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Zeta function and regularized determinant on a disc and on a cone

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Abstract

We give formulas for the analytic extension of the zeta function of the induced Laplacian L on a disc and on a cone. This allows the explicit computation of the value of the zeta function and of its derivative at the origin, and hence we get a formula for the regularized determinant of L.

MSC: 11M36; 58J52

Keywords: Zeta function; Elliptic operators; Regularization

1. Introduction

The regularized determinant of an elliptic differential operator was originally introduced in geometric analysis to deal with the heat equation and the index theorem in [2] and with the analytic torsion in [27], and soon became object of strong interest and intensive study in differential geometry [12,22] and conformal geometry, where in particular it is studied as a function of the metric for suitable classes of operators (see e.g. [4,26], and also [5] where a formula is developed for the quotient of the determinant of two conformally related conformal operators), but also in mathematical physics, where it provides a regularization

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of the functional integral [19,36]. Despite all these efforts, few explicit calculations are available (see [30] for a review). In particular, a complete answer has been provided for the one-dimensional case, namely for regular operators on the circle in [9] and for Sturm–Liouville operators on the line segment in [10,20]. The main feature of the analysis of Sturm–Liouville operators is that it allows to deal not just with the regular case, but also with the singular one. This is obtained by using methods of functional analysis, originally developed in the study of the asymptotic expansion of the heat kernel and the resolvent of an elliptic operator [23,29] and subsequently generalized to some classes of singular ones [11,7]. The principal example of such singular operators arises from a very natural geometrical problem: namely, the Laplace operator for a manifold with a conical singularity. The analysis and the geometry of spaces with singularities of conical type were developed in the classical works of Cheeger (see e.g. [13–15]), using methods of differential geometry, while formulas for the first terms in the asymptotic expansion of the associated heat operator were given by Brüning and Seeley [7,8] for a larger class of operators. The results mentioned suggest the possibility of tackling with success the problem of the computation of the determinant for a suitable operator on a cone and in particular on a disc. Consequently, some works appeared in the literature, where the problem of getting a description of the regularized determinant for a cone $C(N)$ over some compact manifold N was faced [3,24,16]. The approach is to deal not directly with the zeta function on the manifold $C(N)$, but with the correspondent zeta function on the product $C(N) \times S^1$, that is to say the function that describes the functional determinant for the associated quantum field theory. However, the explicit expression provided for the analytic continuation of the zeta function on $C(N)$ is in general not effective. In this work we give a complete answer to the basic case of the flat cone in $\mathbb{R}^3$, that is to say when $N = S^1_l$, the circle of fixed radius l; this answer is given by providing an effective analytic expression for the zeta function that allows to compute the main zeta invariants. Our result is effective in the sense that we obtain a function of the angle of the cone, and hence a number for the flat disc (both depending on l). More precisely, the approach described in this work follows the line introduced in [30], where the computation of the regularized determinant is pursued by the comprehension of the associated zeta function (recall that for a suitable operator L: $\det L = \exp(-\zeta'(0, L))$). Beside the importance of the particular application, our main motivation is to establish a general effective method to deal with problems where a double sum appears in the definition of the zeta function. In fact, our approach is likely to be generalized to various more general situations, including the case of a general cone. There are works in progress in these directions.

To state our main result, let L_ν be the Laplacian on a cone of angle $\operatorname{arcsec}\nu$ and length l in the Euclidean space, with the metric induced from the immersion and Dirichlet boundary condition. Then we have the following theorem.

Theorem 1. *The zeta function $\zeta(s, L_\nu)$ associated to the Laplacian L_ν can be analytically extended at $s = 0$ with*

$$\zeta(0, L_\nu) = \frac{1}{12}\left(\nu + \frac{1}{\nu}\right),$$

$$\zeta'(0, L_\nu) = \frac{1}{6}\left(\nu + \frac{1}{\nu}\right)\log l + \frac{\nu}{6} - \frac{\nu}{6}\log 2\nu - 2\nu\zeta'_R(-1)$$

$$+\frac{1}{6}\sum_{n=1}^{\infty}\left[\zeta_H(2,\nu n+1)-\frac{1}{\nu n+1}\right]+\sum_{m=2}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{n,k=1}^{\infty}\frac{1}{(\nu n+k)^{m+1}}$$
$$-\frac{1}{2}\log\nu-\frac{1}{6\nu}\left[\log 2\nu-\frac{5}{2}+\psi\left(1+\frac{1}{\nu}\right)\right].$$

An alternative formula for the derivative using integrals of special functions instead of series is given at the end of Section 5. The case of the disc ($\nu = 1$) follows as a particular case, namely the following corollary.

Corollary 1. *The zeta function associated to the Laplacian on a disc of radius l extends analytically at $s = 0$ with*

$$\zeta(0, L_1) = \tfrac{1}{6}, \qquad \zeta'(0, L_1) = \tfrac{1}{3}(\log l - \log 2) + \tfrac{1}{2}\log 2\pi + \tfrac{5}{12} + 2\zeta_R'(-1).$$

The remaining of these notes is dedicated to the proof of Theorem 1, up to Section 3 where the one-dimensional case is outlined. The work is organized as follows. Our approach starts from the observation that the problem on the cone can be thought as a generalization of the one-dimensional problem of the Laplacian with a singular potential term on the line segment. Under this point of view, the method consists in using an explicit expression for the eigenvalues of the Laplacian to write the formal series representation for the associated zeta function and hence in getting a suitable alternative representation for the same zeta function, that immediately gives an analytic extension at the origin. For this purpose, the analysis of the one-dimensional problem must be performed using the method described in Section 3. There are two clue points: first, we use the spectral decomposition of the operator to construct a regularized 'spectral' decomposition of the zeta function (Eq. (1) in Section 4); second, the regularization introduced allows us to use the method described in Section 3 to get the desired alternative representation of the zeta function. Eventually, some extra work is needed, consisting in producing the right analytic extension of some other functions appearing in the new representation of the zeta function, to get the final result for the determinant. All this is in Section 4. We delay comments and remarks to the last section.

2. Description of the problem

Let C_ν be the cone of angle $\operatorname{arcsec}\nu > 0$ and length l. Choosing the coordinates $(\theta, x) \in S^1 \times (0, l]$, the induced metric is $g = (\mathrm{d}x)^2 + \nu^{-2}x^2(\mathrm{d}\theta)^2$, and (after a Liouville transformation) the Laplacian becomes

$$L_\nu = -\partial_x^2 + \frac{1}{x^2}A_\nu(x),$$

where the operator

$$A_\nu(x) = -\nu^2\partial_\theta^2 - \tfrac{1}{4},$$

on S^1 has the complete system $\{\lambda_{\nu,n} = \nu^2 n^2 - \frac{1}{4}; \phi_{\nu,n}(\theta) = \mathrm{e}^{in\theta}\}$, $\theta \in [0, 2\pi]$, $n \in \mathbb{Z}$, each $\lambda_{\nu,n}$ having multiplicity 2, up to $\lambda_{\nu,0}$ having multiplicity 1. Then, we have a spectral decomposition of L_ν as

$$L_\nu = \sum_{n\in\mathbb{Z}} L_{\nu,n} \Pi_{\lambda_{\nu,n}},$$

where

$$L_{\nu,n} = -\mathrm{d}_x^2 + \frac{1}{x^2}\lambda_{\nu,n} = -\mathrm{d}_x^2 + \frac{1}{x^2}\left(\nu^2 n^2 - \frac{1}{4}\right).$$

With Dirichlet boundary condition at $x = l$, each $L_{\nu,n}$ has the complete system $\{\lambda_{\nu n,k} = \frac{j^2_{\nu|n|,k}}{l^2}; \psi_{\nu n,k}(x) = \frac{\sqrt{2x} J_{\nu|n|}(j_{\nu|n|,k} x/l)}{l J_{\nu|n|+1}(j_{\nu|n|,k})}\}$, $k \in \mathbb{N} - \{0\}$, where $j_{\nu|n|,k}$ are the (positive) zeros of the Bessel function $J_{\nu|n|}(x)$ [35, 15.40], and hence L_ν has the complete system

$$\left\{\lambda_{\nu n,k} = \frac{j^2_{\nu|n|,k}}{l^2}, \phi_{\nu,n}(\theta)\psi_{\nu|n|,k}(x)\right\}, \quad n = 0, \pm 1, \pm 2, \ldots, \quad k = 1, 2, \ldots$$

3. The zeta function on the line segment

In this section we present a method to deal with the zeta function associated to the one-dimensional problem that we will generalize in the next section to treat the two-dimensional problem. The results of this section are not new [20], and also the method uses classical tools [34,33,1], but our approach is different and expressly conceived for the purposes of the general case (see also [31]). For the operator[1]

$$S_\nu = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{4\nu^2 - 1}{4x^2},$$

on the line interval $(0, l]$, with the discrete resolution

$$\left\{\lambda_{\nu,n} = \frac{j^2_{\nu,n}}{l^2}, \phi_{\nu,n}(x) = \frac{\sqrt{2x} J_\nu(\lambda_{\nu,n} x)}{l J_{\nu+1}(j_{\nu,n})}\right\},$$

where $j_{\nu,n}$ are the positive zeros of the Bessel function $J_\nu(z)$, we introduce the zeta function

$$\zeta(s, S_\nu) = \sum_{n=1}^{\infty} \lambda_{\nu,n}^{-s} = l^{2s} \sum_{n=1}^{\infty} j_{\nu,n}^{-2s},$$

[1] With Dirichlet boundary condition at $x = l$.

for $\mathrm{Re}(s) > \frac{1}{2}$, also called Bessel zeta function [32]. Using the Mellin transform, we get the analytic representation

$$\zeta(s, S_\nu) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t, \nu)\, \mathrm{d}t,$$

where the trace of the heat operator is

$$f(t, \nu) = \mathrm{Tr}\, \mathrm{e}^{-tS_\nu} = \sum_{n=1}^\infty \mathrm{e}^{-\lambda_{\nu,n} t},$$

and from this the complex representation

$$f(t, \nu) = \frac{1}{2\pi i} \int_{\Lambda_c} \mathrm{e}^{-\lambda t} R(\lambda, \nu)\, \mathrm{d}\lambda,$$

where the contour[2] is $\Lambda_c = \{\lambda \in \mathbb{C} \| \arg(\lambda - c)| = \pi/4\}$, oriented counter clockwise, for some $0 < c < j_{\nu,1}$, and the trace of the resolvent is

$$R(\lambda, \nu) = \sum_{n=1}^\infty \frac{1}{\lambda - \lambda_{\nu,n}}.$$

We now observe that it is easy to express such function in terms of special functions. In fact, taking logarithmic derivative of the infinite product representation of the Bessel function $I_\nu(z)$ [35, 15.41], we get:

$$R(\lambda, \nu) = \frac{\nu}{2z^2} - \frac{1}{2z} \frac{\mathrm{d}}{\mathrm{d}z} \log I_\nu(lz).$$

Here, $z = \sqrt{-\lambda}$, we set $\arg(-\lambda) = 0$ on the line $(-\infty, 0]$ and we fix the sector $s_+ = \{z \in \mathbb{C} \| \arg z| < \pi/2\}$ for z.

At this point it is worth observing that all information about poles and residua of $\zeta(s, S_\nu)$ can be obtained using the representation introduced, asymptotic expansions for Bessel functions and classical arguments [17,21,28]. This is an easy way for producing the results relative to the so called 'constant case' when studying regular singular operators [11,7,8]. In order to obtain the derivative at $s = 0$, we need more, and precisely we introduce the following lemma.

[2] See for example [37, 12.22] for this type of Hankel's integrals.

Lemma 1. *Suppose the zeta function* $z(s,x)=\sum_{n=1}^{\infty}a_n(x)^{-s}$ *has the following representation (everything smooth in* x*):*

$$z(s,x)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\mathrm{e}^{-\lambda t}R(\lambda,x)\,\mathrm{d}\lambda\,\mathrm{d}t,$$

where the contour is as above, and there is a primitive function $-T(\lambda,x)$ *for the function* $R(\lambda,x)=-\frac{\mathrm{d}}{\mathrm{d}\lambda}T(\lambda,x)$*, satisfying the following properties:*

(a) *T is analytic near* $\lambda=0$*,*
(b) *for fixed x and large* λ *in some domain of the complex plane, the function* $T(\lambda,x)$ *has an asymptotic expansion in terms of powers and logarithms as the one considered in [7] (see also [8, Section 7]), namely* $\sum(-\lambda)^{\alpha}\log^{k}(-\lambda)$*, where* α *runs through a discrete set of real numbers with* $\alpha\to-\infty$*, and* $k=0,1$*. In particular, we will be interested in the constant and logarithmic terms, so we write*

$$T(\lambda,x)=\cdots+A(x)\log(-\lambda)+B(x)+\cdots$$

Then, $z(s,x)$ *can be analytically extended at* $s=0$ *and*

$$z(0,x)=-A(x),\qquad z'(0,x)=-B(x)+T(0,x).$$

Proof. Integrating by part, first in λ and hence in t, the given complex representation for $z(s,x)$, we get

$$z(s,x)=\frac{s}{\Gamma(s)}\int_0^{\infty}t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}T(\lambda,x)\,\mathrm{d}\lambda\,\mathrm{d}t.$$

Next, since by definition the zeta function is well defined for large s

$$z(s,x)=\frac{s^2}{\Gamma(s+1)}\int_0^{1}t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}T(\lambda,x)\,\mathrm{d}\lambda\,\mathrm{d}t+s^2f(s),$$

where f is regular near $s=0$. Because of the pole at $\lambda=0$, we have to split the complex integral as follows to use the expansion for large λ (small t):

$$\int_{\Lambda_c}=\int_{\Lambda_{-c}}-\int_{C_c}=\int_{\Lambda_{-c}}+T(0,x),$$

where C_c is a circle around the origin of radius c. Moreover, by assumption (b)

$$\int_{\Lambda_{-c}}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}T(\lambda,x)\,\mathrm{d}\lambda=\int_{\Lambda_{-c}}\frac{\mathrm{e}^{-\lambda}}{-\lambda}T\left(\frac{\lambda}{t},x\right)\,\mathrm{d}\lambda$$
$$=\cdots+\gamma A(x)+A(x)\,\log t-B(x)+\cdots$$

where we have explicited only the relevant part and the Euler constant $\gamma = -\psi(1)$ [18, 8.366.1] appears applying the following formula

$$\frac{1}{2\pi i}\int_{\Lambda_{-c}} \frac{e^{-\lambda}}{(-\lambda)^a}\log(-\lambda)\,d\lambda = -\frac{d}{da}\frac{1}{2\pi i}\int_{\Lambda_{-c}} \frac{e^{-\lambda}}{(-\lambda)^a}\,d\lambda = \frac{d}{da}\frac{1}{\Gamma(a)} = -\frac{\psi(a)}{\Gamma(a)},$$

where we have used [18, 8.315.1]. This means that we can write

$$z(s,x) = \frac{s}{\Gamma(s+1)}\left[\gamma A(x) - B(x) - \frac{1}{s}A(x) + T(0,x)\right] + s^2 g(s),$$

where again g is regular near $s = 0$, and from that, the thesis follows at once.

Applying this argument to the function $\zeta(s, S_\nu)$, we have

$$T(\lambda,\nu) = \nu\log lz - \log I_\nu(lz) - \log 2^\nu \Gamma(\nu+1),$$

and this $T(\lambda,\nu)$ is precisely the function that we will use in the next section to deal with the general case. For completeness, we conclude the computations for the one-dimensional case (compare with [31]). We have

$$A(\nu) = \tfrac{1}{2}(\nu + \tfrac{1}{2}),$$
$$B(\nu) = \tfrac{1}{2}\log 2\pi + (\nu+\tfrac{1}{2})\log l - \log 2^\nu\Gamma(\nu+1), \qquad T(0,\nu) = 0.$$

and this gives:

$$\mathrm{Res}_0(\zeta(s,S_\nu), s=0) = -\tfrac{1}{2}(\nu+\tfrac{1}{2}),$$
$$\mathrm{Res}_0(\zeta'(s,S_\nu), s=0) = \log\frac{2^{\nu-1/2}\Gamma(\nu+1)}{\sqrt{\pi}\,l^{\nu+1/2}}.$$

4. The zeta function on the cone

We introduce the zeta function associated to the operator L_ν by the formal series

$$\zeta(s, L_\nu) = \sum_{k=1}^{\infty}\sum_{n=-\infty}^{+\infty} \lambda_{\nu n,k}^{-s} = l^{2s}\sum_{k=1}^{\infty} j_{0,k}^{-2s} + 2l^{2s}\sum_{n,k=1}^{\infty} j_{\nu n,k}^{-2s},$$

that, by classical estimate on the zeros of a Bessel functions [35, 15.40] and standard argument on double series [37, 2.5], is well defined for $\mathrm{Re}(s) > 1$. The first term can be treated by the means provided in the previous section, so the problems lay in understanding the function

$$z(s,\nu)=\sum_{n,k=1}^{\infty} j_{\nu n,k}^{-2s}.$$

The remaining of this section is dedicated to the proof of Theorem 1 and its corollary and is split in four parts: in the first we outline a quite general method to deal with the zeta invariants of a double series, in the second we apply the method to the case under study, in the third we complete calculations and in the fourth we give the proof of the corollary.

4.1. Spectral decomposition

The main feature of our approach is the following 'spectral decomposition' of the function $z(s,\nu)$. For $\mathrm{Re}(s)>1$, we can reorder the terms in the double series and write

$$z(s,\nu)=\sum_{n=1}^{\infty}(\nu n)^{-2s}\sum_{k=1}^{\infty}\left(\frac{j_{\nu n,k}}{\nu n}\right)^{-2s}. \tag{1}$$

This decomposition will allow us to deal effectively with the double sum by using the tools introduced in Section 3. In fact, thanks to the uniform convergence of integrals and series, we get the complex integral representation

$$z(s,\nu)=\frac{s^2}{\Gamma(s+1)}\int_0^{\infty} t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}T(s,\lambda,\nu)\,\mathrm{d}\lambda\,\mathrm{d}t,$$

where

$$T(s,\lambda,\nu)=\sum_{n=1}^{\infty}(\nu n)^{-2s}t_n(\lambda,\nu),$$

and $t_n(\lambda,\nu)$ is defined using the correspondent function on the line segment, namely

$$\begin{aligned} t_n(\lambda,\nu)&=-\sum_{k=1}^{\infty}\log\left[1+\frac{(\nu n)^2(-\lambda)}{j_{\nu n,k}^2}\right]\\ &=-\log I_{\nu n}(\nu n z)-\nu n\ \log 2+\nu n\ \log z-\log\Gamma(\nu n+1)+\nu n\ \log(\nu n). \end{aligned}$$

To proceed further, we need a generalization of Lemma 1. In fact, the function $T(s,\lambda,\nu)$ is not necessarily analytic in s at $s=0$. This depends on the behavior of $t_n(\lambda,\nu)$ for large n: more precisely, a singular behavior can appear only from a term $\frac{1}{n}$ in the expansion of t_n. This suggests to split it in two terms as follows. From the definition (see also Section 4.2),

$t_n(\lambda, \nu)$ has a uniform expansion in some domain of the complex λ plane for large νn (recall ν is a fixed parameter). Set $p_n(\lambda, \nu) = t_n(\lambda, \nu) - \frac{1}{\nu n} f(\lambda, \nu)$, where $f(\lambda, \nu)$ is the coefficient of the term in $\frac{1}{\nu n}$ in the above expansion of $t_n(\lambda, \nu)$ for large νn. Then

$$t_n(\lambda, \nu) = p_n(\lambda, \nu) + \frac{1}{\nu n} f(\lambda, \nu),$$

and

$$z(s, \nu) = \frac{s^2}{\Gamma(s+1)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_c} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} P(s, \lambda, \nu)\, \mathrm{d}\lambda\, \mathrm{d}t \\ + \nu^{-2s-1} \frac{s^2}{\Gamma(s+1)} \zeta_R(2s+1) \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_c} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} f(\lambda, \nu)\, \mathrm{d}\lambda\, \mathrm{d}t,$$

where

$$P(s, \lambda, \nu) = \sum_{n=1}^{\infty} (\nu n)^{-2s} p_n(\lambda, \nu),$$

is regular at $s = 0$. Thus, the first term in $z(s, \nu)$ can be treated precisely by the same means as in Lemma 1; this gives

$$\frac{s^2}{\Gamma(s+1)} \int_0^1 t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_c} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} P(s, \lambda, \nu)\, \mathrm{d}\lambda\, \mathrm{d}t \\ = \frac{s^2}{\Gamma(s+1)} \int_0^1 t^{s-1} [\gamma A(s, \nu) + A(s, \nu) \log t - B(s, \nu) + P(s, 0, \nu)]\, \mathrm{d}t \\ = \frac{s}{\Gamma(s+1)} \left[\gamma A(s, \nu) - B(s, \nu) - \frac{1}{s} A(s, \nu) + P(s, 0, \nu) \right],$$

plus a regular term, vanishing with its derivative at $s = 0$; here

$$A(s, \nu) = \sum_{n=1}^{\infty} (\nu n)^{-2s} a_n(\nu), \qquad B(s, \nu) = \sum_{n=1}^{\infty} (\nu n)^{-2s} b_n(\nu),$$

and a_n and b_n are the coefficients appearing in the asymptotic expansion of $p_n(\lambda, \nu)$ for fixed n and large λ in the appropriate domain[3]

$$p_n(\lambda, \nu) = \cdots + a_n(\nu) \log(-\lambda) + b_n(\nu) + \cdots$$

[3] The existence of such an expansion follows from the definition.

We have obtained the following expression for the function $z(s,\nu)$

$$\begin{aligned} z(s,\nu) = {} & \frac{s}{\Gamma(s+1)}\left[\gamma A(s,\nu) - B(s,\nu) - \frac{1}{s}A(s,\nu) + P(s,0,\nu)\right] \\ & + \nu^{-2s-1}\frac{s^2}{\Gamma(s+1)}\zeta_R(2s+1)\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{e^{-\lambda t}}{-\lambda}f(\lambda,\nu)\,d\lambda\,dt \\ & + \frac{s^2}{\Gamma(s+1)}h(s), \end{aligned} \tag{2}$$

where h is analytic at $s=0$. The method outlined is quite general, but to proceed further we need the explicit expressions of f and P; this is in the next subsection.

4.2. *The values of* $z(0,\nu)$ *and* $z'(0,\nu)$

To get explicit expressions for $P(s,\lambda,\nu)$ and $f(\lambda,\nu)$ we use the representation [25, 10.7]of the Bessel function $I_\nu(\nu x)$:

$$I_\nu(\nu z) = \frac{1}{1+\eta_{2,1}(\nu,\infty)}\frac{e^{\nu\xi(z)}}{\sqrt{2\pi\nu}(1+z^2)^{1/4}}\left[1+\frac{1}{\nu}U_1(z)+\eta_{2,1}(\nu,z)\right],$$

where

$$\xi(z) = \sqrt{1+z^2} + \log\frac{z}{1+\sqrt{1+z^2}}, \qquad U_1(z) = \frac{1}{8\sqrt{1+z^2}} - \frac{5}{24(1+z^2)^{3/2}},$$

and $\eta_{2,1}(\nu,z)$ is the error term, bounded for large ν uniformly in z in the opportune domain. Inserting this in the expression of $t_n(\lambda,\nu)$ we get

$$f(\lambda,\nu) = -U_1(\sqrt{-\lambda}),$$

$$p_n(\lambda,\nu) = -\log I_{\nu n}(\nu n z) + \nu n\log(\nu n z) - \nu n\log 2 - \log\Gamma(\nu n+1) + \frac{1}{\nu n}U_1(z).$$

We start by computing

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{e^{-\lambda t}}{-\lambda}f(\lambda,\nu)\,d\lambda\,dt = -\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{e^{-\lambda t}}{-\lambda}U_1(\sqrt{-\lambda})\,d\lambda\,dt.$$

For $c>1$, consider

$$\frac{1}{2\pi i}\int_{\Lambda_c}\frac{e^{-\lambda t}}{-\lambda}\frac{1}{(1-\lambda)^a}\,d\lambda;$$

where $c < 1$; this can be computed in the new variable $z = \lambda - 1$ (see [37, 12.22, 18,8.353.3])

$$-\frac{1}{2\pi i}\mathrm{e}^{-t}\int_{\Lambda_{c-1}}\frac{\mathrm{e}^{-zt}}{z+1}(-z)^{-a}\,\mathrm{d}z = \frac{1}{\pi}\sin(\pi a)\Gamma(1-a)\Gamma(a,t).$$

Recalling the relation between the incomplete Gamma function [18, 8.35] and the probability integral [18, 8.25], this gives

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}f(\lambda,\nu)\,\mathrm{d}\lambda\,\mathrm{d}t = \frac{\Gamma(s+1/2)}{12\sqrt{\pi}}\left(\frac{1}{s}+5\right).$$

Next, we need the asymptotic expansion of $p_n(\lambda,\nu)$ for large λ and fixed n; By classical asymptotics of Bessel and Gamma functions [18, 8.451.5, 8.344], we get

$$p_n(\lambda,\nu) = -\nu n\sqrt{\lambda} + a_n(\nu)\log(-\lambda) + b_n(\nu) + \mathrm{O}((-\lambda)^{-1/2}),$$

where the interesting terms are:

$$a_n(\nu) = \tfrac{1}{2}(\nu n + \tfrac{1}{2}),$$
$$b_n(\nu) = \tfrac{1}{2}\log 2\pi + (\nu n + \tfrac{1}{2})\log\,\nu n - \nu n\log 2 - \log\,\Gamma(\nu n + 1).$$

This gives

$$A(s,\nu) = \sum_{n=1}^\infty(\nu n)^{-2s}a_n(\nu) = \tfrac{1}{2}\nu^{1-2s}\zeta_R(2s-1) + \tfrac{1}{4}\nu^{-2s}\zeta_R(2s),$$

$$\begin{aligned}B(s,\nu) = \sum_{n=1}^\infty(\nu n)^{-2s}b_n(\nu) = -&\sum_{n=1}^\infty(\nu n)^{-2s}\\ &\times\,\log\Gamma(\nu n+1) + \tfrac{1}{2}\nu^{-2s}\log(2\pi\nu)\zeta_R(2s) + \nu^{1-2s}\log\tfrac{\nu}{2}\zeta_R(2s-1)\\ &-\tfrac{1}{2}\nu^{-2s}\zeta_R'(2s) - \nu^{1-2s}\zeta_R'(2s-1).\end{aligned}$$

The last step is $P(s,0,\nu)$. Recalling the behavior of $I_\nu(z)$ for small z [18, 8.445], we evaluate

$$p_n(0,\nu) = -\frac{1}{12}\frac{1}{\nu n},$$

and hence

$$P(s,0,\nu) = \sum_{n=1}^\infty(\nu n)^{-2s}p_n(0,\nu) = -\tfrac{1}{12}\nu^{-2s-1}\zeta_R(2s+1).$$

Now the explicit expressions for all the quantities involved in the definition of the function $z(s,\nu)$ are available. From Eq. (2), we get

$$z(s,\nu)=\frac{s}{\Gamma(s+1)}\left[\gamma A(s,\nu)-B(s,\nu)-\frac{1}{s}A(s,\nu)+P(s,0,\nu)\right]$$
$$+\nu^{-2s-1}\frac{s^2}{\Gamma(s+1)}\zeta_R(2s+1)\frac{\Gamma(s+1/2)}{\sqrt{\pi}}\left(\frac{1}{12s}+\frac{5}{12}\right)+\frac{s^2}{\Gamma(s+1)}h(s).$$

From this, by using the known behavior of the Riemann's zeta function $\zeta_R(z)$ near its singular point $z=1$ [18, 9.533.2], that gives near $s=0$

$$\zeta_R(2s+1)=\frac{1}{2s}+\gamma+\mathrm{o}(s),$$

and the fact that all the other quantities are regular at $s=0$, we can compute

$$z(0,\nu)=-A(0,\nu)+\frac{1}{24\nu}=-\frac{1}{2}\nu\zeta_R(-1)-\frac{1}{4}\zeta_R(0)+\frac{1}{24\nu}=\frac{1}{24}\nu+\frac{1}{8}+\frac{1}{24\nu}.$$

This proves the first part of Theorem 1. Eventually, we can derive with respect to s, and evaluate the derivative at $s=0$. We get

$$z'(0,\nu)=P(0,0,\nu)-A'(0,\nu)-B(0,\nu)+\frac{1}{12\nu}\left(\gamma-\log 2\nu+\tfrac{5}{2}\right)$$
$$=\eta(0,\nu)+\tfrac{1}{4}\log 2\pi-\tfrac{\nu}{12}\log 2+\tfrac{1}{12\nu}\left(\gamma-\log 2\nu+\tfrac{5}{2}\right),$$

where

$$\eta(s,\nu)=\sum_{n=1}^{\infty}(\nu n)^{-2s}\log\Gamma(\nu n+1)-\tfrac{1}{12}\nu^{-2s-1}\zeta_R(2s+1),$$

and to complete the proof of Theorem 1 we need the explicit computation of $\eta(0,\nu)$; this is done in the next subsection.

4.3. *Computation of* $\eta(0,\nu)$

By definition $\eta(s,\nu)$ is regular at $s=0$. On the other side, it is clearly not allowed to get its value at $s=0$ by simple substitution of the value $s=0$ in the defining expression, because the two terms are not regular independently at this value of s. To get rid of the singularity, we use the series representation for the logarithm of the Gamma function [18, 8.343.2]

$$
\begin{aligned}
\eta(s,\nu) &= \sum_{n=1}^{\infty}(\nu n)^{-2s}\left[\log\Gamma(\nu n) - \frac{1}{12}\frac{1}{\nu n}\right] + \nu^{-2s}\log\nu\zeta_R(2s) - \nu^{-2s}\zeta_R'(2s)\\
&= \sum_{n=1}^{\infty}(\nu n)^{-2s}\left[\nu n\log\nu n - \nu n - \tfrac{1}{2}\log\nu n + \tfrac{1}{2}\log 2\pi\right.\\
&\quad\left. + \frac{1}{12}\sum_{k=1}^{\infty}\frac{1}{(\nu n+k)^2} + \frac{1}{2}\sum_{m=2}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{k=1}^{\infty}\frac{1}{(\nu n+k)^{m+1}} - \tfrac{1}{12}\tfrac{1}{\nu n}\right]\\
&\quad + \nu^{-2s}[\log\nu\zeta_R(2s) - \zeta_R'(2s)]\\
&= \nu^{1-2s}\log\nu\zeta_R(2s-1) - \nu^{1-2s}\zeta_R'(2s-1) - \nu^{1-2s}\zeta_R(2s-1)\\
&\quad + \tfrac{1}{2}\nu^{-2s}\log\nu\zeta_R(2s) - \tfrac{1}{2}\nu^{-2s}\zeta_R'(2s) + \tfrac{1}{2}\log 2\pi\nu^{-2s}\zeta_R(2s)\\
&\quad + M(s,\nu) + \delta(s,\nu),
\end{aligned}
$$

where

$$
M(s,\nu) = \frac{1}{12}\sum_{n=1}^{\infty}(\nu n)^{-2s}\left[\sum_{k=1}^{\infty}\frac{1}{(\nu n+k)^2} - \frac{1}{\nu n}\right],
$$

and

$$
\delta(s,\nu) = \frac{1}{2}\sum_{m=2}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{n,k=1}^{\infty}\frac{(\nu n)^{-2s}}{(\nu n+k)^{m+1}}.
$$

Now, all the terms are regular at $s=0$ and we can compute $\eta(0,\nu)$. In particular, for what concerns M, it is convenient to decompose it as follows[4]

$$
M(s,\nu) = \frac{1}{12}\sum_{n=1}^{\infty}(\nu n)^{-2s}\left[\zeta_H(2,\nu n+1) - \frac{1}{\nu n+1}\right] - \frac{1}{12}\sum_{n=1}^{\infty}(\nu n)^{-2s}\left[\frac{1}{\nu n(\nu n+1)}\right]
$$

this gives

$$
M(0,\nu) = \frac{1}{12}\sum_{n=1}^{\infty}\left[\zeta_H(2,\nu n+1) - \frac{1}{\nu n+1}\right] - \frac{1}{12\nu}\left[\gamma + \psi\left(\frac{1+1}{\nu}\right)\right].
$$

[4] Where ζ_H is the Hurwitz zeta function.

We can provide two alternative integral and series representations for $M(0, \nu)$ by using the Plana theorem [35, p. 146]:

$$M(0,\nu)=\frac{1}{12}\left\{\frac{1}{2}\zeta_H(2,\nu+1)+\frac{1}{\nu}[\log\nu-\psi(\nu+1)]+N(\nu)-\frac{\gamma}{\nu}\right\},$$

where

$$N(\nu)=i\int_0^\infty\frac{\zeta_H(2,\nu(1+iy)+1)-\zeta_H(2,\nu(1-iy)+1)}{\mathrm{e}^{2\pi y}-1}\,\mathrm{d}y,$$

or

$$N(\nu)=\sum_{k=0}^\infty\sum_{n=1}^\infty|B_{2k+2}|\frac{\nu^{2k+1}}{(n+\nu)^{2k+3}},$$

and the B_i are the Bernoulli numbers [18, 9.71]. Eventually, the expression for $\eta(0, \nu)$ is

$$\eta(0,\nu)=-\tfrac{1}{12}\nu(\log\nu-1)-\nu\zeta_R'(-1)-\tfrac{1}{4}\log\nu+M(0,\nu)+\delta(0,\nu),$$

where $M(0, \nu)$ is given above and

$$\delta(0,\nu)=\frac{1}{2}\sum_{m=2}^\infty\frac{m}{(m+1)(m+2)}\sum_{n,k=1}^\infty\frac{1}{(\nu n+k)^{m+1}}.$$

This completes the proof of Theorem 1.

4.4. *Proof of Corollary 1*

The direct computation of the limit for $\nu = 1$ of the expression given in Theorem 1 is not easy,[5] so we proceed in the following alternative way. First, introduce the function (see also Lemma 2)

$$\chi(s)=\sum_{n,k=1}^\infty(n+k)^{-s},$$

for $\mathrm{Re}(s) > 2$. The double sum corresponds to the ordinary one

$$\chi(s)=\sum_{n=2}^\infty(n-1)n^{-s},$$

[5] Although, notice that $M(0, 1)$ can be simplified giving $\frac{1}{12}\left(1-\frac{\pi^2}{6}\right)$.

simply by recollecting the terms. Hence:

$$\chi(s) = \zeta_R(s-1) - \zeta_R(s),$$

and $\chi'(0) = \zeta_R'(-1) + \frac{1}{2}\log 2\pi$. On the other side, if we analyze $\chi(s)$ by the same means that we have used to analyze $z(s, \nu)$, we obtain

$$\chi'(0) = \eta(0,1) + \tfrac{1}{4}\log 2\pi + \tfrac{1}{12}\gamma,$$

and therefore

$$\eta(0,1) = \tfrac{1}{4}\log 2\pi + \zeta_R'(-1) - \tfrac{1}{12}\gamma.$$

5. Comments and remarks

The value at the origin of the zeta function associated to the Laplace operator on a compact Riemannian manifold M of dimension m with boundary is well known [6]. Namely, provided the heat kernel operator has the expansion

$$\sum_{n=0}^{\infty} a_n t^{(n-m)/2},$$

for small t, then $\zeta(0, \Delta_M) = a_m - \dim\ker \Delta_M$. The first coefficients can be computed in terms of local geometric quantities [6,17 4.5], in particular on the disc $D = C_1$ of radius l, $L_{xx} = \frac{1}{x}$, and

$$a_2 = \frac{1}{24\pi}\int_{\partial D} 2L_{xx}|_{x=l}\,\mathrm{d}\theta = \tfrac{1}{6},$$

in agreement with Corollary 1. The situation on the cone is more delicate, since a singular term appears. This problem has been studied in [8], by analyzing the asymptotic expansion of the resolvent, and gives for the Laplacian on the cone C_ν

$$a_2 = \tfrac{1}{12}\left(\nu - \frac{1}{\nu}\right) + a_2^{\mathrm{reg}},$$

where the regular coefficient can be computed as above, and is $a_2^{\mathrm{reg}} = \frac{1}{6\nu}$.

We conclude by providing an alternative representation for $\zeta'(0, L_\nu)$. This can be obtained studying the two-dimensional Hurwitz zeta function ($\mathrm{Re}(\alpha) > 0$)

$$\chi(s, \alpha) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(k + \alpha n)^{-s},$$

that generalizes the function χ introduced in Section 4.4. This function can be studied using the Plana theorem as in [30]. As a result we get the following lemma.

Lemma 2. *The function $\chi(s,\alpha)$ has a regular analytic continuation to the whole complex plane up to simple poles at $s=1$ and 2 with residua $-\frac{1}{2}(1+\frac{1}{\alpha})$ and $\frac{1}{\alpha}$ respectively. The point $s=0$ is regular and*

$$\chi(0,\alpha)=\frac{1}{4}+\frac{1}{12\alpha}+\frac{\alpha}{12},$$

$$\chi'(0,\alpha)=\frac{1}{2}\log\Gamma(\alpha+1)+\frac{1}{4}\log 2\pi-1-\frac{1}{\alpha}\zeta_H(-1,\alpha+1)-\frac{1}{\alpha}\zeta'_H(-1,\alpha+1)$$
$$+i\int_0^\infty\log\frac{\Gamma(\alpha(1+it))}{\Gamma(\alpha(1-it))}\frac{\mathrm{d}t}{\mathrm{e}^{2\pi t}-1}.$$

Alternatively, we can apply the method used in Section 4 to compute $\chi'(0,\alpha)$:

$$\chi'(0,\alpha)=\eta(0,\alpha)+\frac{1}{4}\log 2\pi+\frac{1}{12\alpha}(\gamma-\log\alpha).$$

By comparison of the results we get the integral representation for the $\eta(0,\alpha)$

$$\eta(0,\alpha)=\frac{1}{2}(\alpha-1)+i\int_0^\infty\log\frac{\Gamma(\alpha(1+iy))}{\Gamma(\alpha(1-iy))}\frac{\mathrm{d}y}{\mathrm{e}^{2\pi y}-1}+\frac{1}{2}\log\Gamma(\alpha+1)$$
$$+\frac{1}{12\alpha}(1-\gamma+\log\alpha)-\frac{1}{\alpha}\zeta'_H(-1,\alpha+1),$$

and using this expression with $\alpha=\nu$, we obtain the following alternative representation for the derivative of $\zeta(s,L_\nu)$ at $s=0$:

$$\zeta'(0,L_\nu)=\tfrac{1}{6}\left(\nu+\frac{1}{\nu}\right)\log l+\nu-1-\frac{1}{6}\nu\log 2+(7-2\log 2)\frac{1}{12\nu}$$
$$+2i\int_0^\infty\log\frac{\Gamma(\nu(1+iy))}{\Gamma(\nu(1-iy))}\frac{\mathrm{d}y}{\mathrm{e}^{2\pi y}-1}+\log\Gamma(\nu+1)-\frac{2}{\nu}\zeta'_H(-1,\nu+1).$$

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