsion of the quaternionic spherical space forms. Before, we verify exactness of the relevant chain complex.

Proposition 1 *The complex $C(K(4t); \mathbb{H}_{\rho_{a,b,c}})$ is acyclic.*

Proof The complex (10) is semi-exact by Lemma 7. Since submodules and quotients of modules over division rings are free, we have $\dim(C_k) = \dim(\ker \partial_k) + \dim(\mathrm{Im}\partial_k)$, for each k, and the thesis follows if we can show that $\mathrm{Im}(\partial_{4k-j}) \neq \{0\}$, for $j = 1, 2, 3$. But this follows since $1-\zeta^{ar_k} \neq 0$ because $ar_k \neq 0 \bmod 2t$, for all k, since by hypothesis $(a, 2t) = 1$, and $(r_k, 2t) = 1$. □

Proposition 2 *The R torsion of the quaternionic space form $Q(4t; q_1, \ldots, q_n)$ with respect to the representation $\rho_{a,b,c} : \pi_1(Q(4t)) \to \mathbb{H}^\times$ defined in Eq. (9) is the positive real number:*

$$\tau_{\mathrm{R}}(Q(4t; q_1, \ldots, q_n); \rho_{a,b,c}) = \prod_{k=1}^{n} \left|(1-\zeta^{ar_k})\right|,$$

where the r_k are the multiplicative inverses mod $2t$ of the q_k.

Proof Consider the complex in Eq. (10). We choose the following bases for the boundaries (note that they are all one dimensional), with $1 \leq k \leq n$:

$$b_{4k-1} = c_{4k-1}, \quad b_{4k-2} = c_{4k-2,1}, \quad b_{4k-3} = (\zeta^{ar_k}-1)^{-1}c_{4k-3,1}, \quad b_{4k-4} = \emptyset.$$

Then, by definition, the relevant change of bases are in dimensions $4k-2$ and $4k-3$, and the associated matrices are

$$(\partial_{4k-1}(b_{4k-1}), h_{4k-2}, b_{4k-2}/c_{4k-2}) = \begin{pmatrix} 1-\zeta^{ar_k} & 1-\zeta^{ar_k+c}j^b \\ 1 & 0 \end{pmatrix},$$

$$(\partial_{4k-2}(b_{4k-2}), h_{4k-3}, b_{4k-3}/c_{4k-3}) = \begin{pmatrix} N_{\zeta^{ar_k}} & -1-\zeta^c j^b \\ (\zeta^{ar_k}-1)^{-1} & 0 \end{pmatrix},$$

and by Eq. (7) this gives the Whitehead torsion (recall the complex is acyclic by Proposition 1, so $h = \emptyset$ will be omitted)

$$\begin{aligned}\tau_{\mathrm{R}}&(Q(4t; q_1, \ldots, q_n); \rho_{a,b,c}) \\ &= \sum_{k=1}^{n} \left([\partial_{4k-1}(b_{4k-1}), b_{4k-2}/c_{4k-2}] - [\partial_{4k-2}(b_{4k-2}), b_{4k-3}/c_{4k-3}]\right),\end{aligned}$$

where we recall that the notation means that the classes are in $\tilde{K}_1(\mathbb{H})$. Since $\tilde{K}_1(\mathbb{H})$ is isomorphic to $\mathbb{R}^+$ and an isomorphism is the non commutative determinant, by facts (II) and (III), Remark 1 and Sect. 6.1, we identify the R torsion $\tau_{\mathrm{R}}(Q(4t; q_1, \ldots, q_n); \rho_{a,b,c})$ with the positive real number

$$\tau_{\mathrm{R}}(Q(4t; q_1, \ldots, q_n);\ \rho_{a,b,c}) = \prod_{k=1}^{n} \frac{|\mathrm{Det}(\partial_{4k-1}(b_{4k-1}), b_{4k-2}/c_{4k-2})|}{|\mathrm{Det}(\partial_{4k-2}(b_{4k-2}), b_{4k-3}/c_{4k-3})|}.$$

Using the rules given in Eq. (6), we obtain

$$\begin{aligned}\mathrm{Det}(\partial_{4k-1}(b_{4k-1}), b_{4k-2}/c_{4k-2}) &= \mathsf{ab}(-(1-\zeta^{ar_k+c} j^b)),\\ \mathrm{Det}(\partial_{4k-2}(b_{4k-2}), b_{4k-3}/c_{4k-3}) &= \mathsf{ab}((1+\zeta^c j^b)(\zeta^{ar_k}-1)^{-1}),\end{aligned}$$

in $\mathbb{H}^\times/[\mathbb{H}^\times, \mathbb{H}^\times]$. Next, using the isomorphism described in (III), we obtain

$$\begin{aligned}|\mathrm{Det}(\partial_{4k-1}(b_{4k-1}), b_{4k-2}/c_{4k-2})\ | &= |\ (1-\zeta^{ar_k+c} j^b)|,\\ |\mathrm{Det}(\partial_{4k-2}(b_{4k-2}), b_{4k-3}/c_{4k-3})\ | &= |\ (1+\zeta^c j^b)(\zeta^{ar_k}-1)^{-1}|,\end{aligned}$$

and therefore

$$\tau_{\mathrm{R}}(Q(4t; q_1, \ldots, q_n); \rho_{a,b,c}) = \prod_{k=1}^{n} \frac{|(1-\zeta^{ar_k+c} j^b)|}{|(1+\zeta^c j^b)(\zeta^{ar_k}-1)^{-1}|}.$$

Observing that (recall that b is odd for $(b, 4) = 1$)

$$\begin{aligned}|1-\zeta^{ar_k+c} j^b|^2 &= (1-\zeta^{ar_k+c} j^b)\overline{(1-\zeta^{ar_k+c} j^b)}\\ &= (1-\zeta^{ar_k+c} j^b)(1+\zeta^{ar_k+c} j^b) = 1-(\zeta^{ar_k+c} j^b)^2 = 2,\end{aligned}$$

$$|1+\zeta^c j^b|^2 = (1+\zeta^c j^b)\overline{(1+\zeta^c j^b)} = (1+\zeta^c j^b)(1-\zeta^c j^b) = 1-(\zeta^c j^b)^2 = 2.$$

We simplify the above formula as follows:

$$\tau_{\mathrm{R}}(Q(4t; q_1, \ldots, q_n); \rho_{a,b,c}) = \prod_{k=1}^{n} \frac{|(1-\zeta^{ar_k+c} j^b)|}{|(1+\zeta^c j^b)(\zeta^{ar_k}-1)^{-1}|} = \prod_{k=1}^{n} |\zeta^{ar_k}-1|.$$

□

6.4 Classification of quaternionic spherical space forms

The aim of this section is to prove a classification theorem for quaternionic spherical space forms (Theorem 1, compare with [6, Theorem 12.7]). We will see that homeomorphic and simple homotopic classifications coincide. By the result of Proposition 2, the problem reduces to similar problem for the lens spaces. We start by fixing preferred generators for the fundamental group of a quaternionic spherical space form. Namely, recalling the cell decomposition described in Lemma 6, and the action of the fundamental group as described in Sect. 2, we identify the generators x and y of $\pi_1(Q(4t)) = \mathbf{Q}_{4t}$, with the covering transformations $\alpha(x, c_0)$, and $\alpha(y, c_0)$. It is easy to see that these two cycles correspond to the $c_{1,1}$ and $c_{1,2}$.

Proposition 3 *Let $Q(4t; q_1, \ldots, q_n)$ and $Q(4t; q'_1, \ldots, q'_n)$ be two quaternionic spherical space forms, and suppose there exists a simple homotopy equivalence $f : Q \to Q'$, such that the induced homomorphism $f_* : \pi_1(Q) \to \pi_1(Q')$ sends the generator x of the cyclic* *subgroup onto* $f(x) = (x')^a$, *where* a *is an integer with* $(a, 2t) = 1$. *Then there are numbers* $\epsilon_l = \pm 1$, *and some permutation* σ *of* $\{1, \ldots, n\}$ *such that*

$$(q'_1, \ldots, q'_n) = (\epsilon_1 a q_{\sigma(1)}, \ldots, \epsilon_n a q_{\sigma(n)}) \bmod 2t.$$

Proof Since f is a simple homotopy equivalence, the induced homomorphism on the fundamental groups is an isomorphism. Since all automorphisms of $\mathbf{Q}_{4t}$ are known, it follows that $f_* : \pi_1(Q) \to \pi_1(Q')$ is of the form: $f_*(x) = (x')^a$, $f_*(y) = (x')^c(y')^b$, with a, b, c integers with $(a, 2t) = (c, 2t) = (b, 4) = 1$. Commutativity of the diagram of groups (on the left) induces a commutative diagram of rings (on the right)

$$\begin{tikzcd} \pi_1(Q) \arrow[r, "f_*"] \arrow[d, "\rho_{a,b,c}"'] & \pi_1(Q') \arrow[d, "\rho_{1,1,0}"] \\ \mathbb{H}^\times \arrow[r, "="'] & \mathbb{H}^\times, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}\mathbf{Q}_{4t} \arrow[r, "f_*"] \arrow[d, "\rho_{a,b,c}"'] & \mathbb{Z}\mathbf{Q}_{4t} \arrow[d, "\rho_{1,1,0}"] \\ \mathbb{H} \arrow[r, "="'] & \mathbb{H}, \end{tikzcd}$$

fixing the generators of the fundamental group. Since f is a simple homotopy equivalence, Q and Q' have the same torsion, and using the calculation in Proposition 2 with the generators of the fundamental groups fixed in this way, we obtain:

$$\prod_{k=1}^{n} \left|(\zeta^{r_k} - 1)\right| = \prod_{k=1}^{n} \left|(\zeta^{a r_k} - 1)\right|. \tag{11}$$

By the Franz Theorem [6, 12.6], Eq. (11) implies the thesis. □

Theorem 1 *Let* $Q(4t; q_1, \ldots, q_n)$ *and* $Q(4t; q'_1, \ldots, q'_n)$ *be two quaternionic spherical space forms. Then,* Q *and* Q' *are homeomorphic if and only if there are integer numbers* a *and* $\epsilon_1, \ldots, \epsilon_n$, *with* a *prime to* $2t$ *and* $\epsilon_l = \pm 1$, *and some permutation* σ *of* $\{1, \ldots, n\}$, *such that*

$$(q'_1, \ldots, q'_n) = (\epsilon_1 a q_{\sigma(1)}, \ldots, \epsilon_n a q_{\sigma(n)}) \bmod 2t.$$

Proof If Q is homeomorphic to Q', then they are simple homotopy equivalent and therefore the thesis follows from Proposition 3. Conversely, suppose that $q_l = \epsilon_l a q'_{\sigma(l)}$. We know by Lemma 2 that changing the order of the q_l or their sign does not change the homeomorphism class of $Q(4t; q_1, \ldots, q_n)$. Thus, we can assume $q_l = a q'_l$ for all $1 \leq l \leq n$. Let id be the identity map of S^{4n-1}, and consider the following diagram

$$\begin{tikzcd} S^{4n-1} \arrow[r, "id"] \arrow[d, "\pi"'] & S^{4n-1} \arrow[d, "\pi'"] \\ Q \arrow[r, "h"'] & Q' \end{tikzcd}$$

where π and π' are the natural projections. We show that the identity map passes to the quotient. For note that, by definition of the action of the quaternionic group on the sphere, the point $z \in S^{4n-1}$ is mapped to the class $[z] = \{w \in S^{4n-1} \mid w = \alpha(g, z),\ g \in \mathbf{Q}_{4t}\}$, where the action α is described in Eq. (1). Since the subgroup generated by x is finite cyclic and $(a, 2t) = 1$, it is clear that the orbit of x and the orbit of x^a coincide. The same is true for the orbits of y and y^b. □

Corollary 2 *Given a free isometric action of* $\mathbf{Q}_{4t}$ *on* S^{4n-1}*, there are unique* $q_1, \ldots, q_n$ *with* $1 \le q_1 \le \ldots \le q_n < t$*, for which there exists a* $\mathbf{Q}_{4t}$*-equivariant isometry from* S^{4n-1} *equipped with the given action to* S^{4n-1} *equipped with the action* α *defined in Sect. 2.*

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