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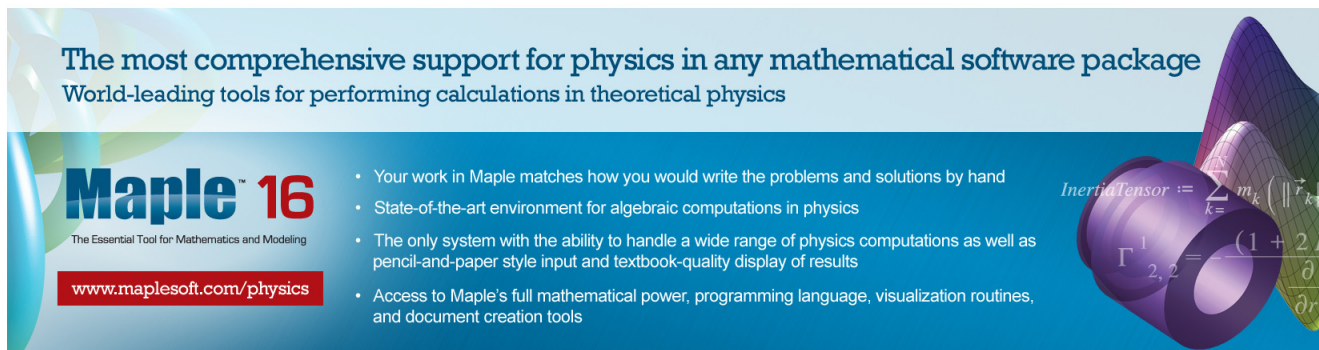
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Inertia Tensor := $\sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \mathbf{1} - \vec{r}_k \vec{r}_k^T \right)$

$\Gamma_{2,2}^1 = \frac{(1 + 2r)}{\partial r}$

Singular perturbations with boundary conditions and the Casimir effect in the half space

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We study the self-adjoint extensions of a class of nonmaximal multiplication operators with boundary conditions. We show that these extensions correspond to singular rank 1 perturbations (in the sense of Albeverio and Kurasov [*Singular Perturbations of Differential Operators* (Cambridge University Press, Cambridge, 2000)]) of the Laplace operator, namely, the formal Laplacian with a singular delta potential, on the half space. This construction is the appropriate setting to describe the Casimir effect related to a massless scalar field in the flat space-time with an infinite conducting plate and in the presence of a pointlike “impurity.” We use the relative zeta determinant (as defined in the works of Müller [“Relative zeta functions, relative determinants and scattering theory,” *Commun. Math. Phys.* **192**, 309 (1998)] and Spreafico and Zerbini [“Finite temperature quantum field theory on noncompact domains and application to delta interactions,” *Rep. Math. Phys.* **63**, 163 (2009)]) in order to regularize the partition function of this model. We study the analytic extension of the associated relative zeta function, and we present explicit results for the partition function and for the Casimir force. © 2010 American Institute of Physics. [doi:10.1063/1.3397551]

I. INTRODUCTION

Recently, there has been a growing interest in the Casimir effect, namely, the manifestation of vacuum energy at experimental as well as at theoretical level (see, for example, Refs. 22 and 23 and references therein). Because of the increasing interest in the Casimir effect and in spite of several results which have already been obtained, “a solvable model” that permits one to obtain systematically explicit results is of the greatest interest. In this paper, we present such a model. Moreover, we put at work mathematical techniques, which are of interest by themselves.

We shall study the Casimir effect related to a massless scalar field in a flat space-time modified by the presence of a pointlike (uncharged) “impurity,” modeled by deltalike potentials, in manifolds with and without boundary.

The boundaryless delta potential case has been already treated (see, for example, Refs. 1, 7, 21, 26, 28, and 29). This case is also referred to as semitransparent boundary conditions (see Refs. 8, 16, 18, and 20 and references therein). Here, we shall deal with the new case of a delta potential on the half space.

On general grounds, from one side, we present a rigorous mathematical description of the

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Schrödinger-like operators with deltalike potentials, and from the other side, we will make use of a technique to regularize the functional determinant of self-adjoint elliptic operators defined on noncompact manifolds associated with continuous spectrum.

In order to start formulating the problem, we use the approach of finite temperature quantum field theory based on the imaginary time formalism (see, for example, Refs. 15, 13, 9, 14, 10, and 25 and references therein). We consider a massless scalar free field in four dimensional Minkowski space-time interacting with an external field represented by a potential V . Thus, one is dealing with the manifold $X(T) = S_{\beta/2\pi}^1 \times M$, where S_r^1 is the circle of radius r , $\beta = 1/T$, the period of the imaginary compactified time, is the inverse of the temperature, and M is a three dimensional manifold. The relevant operator reads as $H = -\Delta_{X(T)} + V = -\partial_u^2 - \Delta_M + V = -\partial_u^2 + L_M$, where Δ_Y is the Laplace–Beltrami operator on a manifold Y defined by some Riemannian structure, and $V: M \rightarrow \mathbb{R}$ is a suitable potential.

The canonical partition function at temperature T of this model may be formally written as

$$\log Z = -\frac{\beta}{2} \sum_{\lambda \in \text{Sp } L_M} \lambda^{1/2} - \sum_{\lambda \in \text{Sp } L_M} (1 - e^{-\beta\sqrt{\lambda}}). \quad (1)$$

Here, $\text{Sp } L_M$ is the spectrum (a self-realization of) L_M . We are assuming M to be a compact manifold and V a smooth potential. The first term on the right-hand side of Eq. (1) corresponds to the vacuum energy contribution (Casimir energy) given by

$$E_c = -\lim_{\beta \rightarrow \infty} \partial_\beta \log Z = \frac{1}{2} \sum_{\lambda \in \text{Sp } L_M} \lambda^{1/2}, \quad (2)$$

while the second one, corresponding to the statistical sum contribution, is vanishing in the zero temperature limit. In order to give a meaning to the divergent first term in Eq. (1), one may make use of the well-known zeta function regularization, namely, one introduces the generalized zeta function, defined for large values of the real part of s by

$$\zeta(s; L_M) = \sum_{\lambda \in \text{Sp } L_M} \lambda^{-s},$$

and by analytic continuation elsewhere, and one replaces Eq. (2) by $E_c = \frac{1}{2} \zeta(-1/2; L_M)$.

Nevertheless, this approach does not work in general because it may happen that $\zeta(s; L_M)$ is singular in $s = -1/2$. A possible approach is to consider $\log Z$ as a regularized functional determinant of the operator $H = -\partial_u^2 + L_M$, namely, $\log Z = -\frac{1}{2} \zeta'(0, H)$ (see, for example, Ref. 12). As a consequence, it is possible to show that it can be expressed in terms of some invariants of the geometric zeta function, i.e., the zeta function of the restriction of H to M , and introducing another spectral function, the generalized Dedekind eta function,²⁵ defined for a positive operator A with discrete spectrum by

$$\eta(\tau; A) = \prod_{\lambda \in \sigma(A)} (1 - e^{-\tau\sqrt{\lambda}}).$$

In fact, assuming that $-\Delta_M + V$ has trivial kernel, by Proposition 3 of Ref. 25 (see also Refs. 9, 14, and 12), we have

$$\zeta(0; H) = \frac{1}{T} \text{Res}_{s=-1/2} \zeta(s; -\Delta_M + V),$$

while by Corollary 1 of Ref. 25,

$$\zeta'(0; H) = -\frac{1}{2T} \operatorname{Res}_0 \zeta(s; -\Delta_M + V) - \frac{1 - \log 2}{T} \operatorname{Res}_1 \zeta(s; -\Delta_M + V) - 2 \log \eta\left(\frac{1}{T}; -\Delta_M + V\right),$$

where, for a meromorphic function $f(s)$, Res_0 and Res_k are defined by means of the Laurent expansion

$$f(s) = \operatorname{Res}_0 f(s) + \sum_{k=1}^{\infty} \frac{1}{(s - s_0)^k} \operatorname{Res}_k f(s).$$

In this paper, we will consider explicitly differential operators with a singular potential on noncompact manifolds with or without boundary. Thus, we shall need to generalize the above results to the noncompact case. This will be done in detail in the next sections.

Now let us introduce the class of models we are going to investigate. In order to describe the class of operators, we shall deal with, we start with an heuristic treatment which will be mathematically justified in the next sections. We first recall the Lipmann–Schwinger equations for an operator $H = H_0 + V$ defined in $L_2(\mathbb{R}^n)$, where $H_0 = -\Delta$ is minus Laplacian and V is a suitable nonconfining potential. They are given by

$$\Psi^\pm(x) = \Psi_0^\pm(x) + \int_{\mathbb{R}^n} G_k^{(0)}(x, y) V(y) \Psi^\pm(y) dy, \quad x \in \mathbb{R}^n,$$

where $G_k^{(0)}(x, y)$ is the Green’s function of the unperturbed operator H_0 , namely,

$$(H_0 - k^2) G_k^{(0)}(x, y) = \delta(x - y).$$

For example, for $n=1$, one has

$$G_k^{(0)}(x, y) = \frac{i}{2k} e^{ik|x-y|}.$$

The above integral equation is the counterpart of the well-known resolvent identity associated with the resolvent of the operator H .

We shall consider singular perturbations of the form

$$V(x) = g \delta(x - a),$$

where g is the real coupling constant, and we limit our analysis to the cases $n=1, 2, 3$, since only within these cases, one may implement deltalike interactions by self-adjoint operators in Hilbert space (see, for example, Ref. 2). Hence, heuristically

$$H = -\Delta + g \delta(x - a). \tag{3}$$

In this case, one formally has as solution

$$\Psi^\pm(x) = \Psi_0^\pm(x) + g G_k^{(0)}(x, a) \Psi^\pm(a).$$

Since $G_k^{(0)}(x, a)$ is singular when $x \rightarrow a$ for $n=2, 3$, the above solution of the original integral equation is *inconsistent* and one has to deal with a regularization and a renormalization procedure, first introduced in Ref. 5. Here, we describe a regularization in the configuration space. First, the regularization may be achieved by making the replacement $g \rightarrow g(\varepsilon)$ and $G_k^{(0)}(x, a) \rightarrow G_k^{(0)}(x + \varepsilon, a)$, for $\varepsilon > 0$. As a result, neglecting terms which vanish as the cutoff ε is removed, i.e., when $\varepsilon \rightarrow 0$, we may solve the above equation and arrive at

$$\Psi^\pm(a) = \Psi_0^\pm(a) + g(\varepsilon) G_k^{(0)}(a, a + \varepsilon) \Psi^\pm(a).$$

Thus, the regularized solution may be written as

$$\Psi^\pm(x) = \Psi_0^\pm(x) + \frac{1}{\frac{1}{g(\varepsilon)} - G_k^{(0)}(a, a + \varepsilon)} G_k^{(0)}(x, a) \Psi_0^\pm(a). \quad (4)$$

Furthermore, the renormalization consists of assuming that $g(\varepsilon)$ vanishes in the limit $\varepsilon \rightarrow 0$ in such a way that

$$\frac{1}{g(\varepsilon)} - G_k^{(0)}(a, a + \varepsilon) = \frac{1}{g_R} - \text{Res}_{\varepsilon=0} G_k^{(0)}(a, a + \varepsilon) + O(\varepsilon)$$

for some $g_R \neq 0$. As a consequence, one may remove the cutoff and one arrives at a finite expression, where renormalized quantities appear, that is,

$$\Psi^\pm(x) = \Psi_0^\pm(x) + \frac{1}{\frac{1}{g_R} - \text{Res}_{z=a} G_k^{(0)}(a, z)} G_k(x, a) \Psi_0^\pm(a).$$

For $n > 3$, formally the above expression is still valid, but the interpretation of $\Psi^\pm(x)$ as scattering states related to a self-adjoint Hamiltonian defined on an Hilbert space no longer holds.

As an example, let us consider $n=3$ and $a=0$. Then, one has

$$G_k^{(0)}(x) = \frac{1}{4\pi|x|} e^{ik|x|}$$

and

$$\Psi^\pm(x) = e^{\pm ikx} + \frac{1}{\frac{1}{g_R} - ik} \frac{e^{ik|x|}}{|x|}.$$

Instead, for $n=2$, one has

$$G_k(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|),$$

with $H_0^{(1)}$ being a Hankel function. Due to the presence of a logarithmic singularity for $x=y$, the regularization procedure leads to the appearance of an arbitrary dimensional scale ℓ and the regularized coupling constant has to be “running” in order to ensure the independence of the physical observables from ℓ . The result is

$$\Psi^\pm(x) = \Psi_0^\pm(x) + \frac{(i/4)H_0^{(1)}(k|x|)}{\frac{1}{g_R(\ell)} + \frac{1}{2\pi}(\ln(k\ell/2i) - \Psi(1))}.$$

Going back to the $n=3$ case, one may obtain the physical meaning of g_R , considering the nonrelativistic scattering of a particle of mass $m > 0$. In this case, the operator H_0 is the kinetic energy (the Planck constant being taken to be 1) and we have for the scattering wave functions $\Psi^\pm(x)$

$$\Psi^\pm(\vec{x}) = e^{\pm ikx} + \frac{2mg_R}{1 - 2img_R k} \frac{e^{ik|x|}}{|x|}.$$

The scattering amplitude can be read off and is

$$f(k) = \frac{2mg_R}{1 - 2img_R k},$$

and the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(k)|^2 = \frac{4m^2 g_R^2}{1 + 4m^2 g_R^2 k^2}.$$

The scattering length may be defined as

$$a^2 = \lim_{k \rightarrow 0} |f(k)|^2,$$

in such a way that $\lim_{k \rightarrow 0} \sigma(k) = 4\pi a^2$. Thus,

$$a^2 = 4m^2 g_R^2,$$

namely, the regularized coupling constant is proportional to the scattering length of the related nonrelativistic three dimensional scattering process.

It is easy to show that Eq. (4) is equivalent to the following expression for the kernels of the resolvents:

$$G_\lambda(x, y) = G_\lambda^{(0)}(x, y) + \frac{1}{\frac{1}{g_R} - \text{Res}_0 G_k^{(0)}(a, z)} G_\lambda^{(0)}(x, a) G_\lambda^{(0)}(y, a). \quad (5)$$

This formula is valid in general. For example, when the unperturbed operator H_0 is the opposite of the Laplace operator defined in the manifold with boundary $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, we may repeat the above arguments and arrive at Eq. (5), in which $G_\lambda^{(0)}(x, y)$ and $G_\lambda(x, y)$ now satisfy a suitable boundary condition, for example, the Dirichlet boundary condition

$$G_\lambda^{(0)}(0, y) = 0.$$

In the physically relevant case of $n=3$, we have

$$G_\lambda^{(0)}(x, y) = \frac{1}{4\pi} \left(\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} - \frac{e^{i\sqrt{\lambda}|x-Ry|}}{|x-Ry|} \right),$$

where R is the spatial reflection with respect to the plane that forms the boundary \mathbb{R}^2 . For example, in this case, the renormalization leads to

$$\frac{1}{g(\varepsilon)} - G_\lambda^{(0)}(\varepsilon) = \frac{1}{g_R} - \frac{\lambda}{4\pi} - \frac{e^{-2\lambda a}}{2a} + O(\varepsilon). \quad (6)$$

Again, for $n > 3$, one may formally consider the above expressions, but without any references to some Hilbert space.

In the following sections, making use of the method of self-adjoint extensions and the general theory of singular perturbations, we will present a rigorous mathematical derivation of the above heuristic results.

II. SELF-ADJOINT EXTENSIONS OF NONMAXIMAL MULTIPLICATION OPERATORS

A. General setting

Let \mathcal{H} be an Hilbert space (complete and separable) and A a self-adjoint operator in \mathcal{H} . Fixing a suitable restriction \dot{A} of A , it is possible to construct a one parameter family of self-adjoint operators A_α containing the initial operator A . This quite general setting was developed in Ref 3,

Sec. 1.2.2. We recall here the main points of the construction, and we give a new proof of the main result, stated in Lemma 1.2.3 of Ref. 3. Let $|A|=(A^\dagger A)^{1/2}$, and for $s \geq 0$, let

$$\mathcal{H}_s = \text{dom}(|A|^{s/2}) = \{v \in \mathcal{H} \mid (|A| + I)^{s/2} v \in \mathcal{H}\}.$$

Note that $\mathcal{H}_2 = \text{dom}(A)$ and $\mathcal{H}_0 = \mathcal{H}$. \mathcal{H}_s is a complete Hilbert space with scalar product

$$(u, v)_s = ((|A| + I)^s u, v)_0.$$

Obviously, $(|A| + zI)^{s/2}$ is an isometry of \mathcal{H}_s onto \mathcal{H} for all s and for all z that is neither zero nor a negative real number. Let $\mathcal{H}_{-s} = \mathcal{H}_s^\dagger$ be the adjoint space of \mathcal{H}_s . We define the mapping

$$(|A| + zI)^{s/2} : \mathcal{H} \rightarrow \mathcal{H}_{-s},$$

$$(|A| + zI)^{s/2} : u \mapsto (|A| + zI)^{s/2} u,$$

by

$$((|A| + zI)^{s/2} u)(v) = (u, (|A| + zI)^{s/2} v)_0$$

for all $v \in \mathcal{H}_s$, and $\text{Re}(z) > 0$. On the other hand, for each $u \in \mathcal{H}_{-s}$, there exists a vector $u' \in \mathcal{H}_s$ such that

$$u(v) = (u', v)_s$$

for all $v \in \mathcal{H}_s$. It follows from the definition of \mathcal{H}_s that $(|A| + zI)^{s/2} u' \in \mathcal{H}$ and

$$u(v) = (u', v)_s = ((|A| + zI)^{s/2} u', (|A| + zI)^{s/2} v)_0,$$

for $\text{Re}(z) > 0$. Therefore, we have a map

$$(|A| + zI)^{-s/2} : \mathcal{H}_{-s} \rightarrow \mathcal{H},$$

$$(|A| + zI)^{-s/2} : u \mapsto (|A| + zI)^{s/2} u'.$$

It is easy to see that the mapping $(|A| + zI)^{-s/2}$ and $(|A| + zI)^{s/2}$ are inverse to each other. If we define the scalar product

$$(u, v)_{-s} = ((|A| + zI)^{-s/2} u, (|A| + zI)^{-s/2} v)_0,$$

both maps preserve this scalar product. We have proved that $(|A| + zI)^s$ is an isometry of \mathcal{H}_{t+2s} onto \mathcal{H}_t for all real s and t , and all z with $\text{Re}(z) > 0$, and $|A| = |A|$ when both $s, t \geq 0$. Beside these isometries, we have the obvious inclusion $\mathcal{H}_s \subseteq \mathcal{H}_t$ for all $s \geq t$. We also note that using the scalar product in \mathcal{H}_{-s} , the action of the functional $u \in \mathcal{H}_{-s}$ is

$$u(v) = (u, (|A| + zI)^s v)_{-s}$$

for all $v \in \mathcal{H}_s$, $\text{Re}(z) > 0$.

Lemma 2.1: Let A be a self-adjoint operator in the Hilbert space \mathcal{H} , and $\mathbf{e} \in \mathcal{H}_{-2} - \mathcal{H}$. Then, the restriction \dot{A} of A defined by the domain

$$\text{dom}(\dot{A}) = \{v \in \mathcal{H} \mid Av \in \mathcal{H}, \mathbf{e}(v) = 0\}$$

is symmetric and has deficiency indices (1,1). The solutions of the equation

$$(\dot{A}^\dagger \mp iI)v = 0, \tag{7}$$

are all given by

$$u_{\pm} = c(|\mathcal{A}| \mp iI)^{-1}\mathbf{e},$$

with $c \in \mathbb{C}$.

Proof: Note that the domain is well defined. $Av \in \mathcal{H}$ if and only if $v \in \text{dom}(A)$. First, we prove that \dot{A} is symmetric. We show that $\text{dom}(\dot{A})^{\perp} = \{0\}$, where the orthogonal complement is in \mathcal{H} . In fact, if this is the case, then

$$(\text{dom}(\dot{A})^{\perp})^{\perp} = \mathcal{H},$$

and $(\text{dom}(\dot{A})^{\perp})^{\perp} = \overline{\text{dom}(\dot{A})}$ since $\text{dom}(\dot{A})$ is a subspace, and the thesis follows. By definition,

$$\text{dom}(\dot{A})^{\perp} = \{v \in \mathcal{H} | (v, w)_0 = 0, \forall w \in \text{dom}(\dot{A})\}.$$

We show that a vector v satisfies the equation

$$(v, w)_0 = 0 \tag{8}$$

for all $w \in \text{dom}(\dot{A})$ if and only if v is a multiple of \mathbf{e} under the inclusion of \mathcal{H} in \mathcal{H}_{-2} [one implication is obvious since $w \in \text{dom}(\dot{A})$]. This implies that the unique solution v in \mathcal{H} of the above equation is $v=0$. We have the following facts:

(a) if $w \in \text{dom}(\dot{A}) \leq \mathcal{H}_2$, then

$$(|\mathcal{A}| + I)w = (|\mathcal{A}| + I)^{-1}(|\mathcal{A}| + I)^2w \in \mathcal{H},$$

and hence $(|\mathcal{A}| + I)^2w \in \mathcal{H}_{-2}$;

(b) by definition, $w \in \text{dom}(\dot{A})$ if and only if

$$0 = \mathbf{e}(w) = (\mathbf{e}, (|\mathcal{A}| + I)^2w)_{-2}, \tag{9}$$

(c) $\text{dom}(\dot{A}) \leq \mathcal{H}_2$, and hence it is isometric to a subspace D of \mathcal{H}_{-2} , $D = (|\mathcal{A}| + I)^2\text{dom}(\dot{A})$; and

(d) each $v \in \mathcal{H}$ defines a functional \mathbf{v} on \mathcal{H}_2 by $\mathbf{v}(w) = (v, w)_0$; thus, we rewrite Eq. (8) as

$$0 = (v, w)_0 = \mathbf{v}(w) = ((|\mathcal{A}| + I)\mathbf{v}, (|\mathcal{A}| + I)w)_{-2} = (\mathbf{v}, (|\mathcal{A}| + I)^2w)_{-2} \tag{10}$$

for all $w \in \text{dom}(\dot{A})$, with $v = \mathbf{v} \in \mathcal{H} \leq \mathcal{H}_{-2}$.

By point (c), Eq. (9) means that $D = L(\mathbf{e})^{\perp_{-2}}$, where \perp_{-2} means that the orthogonal complement is in \mathcal{H}_{-2} . By point (d), \mathbf{v} satisfies Eq. (8) if and only if $\mathbf{v} \in D^{\perp_{-2}}$. Since $\mathcal{H}_{-2} = L(\mathbf{e}) \oplus L(\mathbf{e})^{\perp_{-2}}$, it follows that $D^{\perp_{-2}} = (L(\mathbf{e})^{\perp_{-2}})^{\perp_{-2}} = L(\mathbf{e})$, and hence \mathbf{v} satisfies Eq. (8) if and only if $\mathbf{v} \in L(\mathbf{e})$, as required.

Next, we prove that the vectors u_{\pm} are the unique solutions of the deficiency equation (7). Note that since $\mathbf{e} \in \mathcal{H}_{-2}$, it follows that $u_{\pm} \in \mathcal{H}$ and $\dot{A}u_{\pm} \notin \mathcal{H}$. We show that $u_{\pm} \in D(\dot{A}^{\dagger})$. By definition,

$$\text{dom}(\dot{A}^{\dagger}) = \{v \in \mathcal{H} | \exists u \in \mathcal{H}, (\dot{A}w, v)_0 = (w, u)_0, \forall w \in \text{dom}(\dot{A})\}.$$

If we take $u = \pm iu_{\pm}$, then

$$\begin{aligned} (\dot{A}w, u_{\pm})_0 - (w, \pm iu_{\pm})_0 &= (\dot{A}w, u_{\pm})_0 - (\mp iw, u_{\pm})_0 = ((\dot{A} \pm iI)w, u_{\pm})_0 \\ &= ((A \pm iI)w, u_{\pm})_0 = (w, (A \mp iI)u_{\pm})_0 = \overline{\mathbf{e}(w)} = 0 \end{aligned}$$

since $A = \dot{A}$ on $\text{dom}(\dot{A})$ and A is self-adjoint, and hence u_{\pm} belong to $\text{dom}(\dot{A}^{\dagger})$. This also means that $\dot{A}^{\dagger}u_{\pm} = \pm iu_{\pm}$, and therefore, the u_{\pm} are solutions of Eq. (7). It remains to show that these are the unique solutions. Note that the solutions of Eq. (7) are elements of the space

$$\ker(iI - \dot{A}^\dagger) = (\text{ran}(-iI - \dot{A}))^\perp,$$

and $u \in (\text{ran}(-iI - \dot{A}))^\perp$ if and only if

$$(u, (-iI - \dot{A})w)_0 = 0 \tag{11}$$

for all $w \in \text{dom}(\dot{A})$. By point (c) above, if $w \in \text{dom}(\dot{A})$, then $(|\mathcal{A}| + I)^2 w \in \mathcal{H}_{-2}$, so Eq. (11) means that

$$\begin{aligned} 0 &= (u, (-iI - \dot{A})w)_0 = (u, (-iI - A)w)_0 = (u, (-iI - A)(|\mathcal{A}| + I)^{-2}(|\mathcal{A}| + I)^2 w)_0 \\ &= ((iI - A)u, (|\mathcal{A}| + I)^2 w)_{-2}, \end{aligned}$$

since A is self-adjoint. This implies that $(iI - A)u \in D^{\perp-2}$ [where the space D was defined in point (c) above]. Since $D^{\perp-2} = L(\mathbf{e})$, this completes the proof.

Using the standard von Neumann theory of self-adjoint extensions, we characterize the adjoint and the self-adjoint extensions of \dot{A} as follows.^{6,11,27}

Lemma 2.2: The adjoint operator \dot{A}^\dagger is

$$\text{dom}(\dot{A}^\dagger) = \{w \oplus c_+ u_+ \oplus c_- u_-, w \in \text{dom}(\dot{A}), c_\pm \in \mathbb{C}\},$$

$$\dot{A}^\dagger(w + u_+ + u_-) = \dot{A}w + iu_+ - iu_-.$$

Lemma 2.3: All the self-adjoint extensions A_θ , $0 \leq \theta < 2\pi$, of the operator \dot{A} are

$$\text{dom}(A_\theta) = \{w \oplus c_+(u_+ \oplus e^{i\theta} u_-), w \in \text{dom}(\dot{A}), c_+ \in \mathbb{C}\},$$

$$A_\theta(w + u_+ + u_-) = \dot{A}^\dagger(w + u_+ + u_-) = \dot{A}w + iu_+ - iu_-.$$

For proofs of these lemmas, see, for example, Refs. 3 and 4. An equivalent description of the self-adjoint extensions can be given by boundary conditions on the domain of the adjoint operator as

$$\text{dom}(A_\theta) = \{u \in \text{dom}(\dot{A}^\dagger) | (u_-, u)_0 = e^{i\theta} (u_+, u)_0\}.$$

Remark 2.4: Note that the case $\theta = \pi$ gives the maximal operator, namely, $A_\theta \subseteq A_\pi$, for all θ . $v \in \text{dom}(Q_\pi)$ if and only if $v = w + c_+(u_+ - u_-)$, with $w \in \text{dom}(\dot{A})$, and $c_+ \in \mathbb{C}$. However it is easy to see that the function $u = u_+ - u_- = 2i(\mathcal{A}^2 + I)^{-1}e$ is such that $u \in L^2(\Omega^n)$ and $\dot{A}u \in L^2(\Omega^n)$. This means that $u \in \text{dom}(\dot{A})$, and the statement follows.

Next we characterize the resolvent of the self-adjoint extensions A_θ of \dot{A} . This should be compared with Theorem 1.2.1 of Ref. 3.

Lemma 2.5: Let A_θ be one of the self-adjoint extensions of the operator \dot{A} described in Lemma 2.3. Let $\lambda \in \rho(A_\theta) \cap \rho(A_\pi)$, then the resolvent of A_θ of \dot{A} is

$$R(\lambda, A_\theta)v = R(\lambda, A_\pi)v + c_\theta(\lambda)(u_{\bar{\lambda}}, v)_0 u_\lambda,$$

where $c_\theta(\lambda)$ is some function of λ , and

$$u_\lambda = (|\mathcal{A}| - \lambda I)^{-1}e.$$

Moreover, the difference of the resolvents $R(\lambda, A_\theta) - R(\lambda, A_\pi)$ is of trace class, and

$$\text{Tr}(R(\lambda, A_\theta) - R(\lambda, A_\pi)) = c_\theta(\bar{\lambda})(u_{\bar{\lambda}}, u_\lambda)_0.$$

Proof: Let $\lambda \in \rho(A_\theta) \cap \rho(A_\pi)$. Consider

$$(R(\lambda, A_\theta) - R(\lambda, A_\pi))v = ((\lambda I - A_\theta)^{-1} - (\lambda I - A_\pi)^{-1})v.$$

Since for all $u \in \text{ran}(\lambda I - \dot{A})$, we have that $(\lambda I - A_\theta)^{-1}u = (\lambda I - \dot{A})^{-1}u$, it follows that

$$(R(\lambda, A_\theta) - R(\lambda, A_\pi))v = ((\lambda I - A_\theta)^{-1} - (\lambda I - A_\pi)^{-1})P_{(\text{ran}(\lambda I - \dot{A}))^\perp}v,$$

where $P_{(\text{ran}(\lambda I - \dot{A}))^\perp}$ denotes the projection onto $(\text{ran}(\lambda I - \dot{A}))^\perp$. However, it is easy to see that the proof of Lemma 2.6 generalizes for any $\lambda \neq i$ in the resolvent set, thus $(\text{ran}(\lambda I - \dot{A}))^\perp = \langle u_\lambda^- \rangle$, with

$$u_\lambda = (|\mathcal{A}| - \lambda I)^{-1}e.$$

It follows that

$$(R(\lambda, A_\theta) - R(\lambda, A_\pi))v = ((\lambda I - A_\theta)^{-1} - (\lambda I - A_\pi)^{-1})(u_\lambda^-, v)_0 u_\lambda^-.$$

Now, the vector $y = ((\lambda I - A_\theta)^{-1} - (\lambda I - A_\pi)^{-1})u_\lambda^-$ itself belongs to $(\text{ran}(\bar{\lambda} I - \dot{A}))^\perp$. For, since \dot{A}^\dagger is an extension of A_θ for all θ ,

$$(\lambda I - \dot{A}^\dagger)y = (\lambda I - \dot{A}^\dagger)((\lambda I - A_\theta)^{-1} - (\lambda I - A_\pi)^{-1})u_\lambda^- = 0,$$

implies that $y \in \ker(\lambda I - \dot{A}^\dagger)$. Therefore, we have proved that for all $v \in \text{dom}(\lambda I - A_\theta)^{-1}$,

$$(R(\lambda, A_\theta) - R(\lambda, A_\pi))v = (u_\lambda^-, v)_0 c_\theta(\lambda) u_\lambda^-.$$

This means that

$$\text{Tr}(R(\lambda, A_\theta) - R(\lambda, A_\pi)) = c_\theta(\bar{\lambda})(u_\lambda^-, u_\lambda^-)_0 < \infty,$$

by the definition of u_λ since $e \in \mathcal{H}_{-2}$.

B. Multiplication operators

We pass now to consider a more concrete situation, namely, multiplication operators. These operators provide the most natural setting where the results given in the previous section for abstract operators apply. Dually, all the results of the present section can be proven independent of the theory developed in Sec. II A, but working directly in the concrete Sobolev spaces described below. We will not give complete proofs in this concrete setting since they are precisely the same as the one provided in the abstract presentation of Sec. II A. The main advantage working in this concrete setting is that all the spaces \mathcal{H}_s of Sec. II A are subspaces of some large function space, and therefore, all the functionals can be identified with some concrete functions in these spaces. First, some preliminaries and notations. The measure appearing in all integrals is going to be Lebesgue's measure. Correspondingly, measurable sets and functions are understood in the sense of Lebesgue's integration theory. Let Ω^n be some unbounded measurable subset of \mathbb{R}^n . Let s be real, $q: \Omega^n \rightarrow \mathbb{C}$ be a measurable function, and

$$m_s(x) = (1 + |q(x)|)^{s/2}.$$

We define the spaces

$$L^{2,s}(\Omega^n) = \{f \in \text{map}(\Omega^n, \mathbb{C}) \mid f m_s \in L^2(\Omega^n)\}.$$

Note that $L^2(\Omega^n) = L^{2,0}(\Omega^n)$ and $L^{2,s}(\Omega^n) \subset L^{2,s'}(\Omega^n)$ if $s > s'$. In $L^{2,s}(\Omega^n)$, we have the scalar product

$$(f, g)_{L^2, s(\Omega^n)} = \int_{\Omega^n} \bar{f}(x)g(x)(m_s(x))^2 d^n x,$$

and with this scalar product, the spaces $L^{2, s}(\Omega^n)$ are complete separable Hilbert spaces and are the Fourier images of the Sobolev spaces $W^{2, s}(\Omega^n)$. We will use the notation B_N^n for the intersection of Ω^n and the closed disk D_N^n of radius N centered at the origin of \mathbb{R}^n .

The maximal multiplication operator Q associated with q is the operator in $L^2(\Omega^n)$ defined by

$$\text{dom}(Q) = \{f \in L^2(\Omega^n) | qf \in L^2(\Omega^n)\},$$

$$Qf = qf.$$

The operator Q is a (closed) self-adjoint operator with core $C_0^\infty(\Omega^n)$. If $q(x) \neq 0$ almost every in Ω^n , then Q is injective. If $q(x) \geq c$, for some c , almost every in Ω^n , then $\text{ran}(Q) = L^2(\Omega^n)$, so $Q: \text{dom}(Q) \rightarrow L^2(\Omega^n)$ is bijective.

C. Nonmaximal multiplication operators and their extensions

Let $e: \Omega^n \rightarrow \mathbb{C}$ be a fixed measurable function. Assume that the functions q and e decrease faster than some power, namely, there exist constants μ, β, c , and c' such that

$$|x^{-\mu}q(x)| \leq c,$$

$$|x^{-\beta}e(x)| \leq c'.$$

Consider in $L^2(\Omega^n)$ the operator

$$\text{dom}(\dot{Q}) = \left\{ f \in L^2(\Omega^n) | qf \in L^2(\Omega^n), \int_{\Omega^n} \bar{e}(x)f(x)d^n x = 0 \right\},$$

$$\dot{Q}f = qf.$$

It is clear that $\text{dom}(\dot{Q})$ is a subspace of $L^2(\Omega^n)$, so the definition is well posed. Also, $\text{dom}(\dot{Q}) \subset \text{dom}(Q)$, so \dot{Q} is a multiplication operator, but, in general, it is not maximal.

Lemma 2.6: *If $\mu - \beta > n/2$, then \dot{Q} is symmetric.*

Proof: This follows from the first statement of Lemma 2.1 provided that $e \in \mathcal{H}_{-2}$. In the present case, $|A| = |Q|$, and therefore, $\mathcal{H}_{-2} = L^{2, -2}(\Omega^n)$. Thus, $e \in \mathcal{H}_{-2}$ if

$$\int_{\Omega^n} |e(x)|^2 (1 + |q(x)|)^{-2} d^n x < \infty.$$

We only need to check the convergence for large $r = |x|$. For large r , the integral behaves like $r^{2\beta - 2\mu + n - 1}$, and therefore, it is convergent if $2\beta - 2\mu + n < 0$.

Lemma 2.7: *Assume that q is a real function, $\mu - \beta > n/2$, and $\beta \geq -n/2$. Then the operator \dot{Q} has deficiency indices (1,1), and the solutions of the equation*

$$(\dot{Q}^\dagger \pm iI)u = 0$$

are all given by the functions

$$\psi_\pm(x) = c \frac{e(x)}{q(x) \mp i},$$

where $c \in \mathbb{C}$, $x \in \Omega^n$.

Proof: This follows from the second statement of Lemma 2.1 provided that $e \in \mathcal{H}_{-2}$ and that $q\psi_{\pm} \in L^2(\Omega^n)$. The first requirement implies that $\mu - \beta > n/2$, as in the proof of the previous lemma. For the second one, consider the integral

$$\int_{\Omega^n} |q(x)\psi_+(x)|^2 d^n x = \int_{\Omega^n} \frac{|q(x)e(x)|^2}{|q(x) + 1|^2} d^n x.$$

This integral behaves for large $r=|x|$ as $r^{2\beta+n}$ and therefore does not converge if $\beta \geq n/2$.

Remark 2.8. Note that the conditions $\mu - \beta > n/2$ and $\beta \geq -n/2$ imply that $\mu > 0$. For $- \beta \leq n/2$, and hence $n/2 < \mu - \beta \leq \mu + n/2$.

We are now in the position of using the results in Lemmas 2.2 and 2.3 to characterize the adjoint of \dot{Q} and to parametrize all the self-adjoint extensions using the parameter θ . With this parametrization, the maximal multiplication operator Q corresponds to the self-adjoint extension defined by $\theta = \pi$ (see Remark 2.4). Using Lemma 2.5, we also have a general formula for the resolvent and we know that the difference between the resolvents of a self-adjoint extension and of the maximal operator is of trace class.

We proceed by studying the particular case where $q(x) = |x|^2$ and e is a bounded function. In this case, we give a more useful characterization of the self-adjoint extensions of \dot{Q} by some explicit integral boundary conditions. We will parametrize the self-adjoint extensions by a real non-negative parameter α . By the assumptions on q and e , we have $\mu = 2$ and $\beta = 0$, and the conditions in Lemmas 2.6 and 2.7 are satisfied if and only if $n = 1, 2$, or 3 . Therefore, we proceed assuming n to be in this range. In this case, if

$$\psi_{\lambda}(x) = \frac{e(x)}{|x|^2 - \lambda},$$

and assuming $\text{Im } \lambda \neq 0$, we have the following expansion for large N :

$$\int_{B_N^n} \bar{e}(x)\psi_{\lambda}(x)d^n x = \int_{B_N^n} \frac{|e(x)|^2}{|x|^2 - \lambda} d^n x = d_n(N) + g_n(\lambda) + o(N^{[n/2]-1}), \quad n = 1, 2, 3. \tag{12}$$

Note that $d_n(N)$ does not depend on λ . The functions $d_n(N)$ and $g_n(\lambda)$ depend on the explicit form of $e(x)$. The values for the relevant choices of $e(x)$ are given in Lemma A.1 in the Appendix. In particular, it is always true that

$$d_1(N) = O\left(\frac{1}{N}\right), \quad d_2(N) = O(\log N), \quad d_3(N) = O(N).$$

From Eq. (12), it follows that

$$\begin{aligned} \int_{B_N^n} \bar{e}(x)(\psi_+(x) + e^{i\theta}\psi_-(x))d^n x &= (1 + e^{i\theta})d_n(N) + g_n(i) + e^{i\theta}g_n(-i) + o(N^{[n/2]-1}) \\ &= i(1 - e^{i\theta})(\alpha d_n(N) + \alpha b_n + a_n) + o(N^{[n/2]-1}), \end{aligned}$$

where α , a_n , and b_n are real numbers given by

$$\alpha = \text{ctg} \frac{\theta}{2},$$

$$a_n = \|\psi_+\|^2 = \|\psi_-\|^2 = \int_{\Omega^n} \frac{|e(x)|^2}{1 + x^4} d^n x = \frac{g_n(i) - g_n(-i)}{2i},$$

$$b_n = \frac{g_n(i) + g_n(-i)}{2}.$$

Therefore, α is the real number which specifies the chosen self-adjoint extension.

This suggests the following description of the boundary conditions of the self-adjoint extensions of \dot{Q} in all the cases $n=1, 2, 3$.

Proposition 2.9: All the self-adjoint extensions of the operator \dot{Q} are given by the following integral boundary condition:

$$\text{dom}(Q_\alpha) = \left\{ f \in L^2(\Omega^n) \mid qf - \alpha ce \in L^2(\Omega^n), \int_{B_N^n} \bar{e}(x)f(x)d^n x = c(\alpha d_n(N) + \alpha b_n + a_n) + o(1) \text{ for large } N \right\},$$

with c being an arbitrary complex constant, $d_n(N)$ the functions given in Eq. (12), and α a real number. The operator Q_α acts on $f \in \text{dom}(Q_\alpha)$ as follows:

$$\begin{aligned} (Q_\alpha f)(x) &= |x|^2 f(x) - \alpha ce(x) \\ &= |x|^2 f(x) - \frac{2\alpha}{\alpha - i} (\psi_+, f) e(x) = |x|^2 f(x) - \lim_{N \rightarrow \infty} \frac{\alpha e(x)}{\alpha d_n(N) + \alpha b_n + a_n} \int_{B_N^n} \bar{e}(x) f(x) d^n x. \end{aligned}$$

Remark 2.10: In dimensions 2 and 3, $d_n(N)$ are divergent quantities and so the constants a_n , and b_n can be dropped in the latter equation, while in dimension $n=1$, one can drop $d_1(N)$ and take the integral on the whole space.

Proof: Let $f \in \text{dom}(Q_\alpha)$, that is, $f = h + c_+(\psi_+ + e^{i\theta}\psi_-)$ with $h \in \text{dom}(\dot{Q})$. Then,

$$\begin{aligned} \int_{B_N^n} \bar{e}(x) f(x) d^n x &= \int_{B_N^n} \bar{e}(x) h(x) d^n x + c_+ \int_{B_N^n} \bar{e}(x) (\psi_+(x) d^n x + e^{i\theta} \psi_-(x) d^n x) \\ &= \int_{B_N^n} \bar{e}(x) h(x) d^n x + c(\alpha d_n(N) + \alpha b_n + a_n) + o(N^{\lfloor n/2 \rfloor - 1}), \end{aligned}$$

with $c = i(1 - e^{i\theta})c_+$. Now we have to show that for large N , the integral in the latter equation always gives contributions, which are negligible with respect $\alpha(d_n(N) + b_n) + a_n$. This is, however, a direct consequence of the fact that $h \in \text{dom}(\dot{Q})$. For by hypothesis, we have

$$\int_{\Omega^n} \bar{e}(x) h(x) d^n x = 0,$$

since $\Omega^n = \cup_N B_N^n$, this implies that

$$\lim_{N \rightarrow \infty} \int_{B_N^n} \bar{e}(x) h(x) d^n x = 0,$$

namely, that

$$\int_{B_N^n} \bar{e}(x) h(x) d^n x = o(1)$$

for large N .

The formula for the action of Q_α easily follows.

Next, we give the resolvent of each extension.

Proposition 2.11: Let Q_α be one of the self-adjoint extensions of the nonmaximal multiplication operator \dot{Q} described in Proposition 2.9. Then, for all $\lambda \in \rho(Q_\alpha) \cap \rho(Q_0)$, Q_α has the resolvent

$$R(\lambda, Q_\alpha)\phi = R(\lambda, Q_0)\phi + \frac{1}{\frac{a_n}{\alpha} + b_n - g_n(\lambda)} \int_{\Omega^n} \bar{e}(x)(R(\lambda, Q_0)\phi)(x)d^n x \psi_\lambda,$$

$$\psi_\lambda(x) = \frac{e(x)}{|x|^2 - \lambda},$$

where the $g_n(\lambda)$ are the functions appearing in the asymptotic expansion of the integral boundary condition defining Q_α applied to the function ψ_λ , as given in Eq. (12), and Q_0 is the maximal multiplication operator Q . Moreover, the difference of the resolvents $R(\lambda, Q_\alpha) - R(\lambda, Q_0)$ is of trace class.

Proof: By Lemma 2.5,

$$R(\lambda, Q_\alpha)\phi = R(\lambda, Q_0)\phi + c_\alpha(\lambda, \phi)\psi_\lambda.$$

In order to find the value of $c_\alpha(\lambda, \phi)$, first we note that $f = R(\lambda, Q_\alpha)\phi \in \text{dom}(Q_\alpha)$; therefore, it must satisfy the conditions given in Proposition 2.9, namely,

$$\int_{B_N^n} \bar{e}(x)f(x)d^n x = \int_{B_N^n} \bar{e}(x)(R(\lambda, Q_\alpha)\phi)(x)d^n x = c(a_n + \alpha b_n + \alpha d_n(N)) + o(1). \tag{13}$$

On the other hand, using Eq. (12) we explicitly have

$$\begin{aligned} \int_{B_N^n} \bar{e}(x)(R(\lambda, Q_\alpha)\phi)(x)d^n x &= \int_{B_N^n} \bar{e}(x)(R(\lambda, Q_0)\phi)(x)d^n x + c_\alpha(\lambda, \phi) \int_{B_N^n} \bar{e}(x)\psi_\lambda(x)d^n x \\ &= \int_{B_N^n} \bar{e}(x)(R(\lambda, Q_0)\phi)(x)d^n x + c_\alpha(\lambda, \phi)(d_n(N) + g_n(\lambda)) + o(N^{\lfloor n/2 \rfloor - 1}). \end{aligned} \tag{14}$$

Now, since $R(\lambda, Q_0)\phi \in \text{dom}(Q_0)$, by Proposition 2.9,

$$\int_{B_N^n} \bar{e}(x)(R(\lambda, Q_0)\phi)(x)d^n x = ca_n + o(1)$$

for large N . Note that the constant a_n does not depend on the extension by definition.

This means that we are able to make the comparison between Eqs. (13) and (14). We have

$$\int_{\Omega^n} \bar{e}(x)(R(\lambda, Q_0)\phi)(x)d^n x + c_\alpha(\lambda, \phi)(d_n(N) + g_n(\lambda)) - c(a_n + \alpha b_n + \alpha d_n(N)) = o(1),$$

and this implies that $c_\alpha(\lambda, \phi) = \alpha c$ and

$$c_\alpha(\lambda, \phi) = \frac{\alpha}{a_n + \alpha b_n - \alpha g_n(\lambda)} \int_{\Omega^n} \bar{e}(x)(R(\lambda, Q_0)\phi)(x)d^n x = \frac{1}{\frac{a_n}{\alpha} + b_n - g_n(\lambda)} \int_{\Omega^n} \frac{\bar{e}(x)\phi(x)}{\lambda - |x|^2} d^n x.$$

Note that in the case $n=1$, we are comparing constants since $d_1(N)$ is small in this case (see Remark 2.10).

In order to give the kernel of the resolvent, we need a suitable delta function in the space Ω^n . This will be denoted by δ_{Ω^n} and is defined by the property

$$\int_{\Omega^n} \delta_{\Omega^n}(x-a)f(x)d^n x = f(a)$$

in the appropriate space of test functions over \mathbb{R}^n . Explicit formulas will be given in the concrete examples studied below.

Corollary 2.12: The operator $R(\lambda, Q_\alpha)$ of Proposition 2.11 is an integral operator with kernel

$$\ker(x, y; R(\lambda, Q_\alpha)) = \frac{\delta_{\Omega^n}(x-y)}{\lambda - |x|^2} - \frac{1}{\frac{a_n}{\alpha} + b_n - g_n(\lambda)} \frac{e(x)\bar{e}(y)}{(\lambda - |x|^2)(\lambda - |y|^2)}.$$

Proof: Since

$$(R(\lambda, Q_0)\phi)(x) = \frac{\phi(x)}{\lambda - |x|^2},$$

we have that the operator

$$\begin{aligned} (A_\lambda \phi)(x) &= \psi_\lambda(x) \int_{\Omega^n} \bar{e}(y)(R(\lambda, Q_0)\phi)(y)d^n y \\ &= \psi_\lambda(x) \int_{\Omega^n} \bar{e}(y) \frac{\phi(y)}{\lambda - |y|^2} d^n y \\ &= \int_{\Omega^n} \frac{e(x)\bar{e}(y)}{(|x|^2 - \lambda)(\lambda - |y|^2)} \phi(y) d^n y \end{aligned}$$

is an integral operator with kernel

$$\ker(x, y; A(\lambda)) = - \frac{e(x)}{\lambda - |x|^2} \frac{\bar{e}(y)}{\lambda - |y|^2}.$$

Corollary 2.13: The difference of the resolvents $R(\lambda, Q_\alpha) - R(\lambda, Q_0)$ of Proposition 2.11 is a trace class operator with trace

$$\text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0)) = - \frac{1}{\frac{a_n}{\alpha} + b_n - g_n(\lambda)} \int_{\Omega^n} \frac{|e(x)|^2}{(|x|^2 - \lambda)^2} d^n x.$$

Remark 2.14: Assuming that $\text{Im } \lambda \neq 0$, we have the bound

$$|\text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0))| \leq K \frac{1}{\left| \frac{a_n}{\alpha} + b_n - g_n(\lambda) \right|} |\lambda^{n/2-2}|,$$

with some positive constant K . By Corollary 2.12,

$$\ker(x, y; R(\lambda, Q_\alpha) - R(\lambda, Q_0)) = - \frac{1}{\frac{a_n}{\alpha} + b_n - g_n(\lambda)} \frac{e(x)\bar{e}(y)}{(\lambda - |x|^2)(\lambda - |y|^2)}.$$

Since $\text{Im } \lambda \neq 0$, we can integrate obtaining

$$\begin{aligned}
 |\text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0))| &= \frac{1}{\left| \frac{a_n}{\alpha} + b_n - g_n(\lambda) \right|} \int_{\Omega^n} \frac{|e(x)|^2}{(|x|^2 - \lambda)^2} d^n x \\
 &\leq \frac{1}{\left| \frac{a_n}{\alpha} + b_n - g_n(\lambda) \right|} \int_{\mathbb{R}^n} \frac{|e(x)|^2}{(|x|^2 - \lambda)^2} d^n x \\
 &\leq \frac{K}{\left| \frac{a_n}{\alpha} + b_n - g_n(\lambda) \right|} |\lambda^{n/2-2}|.
 \end{aligned}$$

In the latter line we have used Lemma A.4 in the Appendix.

We conclude this section, giving the continuum spectrum of the operators Q_α . Possible isolated eigenvalues of finite multiplicity will be detected by an explicit study of the resolvent of particular examples in the following sections.

Lemma 2.15: The continuum spectrum of the operator Q_α coincides with the non-negative real axis, i.e., $\text{Sp}_c Q_\alpha = [0, \infty)$.

Proof: We recall that $\text{Sp}_c \subseteq \text{Sp}_e$ for closed operators, and $\text{Sp}_e = \overline{\text{Sp}_c} \cup \overline{\text{Sp}_p} \cup \text{Sp}_p^\infty$ for self-adjoint operators. Then, the thesis follows since all self-adjoint extensions have the same essential spectrum, and the maximal operator Q_0 is known to have the pure continuous spectrum $\text{Sp } Q_0 = \text{Sp}_c Q_0 = [0, \infty)$.

D. The basic example in the whole space

Let $\Omega^n = \mathbb{R}^n$, $e(x) = e^{iax}$, where $a \in \mathbb{R}^n$, and $n = 1, 2, 3$. We study in this section the self-adjoint extensions of the (closure of the) operator \dot{Q} in $L^2(\mathbb{R}^n)$ defined by

$$\text{dom}(\dot{Q}) = \left\{ f \in L^2(\mathbb{R}^n) \mid qf \in L^2(\mathbb{R}^n), \int_{\mathbb{R}^n} e^{-iax} f(x) d^n x = 0 \right\},$$

$$\dot{Q}f = qf,$$

where $q(x) = |x|^2$. We have $|e| = 1$ and $B_N^n = D_N^n$. In order to apply Proposition 2.9, we have to compute the functions $d_n(N)$ and $g_n(\lambda)$ appearing in Eq. (12). They are explicitly given in Lemma A.1 in the Appendix.

In the following, we shall assume that $\text{Im } \sqrt{\lambda} > 0$, then from Lemma A.1 in the Appendix, for $n = 3, 2, 1$, we have, respectively,

$$\begin{aligned}
 n = 3, & \quad \begin{cases} \int_{D_N^3} \frac{1}{|x|^2 - \lambda} d^3 x = 4\pi N + 2\pi^2 i \sqrt{\lambda} \text{sgn}(\text{Im } \sqrt{\lambda}) + o(1), \\ d_3(N) = 4\pi N, \quad g_3(\lambda) = 2\pi^2 i \sqrt{\lambda}, \quad a_3 = \sqrt{2}\pi^2, \quad b_3 = -\sqrt{2}\pi^2, \end{cases} \\
 n = 2, & \quad \begin{cases} \int_{D_N^2} \frac{1}{|x|^2 - \lambda} d^2 x = 2\pi \log N - \pi \log(-\lambda) + o(1), \\ d_2(N) = 2\pi \log N, \quad g_2(\lambda) = -\pi \log(-\lambda), \quad a_2 = \frac{\pi^2}{2}, \quad b_2 = 0, \end{cases}
 \end{aligned}$$

$$n = 1, \quad \begin{cases} \int_{D_N^1} \frac{1}{|x|^2 - \lambda} dx = \frac{i\pi}{\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})} - \frac{2}{N} + o\left(\frac{1}{N}\right), \\ d_1(N) = -\frac{2}{N}, \quad g_1(\lambda) = \frac{i\pi}{\sqrt{\lambda}}, \quad a_1 = \frac{\pi}{\sqrt{2}}, \quad b_1 = \frac{\pi}{\sqrt{2}}. \end{cases}$$

Applying Propositions 2.9 and 2.11 and Corollary 2.13, we easily prove the following results.

Lemma 2.16: The self-adjoint extensions of \dot{Q} are

$$n = 3, \quad \begin{cases} \operatorname{dom}(Q_\alpha) = \left\{ f \in L^2(\mathbb{R}^3) \mid Q_\alpha f \in L^2(\mathbb{R}^3), \right. \\ \left. \int_{D_N^3} e^{-iax} f(x) d^3x = c(4\pi N\alpha - \sqrt{2}\pi^2\alpha + \sqrt{2}\pi^2) + o(1), c \in \mathbb{C} \right\}, \\ (Q_\alpha f)(x) = |x|^2 f(x) - \lim_{N \rightarrow \infty} \frac{e^{iax}}{4\pi N} \int_{D_N^3} e^{-iax} f(x) d^3x, \end{cases}$$

$$n = 2, \quad \begin{cases} \operatorname{dom}(Q_\alpha) = \left\{ f \in L^2(\mathbb{R}^2) \mid Q_\alpha f \in L^2(\mathbb{R}^2), \right. \\ \left. \int_{D_N^2} e^{-iax} f(x) d^2x = c\left(2\pi\alpha \log N + \frac{\pi^2}{2}\right) + o(1), c \in \mathbb{C} \right\}, \\ (Q_\alpha f)(x) = |x|^2 f(x) - \lim_{N \rightarrow \infty} \frac{e^{iax}}{2\pi \log N} \int_{B_N^2} e^{-iax} f(x) d^2x, \end{cases}$$

$$n = 1, \quad \begin{cases} \operatorname{dom}(Q_\alpha) = \left\{ f \in L^2(\mathbb{R}) \mid Q_\alpha f \in L^2(\mathbb{R}), \right. \\ \left. \int_{D_N^1} e^{-iax} f(x) dx = c\left(-\frac{2\alpha}{N} + \frac{\pi\alpha}{\sqrt{2}} + \frac{\pi}{\sqrt{2}}\right) + o(1/N), c \in \mathbb{C} \right\}, \\ (Q_\alpha f)(x) = |x|^2 f(x) - \frac{\sqrt{2}\alpha e^{iax}}{\pi(\alpha + 1)} \int_{\mathbb{R}} e^{-iax} f(x) dx. \end{cases}$$

Lemma 2.17: Let Q_α be any of the self-adjoint extensions of the nonmaximal multiplication operator \dot{Q} described in Lemma 2.16. Then, for all $\lambda \in \rho(Q_\alpha) \cap \rho(Q_0)$, Q_α has the resolvent

$$n = 3, \quad R(\lambda, Q_\alpha)\phi = R(\lambda, Q_0)\phi + \frac{1}{\sqrt{2}\pi^2\left(\frac{1}{\alpha} - 1 - i\sqrt{2\lambda}\right)} \int_{\mathbb{R}^3} e^{-iax} (R(\lambda, Q_0)\phi)(x) d^3x \psi_\lambda,$$

$$n = 2, \quad R(\lambda, Q_\alpha)\phi = R(\lambda, Q_0)\phi + \frac{1}{\frac{\pi^2}{2\alpha} + \pi \log(-\lambda)} \int_{\mathbb{R}^2} e^{-iax} (R(\lambda, Q_0)\phi)(x) d^2x \psi_\lambda,$$

$$n = 1, \quad R(\lambda, Q_\alpha)\phi = R(\lambda, Q_0)\phi + \frac{1}{\frac{\pi}{\sqrt{2\alpha}} + \frac{\pi}{\sqrt{2}} - \frac{i\pi}{\sqrt{\lambda}}} \int_{\mathbb{R}} e^{-iax} (R(\lambda, Q_0)\phi)(x) dx \psi_\lambda,$$

where $\psi_\lambda(x) = e^{iax}/|x|^2 - \lambda$ for all the cases.

Corollary 2.18: The operator $R(\lambda, Q_\alpha)$ of Lemma 2.17 is an integral operator with kernel

$$n = 3, \quad \ker(x, y; R(\lambda, Q_\alpha)) = \frac{\delta(x-y)}{\lambda - |x|^2} - \frac{1}{\sqrt{2}\pi^2 \left(\frac{1}{\alpha} - 1 - i\sqrt{2\lambda} \right)} \frac{e^{ia(x-y)}}{(\lambda - |x|^2)(\lambda - |y|^2)},$$

$$n = 2, \quad \ker(x, y; R(\lambda, Q_\alpha)) = \frac{\delta(x-y)}{\lambda - |x|^2} - \frac{1}{\frac{\pi^2}{2\alpha} + \pi \log(-\lambda)} \frac{e^{ia(x-y)}}{(\lambda - |x|^2)(\lambda - |y|^2)},$$

$$n = 1, \quad \ker(x, y; R(\lambda, Q_\alpha)) = \frac{\delta(x-y)}{\lambda - |x|^2} - \frac{1}{\frac{\pi}{\sqrt{2}\alpha} + \frac{\pi}{\sqrt{2}} - \frac{i\pi}{\sqrt{\lambda}}} \frac{e^{ia(x-y)}}{(\lambda - |x|^2)(\lambda - |y|^2)},$$

where $\lambda \in \rho(Q_\alpha) \cap \rho(Q_0)$ and $x, y \in \mathbb{R}^n$, $n=3, 2, 1$, respectively.

Moreover, we have

Lemma 2.19: Let Q_α be any of the self-adjoint extensions of the nonmaximal multiplication operator \dot{Q} described in Lemma 2.16, then the difference $R(\lambda, Q_\alpha) - R(\lambda, Q_0)$ is a trace-class operator and, assuming that $\text{Im} \sqrt{\lambda} > 0$,

$$n = 3, \quad \text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0)) = \frac{1}{i\sqrt{2\lambda} \left(\frac{1}{\alpha} - 1 - i\sqrt{2\lambda} \right)},$$

$$n = 2, \quad \text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0)) = \frac{\pi}{\lambda} \frac{1}{\frac{\pi^2}{2\alpha} + \pi \log(-\lambda)},$$

$$n = 1, \quad \text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0)) = \frac{i}{2\lambda^{3/2}} \frac{1}{\frac{1}{\sqrt{2}\alpha} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{\lambda}}}.$$

Proof: One has to compute the L^2 -trace using the kernel given in Corollary 2.18 and the results of Lemma A.4 in the Appendix.

E. The basic example in the half space

Let $\Omega^n = \mathbb{H}^n = [0, \infty) \times \mathbb{R}^{n-1}$, $e(x) = \sin(ax)$, where $a \in \mathbb{H}^n$, and $n=1, 2, 3$. We study in this section the self-adjoint extensions of the (closure of the) operator \dot{Q} in $L^2(\mathbb{H}^n)$ defined by

$$\text{dom}(\dot{Q}) = \left\{ f \in L^2(\mathbb{H}^n) \mid qf \in L^2(\mathbb{H}^n), f(\{0\} \times \mathbb{R}^{n-1}) = 0, \int_{\mathbb{H}^n} \sin(ax) f(x) d^n x = 0 \right\},$$

$$\dot{Q}f = qf,$$

where $q(x) = |x|^2$. We have $|e| = 1$, and $B_N^n = D_N^n \cap \mathbb{H}^n$, and the functions $d_n(N)$ and $g_n(\lambda)$ appearing in Eq. (12) are given in Lemma A.3 in the Appendix.

Recalling that the space of the functions satisfying Dirichlet boundary condition on the boundary of the half space naturally identifies with the space of the odd functions on the whole space, we realize the delta function in the half space as

$$\delta_{\mathbb{H}^n}(x - a) = \frac{1}{2}(\delta(x - a) - \delta(x - P_n(a))),$$

where P_n is the reflection on the last coordinate.

Now we are able to write down explicitly all relevant quantities concerning the operator \hat{Q} in the half space \mathbb{H}^n for $n=3, 2, 1$.

Assuming that $\text{Im } \sqrt{\lambda} > 0$, from Lemma A.3 in the Appendix, we have

$$n = 3, \quad \begin{cases} \int_{B_N^3} \frac{\sin^2(ax)}{|x|^2 - \lambda} d^3x = \pi N + \frac{i\pi^2}{2} \sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda}) - \frac{\pi^2}{4a} e^{2ia\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})} + o(1), \\ \hat{d}_3(N) = \pi N, \quad \hat{g}_3(\lambda) = \frac{i\pi^2}{2} \sqrt{\lambda} - \frac{\pi^2}{4a} e^{2ia\sqrt{\lambda}}, \\ \hat{a}_3 = \frac{\pi^2}{4} \left(\sqrt{2} + \frac{e^{-\sqrt{2}a} \sin(\sqrt{2}a)}{2a} \right), \quad \hat{b}_3 = -\frac{\pi^2}{4} \left(\sqrt{2} + \frac{e^{-\sqrt{2}a} \cos(\sqrt{2}a)}{2a} \right), \end{cases}$$

$$n = 2, \quad \begin{cases} \int_{B_N^2} \frac{1}{|x|^2 - \lambda} d^2x = \frac{\pi}{2} \log N - \frac{\pi}{4} \log(-\lambda) + \frac{\pi}{2} K_0(2a\sqrt{-\lambda}) + o(1), \\ \hat{d}_2(N) = \frac{\pi}{2} \log N, \quad \hat{g}_2(\lambda) = -\frac{\pi}{4} \log(-\lambda) + \frac{\pi}{2} K_0(2a\sqrt{-\lambda}), \\ \hat{a}_2 = \frac{\pi}{4} \left(\frac{\pi}{2} + iK_0(2a\sqrt{i}) - iK_0(2a\sqrt{-i}) \right), \quad \hat{b}_2 = -\frac{i\pi}{4} (K_0(2a\sqrt{i}) + K_0(2a\sqrt{-i})), \end{cases}$$

$$n = 1, \quad \begin{cases} \int_{B_N^1} \frac{1}{|x|^2 - \lambda} dx = \frac{i\pi}{4\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})} (1 - e^{2ia\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})}) - \frac{1}{2N} + o\left(\frac{1}{N}\right), \\ \hat{d}_1(N) = -\frac{2}{N}, \quad \hat{g}_1(\lambda) = \frac{i\pi(1 - e^{2ia\sqrt{\lambda}})}{4\sqrt{\lambda}}, \\ \hat{a}_1 = \frac{\pi}{4\sqrt{2}} (1 - e^{-\sqrt{2}a} (\cos(\sqrt{2}a) + \sin(\sqrt{2}a))), \\ \hat{b}_1 = \frac{\pi}{4\sqrt{2}} (1 - e^{-\sqrt{2}a} (\cos(\sqrt{2}a) - \sin(\sqrt{2}a))), \end{cases}$$

with $K_0(z)$ being a Bessel function. Here we use the ‘‘hat’’ to distinguish the latter quantities with respect to the ones appearing in the example discussed in Sec. II D, concerning the case of the whole space.

Now, applying Propositions 2.9 and 2.11 and Corollary 2.13, we prove the following results.

Lemma 2.20: The self-adjoint extensions of \hat{Q} are

$$n = 3, \quad \left\{ \begin{aligned} \operatorname{dom}(Q_\alpha) &= \left\{ f \in L^2(\mathbb{R}^3) \mid Q_\alpha f \in L^2(\mathbb{R}^3), f(\{0\} \times \mathbb{R}^2) = 0, \right. \\ &\quad \left. \int_{B_N^3} \sin(ax) f(x) d^3x = c(\pi N \alpha + \alpha \hat{b}_3 + \hat{a}_3) + o(1); c \in \mathbb{C} \right\}, \\ Q_\alpha f &= |x|^2 f(x) - \lim_{N \rightarrow \infty} \frac{\sin(ax)}{\pi N} \int_{B_N^3} \sin(ax) f(x) d^3x, \end{aligned} \right.$$

$$n = 2, \quad \left\{ \begin{array}{l} \text{dom}(Q_\alpha) = \left\{ f \in L^2(\mathbb{H}^2) \mid Q_\alpha f \in L^2(\mathbb{H}^2), f(\{0\} \times \mathbb{R}) = 0, \right. \\ \left. \int_{B_N^2} \sin(ax)f(x)d^2x = c \left(\frac{\pi\alpha \log N}{2} + \alpha\hat{b}_2 + \hat{a}_2 \right) + o(1), c \in \mathbb{C} \right\}, \\ (Q_\alpha f)(x) = |x|^2 f(x) - \lim_{N \rightarrow \infty} \frac{2 \sin(ax)}{\pi \log N} \int_{B_N^2} \sin(ax)f(x)d^2x, \end{array} \right.$$

$$n = 1, \quad \left\{ \begin{array}{l} \text{dom}(Q_\alpha) = \left\{ f \in L^2(\mathbb{R}) \mid Q_\alpha f \in L^2(\mathbb{R}), f(0) = 0, \right. \\ \left. \int_{B_N^1} \sin(ax)f(x)dx = c \left(-\frac{2\alpha}{N} + \alpha\hat{b}_1 + \hat{a}_1 \right) + o(1/N), c \in \mathbb{C} \right\}, \\ (Q_\alpha f)(x) = |x|^2 f(x) - \frac{4\sqrt{2}\alpha \sin(ax) \int_{\mathbb{H}^1} \sin(ax)f(x)dx}{\pi(1+\alpha)(1-e^{-\sqrt{2}a} \cos(\sqrt{2}a)) - \pi(1-\alpha)e^{-\sqrt{2}a} \sin(\sqrt{2}a)}. \end{array} \right.$$

Lemma 2.21: Let Q_α be any of the self-adjoint extensions of the nonmaximal multiplication operator \dot{Q} described in Lemma 2.20. Then, for all $\lambda \in \rho(Q_\alpha) \cap \rho(Q_0)$, Q_α has the resolvent (we assume $\text{Im} \sqrt{\lambda}$ to be positive)

$$n = 3, \quad R(\lambda, Q_\alpha)\phi = R(\lambda, Q_0)\phi + \frac{\int_{\mathbb{H}^3} \sin(ax)(R(\lambda, Q_0)\phi)(x)d^3x}{\frac{\hat{a}_3}{\alpha} + \hat{b}_3 - \left(\frac{i\pi^2\sqrt{\lambda}}{2} - \frac{\pi^2 e^{2ia\sqrt{\lambda}}}{4a} \right)} \psi_\lambda,$$

$$n = 2, \quad R(\lambda, Q_\alpha)\phi = R(\lambda, Q_0)\phi + \frac{\int_{\mathbb{H}^2} \sin(ax)(R(\lambda, Q_0)\phi)(x)d^2x}{\frac{\hat{a}_2}{\alpha} + \hat{b}_2 + \frac{\pi \log(-\lambda)}{4} - \frac{\pi K_0(2a\sqrt{-\lambda})}{2}} \psi_\lambda,$$

$$n = 1, \quad R(\lambda, Q_\alpha)\phi = R(\lambda, Q_0)\phi + \frac{\int_{\mathbb{H}^1} \sin(ax)(R(\lambda, Q_0)\phi)(x)dx}{\frac{\hat{a}_1}{\alpha} + \hat{b}_1 - \frac{i\pi(1 - e^{2ia\sqrt{\lambda}})}{4\sqrt{\lambda}}} \psi_\lambda,$$

where $\psi_\lambda(x) = \sin(ax)/|x|^2 - \lambda$ for all the cases.

Corollary 2.22: The operator $R(\lambda, Q_\alpha)$ of Lemma 2.21 is an integral operator with kernel

$$n = 3, \quad \text{ker}(x, y; R(\lambda, Q_\alpha)) = \frac{\delta_{\mathbb{H}^3}(x-y)}{\lambda - |x|^2} - \frac{1}{\frac{\hat{a}_3}{\alpha} + \hat{b}_3 - \left(\frac{i\pi^2\sqrt{\lambda}}{2} - \frac{\pi^2 e^{2ia\sqrt{\lambda}}}{4a} \right)} \frac{\sin(ax)\sin(ay)}{(\lambda - |x|^2)(\lambda - |y|^2)},$$

$$n = 2, \quad \text{ker}(x, y; R(\lambda, Q_\alpha)) = \frac{\delta_{\mathbb{H}^2}(x-y)}{\lambda - |x|^2} - \frac{1}{\frac{\hat{a}_2}{\alpha} + \hat{b}_2 + \frac{\pi \log(-\lambda)}{4} - \frac{\pi K_0(2a\sqrt{-\lambda})}{2}} \frac{\sin(ax)\sin(ay)}{(\lambda - |x|^2)(\lambda - |y|^2)},$$

$$n = 1, \quad \ker(x, y; R(\lambda, Q_\alpha)) = \frac{\delta_{\mathbb{H}^1}(x - y)}{\lambda - |x|^2} - \frac{1}{\frac{\hat{a}_1}{\alpha} + \hat{b}_1 - \frac{i\pi(1 - e^{2ia\sqrt{\lambda}})}{4\sqrt{\lambda}}} \frac{\sin(ax)\sin(ay)}{(\lambda - |x|^2)(\lambda - |y|^2)},$$

where $\lambda \in \rho(Q_\alpha) \cap \rho(Q_0)$, $x, y \in \mathbb{H}^n$, $n=3, 2, 1$.

Moreover, we have the following.

Lemma 2.23: Let Q_α be any of the self-adjoint extensions of the nonmaximal multiplication operator \dot{Q} described in Lemma 2.16, then the difference $R(\lambda, Q_\alpha) - R(\lambda, Q_0)$ is a trace-class operator and, assuming that $\text{Im } \sqrt{\lambda} > 0$,

$$n = 3, \quad \text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0)) = \frac{\pi^2(1 - e^{2ia\sqrt{\lambda}})}{2i\sqrt{\lambda}} \frac{1}{\frac{\hat{a}_3}{\alpha} + \hat{b}_3 - \left(\frac{i\pi^2\sqrt{\lambda}}{2} - \frac{\pi^2 e^{2ia\sqrt{\lambda}}}{4a} \right)},$$

$$n = 2, \quad \text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0)) = \frac{\pi(1 + 2iaK_1(-2ia\sqrt{\lambda}))}{2\lambda} \frac{1}{\frac{\hat{a}_2}{\alpha} + \hat{b}_2 + \frac{\pi \log(-\lambda)}{4} - \frac{\pi K_0(2a\sqrt{-\lambda})}{2}},$$

$$n = 1, \quad \text{Tr}(R(\lambda, Q_\alpha) - R(\lambda, Q_0)) = \left(\frac{i\pi(1 - e^{2ia\sqrt{\lambda}})}{4\lambda^{3/2}} - \frac{\pi a e^{2ia\sqrt{\lambda}}}{2\lambda} \right) \frac{1}{\frac{\hat{a}_1}{\alpha} + \hat{b}_1 - \frac{i\pi(1 - e^{2ia\sqrt{\lambda}})}{4\sqrt{\lambda}}},$$

where K_1 is a Bessel function.

Proof: All latter integrals are computed in Lemma A.5 in the Appendix.

III. THE LAPLACE OPERATOR WITH DELTA POTENTIAL

We show in this section how the extensions of the multiplication operators introduced in Sec. II C are used in order to define a self-adjoint differential operator corresponding to the formal Laplacian operator with a delta type potential, discussed in Sec. I. This was the original approach of Berezin and Fadeev.⁵ Let

$$-d = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2},$$

denote the formal Laplace operator on Ω^n , where Ω^n is either \mathbb{R}^n or \mathbb{H}^n . Let a be a point in Ω^n and \mathcal{F} denote the Fourier transform in Ω^n . Then, the operator $-\dot{\Delta} = \mathcal{F}^{-1} \dot{Q} \mathcal{F}$ is a (closed) symmetric operator in $L^2(\Omega^n)$, with deficiency indices $(1, 1)$, for $n \leq 3$, and domain

$$\text{dom}(-\dot{\Delta}) = \{f \in W^{2,2}(\Omega^n) | f(a) = 0\}, \quad -\dot{\Delta}f = -df,$$

unitary equivalent to the operator \dot{Q} defined at the beginning of Sec. II C, with $e(x)$ either e^{iax} when $\Omega^n = \mathbb{R}^n$ or $\sin(ax)$ when $\Omega^n = \mathbb{H}^n$. This follows immediately in both cases by the definition of Fourier transform. Therefore, all the self-adjoint extensions of $-\dot{\Delta}$ are the operators $-\Delta_\alpha = \mathcal{F}^{-1} Q_\alpha \mathcal{F}$, where the operators Q_α were defined, in general, in Proposition 2.9, and in the particular case of dimensions $n=3, 2$, and 1 , in Lemmas 2.16 and 2.20, respectively, when $\Omega^n = \mathbb{R}^n$ or \mathbb{H}^n . In all cases, the maximal operator is $-\Delta_0$ with

$$\text{dom}(-\Delta_0) = W^{2,2}, \quad -\Delta_0 f = -df,$$

It is worth to observe here that the operator $\dot{\Delta}$ can also be introduced directly (this is the approach of Ref. 2) as the closure of the operator

$$\text{dom}(-\tilde{\Delta}) = C_0^\infty(\Omega^n - \{a\}), \quad -\tilde{\Delta}f = -df.$$

This follows adapting the standard proof for the maximal operators (see, for example, Ref. 31, Sec. 10.11). For it is clear that if $f \in \text{dom}(-\dot{\Delta})$, then $f(a) = (\mathcal{F}^{-1}\mathcal{F}f)(a) = 0$, and therefore $f \in \text{dom}(\dot{Q})$. Conversely, given $f \in \text{dom}(\dot{Q})$, we have that $(\mathcal{F}^{-1}f)(a) = 0$. Since the closure of the restriction of \dot{Q} on $C_0^\infty(\Omega^n)$ is closed, given a sequence $\{h_n\}$ in $C_0^\infty(\Omega^n)$ such that $h_n \rightarrow h = \mathcal{F}^{-1}f$, and $\dot{Q}h_n \rightarrow \dot{Q}h$, the condition $h(a) = 0$ can only be satisfied if the functions h_n have support away from a .

Next we interpret the operators $-\Delta_\alpha$ as a perturbation of the maximal operator $-\Delta_0$, using the theory of singular perturbation developed in Sec. 1 of Ref. 3, that we recall here briefly. Let A be a self-adjoint operator in the Hilbert space \mathcal{H} as in Sec. II A. Let γ be a real number and $\mathbf{a} \in \mathcal{H}_{-2}$ with norm 1. A singular rank 1 perturbation of the operator A is the operator defined by the following formula:

$$A^\gamma = A + \gamma \mathbf{a}_c(\cdot) \mathbf{a},$$

where \mathbf{a}_c is either \mathbf{a} or a linear bounded extension of \mathbf{a} (Ref. 3, Sec. 1.3.2), depending whether $\mathbf{a} \in \mathcal{H}_{-1} - \mathcal{H}$ or $\mathbf{a} \in \mathcal{H}_{-2} - \mathcal{H}_{-1}$. A rigorous definition of this type of operators acting on the dual space of functionals \mathcal{H}_{-2} has been given in Ref. 3, Sec. 1.3. The domain and the action of the operator are described in Theorems 1.3.1 and 1.3.2, respectively. Using this approach, we can define singular perturbed operators in the Hilbert space \mathcal{H} , by taking the restrictions of the operators A^γ just defined [see Eq. (1.45) of Ref. 3 for the domain]. We will use this definition and we will use the same notation. The operators defined in this way are self-adjoint.

If \dot{A} is a symmetric operator in \mathcal{H} with deficiency indices (1,1), as in Sec. II A, the self-adjoint extension A_α of \dot{A} described in Proposition 2.3, with $\alpha = \text{ctg } \theta$ (or in Prop. 2.9, when $\dot{A} = \dot{Q}$), coincides with the singular rank 1 perturbation A^γ of the operator A_0 if

$$\frac{1}{\gamma} = \frac{1}{\alpha} - c,$$

by Theorem 1.3.3 and the results of Sec. 1.3 of Ref. 3, and where c is a real number. Now, the mapping

$$\mathbf{d}: f \mapsto \int_{\Omega^n} \delta_{\Omega^n}(x-a) f(x) d^n x,$$

defines a functional on $\mathcal{H}_2 = W^{2,2}(\Omega^n)$, and it is easy to see that $\mathbf{d} \in \mathcal{H}_{-2} - \mathcal{H}_{-1}$ (see also Ref. 3, Sec. 1.5.1). However, $\|\mathbf{d}\|_{-2} \neq 1$, thus we need to take in account a normalization factor and we define $\mathbf{a} = \mathbf{d} / \|\mathbf{d}\|_{-2}$ (note that $\|\mathbf{d}\|_{-2} \neq 0$). Since

$$\mathbf{a}(f) = \frac{f(a)}{\|\mathbf{d}\|_{-2}},$$

the operator $-\Delta_\alpha$ corresponds to the singular rank 1 perturbation of the operator $-\Delta_0 = -\Delta^0$,

$$-\Delta^\gamma = -\Delta^0 + \gamma \mathbf{a}_c(\cdot) \mathbf{a} = -\Delta^0 + \frac{\gamma}{\|\mathbf{d}\|_{-2}^2} \mathbf{d}_c(\cdot) \mathbf{d},$$

if we take $1/\gamma = 1/\alpha - c$, for any real c . If we compare this with the formal regularization $-\Delta^{gR} = -\Delta + g_R \delta$ of the formal perturbed operator $-\Delta^g = -\Delta^0 + g \delta$ introduced in Sec. I [see Eqs. (3) and (5)], we get $g_R = \gamma / \|\mathbf{d}\|_{-2}^2$ and, hence,

$$\frac{\|\mathbf{d}\|_{-2}^2}{\alpha} = \frac{1}{g_R} + c\|\mathbf{d}\|_{-2}^2.$$

Summing up, we have proved that the regularized formal operator $-\Delta^{g_R} = -\Delta + g_R \delta$, describing the Laplace operator with a delta type interaction considered in the introduction, corresponds to the operator $-\Delta^\gamma = -\Delta_{\alpha=\gamma/(1+c\gamma)}$, with $\gamma = \|\mathbf{d}\|_{-2}^2 g_R$, and with any real c , and therefore, it is unitary equivalent to the operator $Q_{\alpha=\gamma/(1+c\gamma)}$, defined explicitly in Secs. II D and II E, respectively, when $\Omega^n = \mathbb{R}^n$ or \mathbb{H}^n . The resolvents are given by taking Fourier transform of the resolvents given in Lemmas 2.17 and 2.21. In all cases, the operator $-\Delta^0$ is the maximal operator, and the difference of the resolvents $R(\lambda, -\Delta^{g_R}) - R(\lambda, -\Delta^0)$ is of trace class. The trace is given in Lemmas 2.19 and 2.23.

In particular, we use this result in the formula for the difference of the resolvents given in Proposition 2.11. Since e is the Fourier transform of \mathbf{d} , $\|\mathbf{d}\|_{-2}^2 = \|e\|_{-2}^2 = \|\psi_+\|_0^2 = a_n$, we obtain

$$(R(\lambda, -\Delta^{g_R}) - R(\lambda, -\Delta^0))\phi = \frac{1}{\frac{1}{g_R} + a_n c + b_n - g_n(\lambda)} \int_{\Omega^n} \bar{e}(x) (R(\lambda, -\Delta^0)\phi)(x) d^n x \psi_\lambda. \quad (15)$$

As observed in Ref. 3, we have two free constants in this formula, and therefore, a two parameter family of operators. While the constant g_R has a physical meaning, since it is the coupling constant discussed in Sec. I, the constant c should be fixed. However, a prescription to fix the constant c has been introduced in Sec. 1.3.3 of Ref. 3 for the class of the *homogeneous operators*, defined as follows. Suppose there exists a group G of unitary transformations of the Hilbert space \mathcal{H} . An operator A is said to be *homogeneous* if it rescales in an appropriate way under the action of G , as in Lemma 1.3.2 of Ref. 3. Now, suppose the operator A^0 is homogeneous according to this definition. If this is the case, in the same lemma a condition is given for the existence of a singular rank 1 perturbation A^γ of A^0 , and its unicity is proved. The proof is based on the fact that the self-adjoint operator A^γ satisfies the same symmetry property as A^0 for one and only one value of the constant c . This condition fixes the value of c . The case of the Laplace operator $-\Delta$ in \mathbb{R}^3 was discussed in Sec. 1.5.5 of Ref. 3, where it is shown that $-\Delta$ is an homogeneous operator with respect to the group of the scaling transformations of $L^2(\mathbb{R}^3)$. We review this case and we also investigate the one dimensional case in Sec. IV B. However, as observed at the end of Sec. 1.3.3 of Ref. 3, in the two dimensional case, the Laplace operator is homogeneous but the condition for the existence of the singular rank 1 perturbation is not satisfied. It follows that singular rank 1 perturbations of the Laplace operator in dimension 2 do not exist.

The situation is more difficult for the case of the Hilbert space $L^2(\mathbb{H}^n)$, $n=1, 2, 3$. In this case, we do not have the group of symmetry given by scaling transformations and, consequently, the prescription described above does not apply. On the other side, the situation is more delicate because of the following reason. If we compare the formula for the resolvent given in Eq. (15) with the heuristic formula given in Eq. (5), a straightforward calculation shows that the two coincide if and only if we identify

$$\frac{1}{g_R} = \frac{1}{g_R} + a_n c + b_n.$$

This condition can be satisfied either reregularizing the coupling constant, or assuming the condition $c = -b_n/a_n$. Since the constants a_n and b_n depend on the geometry through the parameter a , the first possibility contradicts the plausible physical requirement that the coupling constant should not depend on the geometry. Therefore, we will assume the second possibility. With this choice, the self-adjoint extension associated with the coupling constant g_R is characterized in the following proposition and its corollary, whose proofs follow by the results of the previous sections.

Proposition 3.1: The operator $-\Delta^{g_R}$ is the self-adjoint extension of the nonmaximal multiplication operator $-\hat{\Delta}$ with resolvent

$$R(\lambda, -\Delta^{g_R})\phi = R(\lambda, -\Delta^0)\phi + \frac{1}{\frac{1}{g_R} - g_n(\lambda)} \int_{\Omega^n} \bar{e}(x)(R(\lambda, -\Delta^0)\phi)(x) d^n x \psi_\lambda,$$

$$\psi_\lambda(x) = \frac{e(x)}{|x|^2 - \lambda},$$

where $\lambda \in \rho(-\Delta^{g_R}) \cap \rho(-\Delta^0)$, and either $e(x) = e^{iax}$ and the functions $g_n(\lambda)$ are given for $n = 3, 2, 1$ and $\Omega^n = \mathbb{R}^n$ in Sec. II D or $e(x) = \sin(ax)$ and the $g_n(\lambda)$ are given for $n = 3, 2, 1$ and $\Omega^n = \mathbb{H}^n$ in Sec. II E. We assume $g_R g_n(\lambda) \neq 1$, which corresponds to pure continuum spectrum (see Lemma 3.4).

Corollary 3.2: The difference of the resolvent $R(\lambda, -\Delta^{g_R}) - R(\lambda, -\Delta^0)$ is trace class with trace

$$\text{Tr}(R(\lambda, -\Delta^{g_R}) - R(\lambda, -\Delta^0)) = - \frac{1}{\frac{1}{g_R} - g_n(\lambda)} \int_{\Omega^n} \frac{|e(x)|^2}{(|x|^2 - \lambda)^2} d^n x.$$

Remark 3.3: In other words, we obtain formulas for the trace of the difference of the resolvents $\text{Tr}(R(\lambda, -\Delta^{g_R}) - R(\lambda, -\Delta^0))$ simply by taking the correspondent formulas given in Lemmas 2.19 and 2.23 and making the substitution $1/g_R = 1/(a_n/\alpha + b_n)$ or $1/g_R = 1/(\hat{a}_n/\alpha + \hat{b}_n)$, respectively.

Explicit formulas for the trace for the cases of interest will be given in Sec. IV.

We conclude this section by studying the eigenvalues of the operators $-\Delta^{g_R}$.

Lemma 3.4: Let $-\Delta^{g_R}$ be the operator with resolvent given in Proposition 3.1. Then, the point spectrum of $-\Delta^{g_R}$, $\text{Sp}_p(-\Delta^{g_R}) = \text{Sp}_d(-\Delta^{g_R})$, is given as follows:

- if $\Omega^n = \mathbb{R}^3$ or $\Omega^n = \mathbb{R}^1$, then $\text{Sp}_p(-\Delta^{g_R}) = \emptyset$ if $g_R \geq 0$, while there is one negative eigenvalue otherwise;
- if $\Omega^n = \mathbb{H}^3$, then $\text{Sp}_p(-\Delta^{g_R}) = \emptyset$ if $a \geq -2g_R/\pi^2$, while there is one negative eigenvalue otherwise; and
- if $\Omega^n = \mathbb{H}^1$, we distinguish two cases: if g_R is finite, then $\text{Sp}_p(-\Delta^{g_R}) = \emptyset$ if $a \leq 2g_R/\pi$, while there is one negative eigenvalue otherwise; if $g_R = \infty$, then $\text{Sp}_p(-\Delta^{g_R}) = \{\pi^2 k^2/a^2\}_{k \in \mathbb{Z}_0}$.

Proof: Assume that $\text{Im} \sqrt{\lambda} \geq 0$. Consider first the case of $\Omega^n = \mathbb{R}^n$. Then, the first statement follow from Theorems 1.1.4 and 3.1.4 of Ref. 2.

Next, consider the case of $\Omega^n = \mathbb{H}^n$. If $n = 3$, the possible eigenvalues are the solutions of the equation

$$\frac{1}{g_R} - \frac{\pi^2}{2} i \sqrt{\lambda} + \frac{\pi^2}{4a} e^{2ia\sqrt{\lambda}} = 0. \tag{16}$$

Let $\sqrt{\lambda} = x + iy$. Then, Eq. (16) becomes

$$b - i(x + iy) + \frac{e^{2ia(x+iy)}}{2a} = b + y - ix + \frac{e^{-2ay}}{2a} (\cos(2ax) + i \sin(2ax)) = 0,$$

with $b = 2/\pi^2 g_R$, and separating the real and imaginary parts

$$\begin{cases} b + y + \frac{e^{-2ay}}{2a} \cos(2ax) = 0 \\ x - \frac{e^{-2ay}}{2a} \sin(2ax) = 0. \end{cases} \tag{17}$$

Since $\lambda = x^2 - y^2 + 2ixy$ must be real, $\sqrt{\lambda} = x + iy$ must be purely real or purely imaginary. Thus, we look for solutions with $x=0$ or $y=0$. In the first case, $x=0$, the system in Eq. (17) reduces to the equation

$$y = -b - \frac{e^{-2ay}}{2a},$$

that can be solved graphically with $f_1(y) = y$, $f_2(y) = -b - e^{-2ay}/2a$. Since we have assumed $y \geq 0$, the existence of solutions depends on the value of $f_2(0)$. If $f_2(0) < 0$, then there are no solutions. Since

$$f_2(0) = -b - \frac{1}{2a},$$

and this quantity is negative for all a if $g_R \geq -2a/\pi^2$, it follows that there are no solutions for all a and $g_R \geq -2a/\pi^2$, and there is one negative eigenvalue otherwise. In the second case, $y=0$, and the system becomes

$$\begin{cases} b + \frac{1}{2a} \cos(2ax) = 0 \\ x - \frac{1}{2a} \sin(2ax) = 0, \end{cases}$$

which has only the trivial solution, $x=0$ ($b = -2/2a$).

If $n=2$, the possible eigenvalues are the solutions of the equation

$$\frac{1}{g_R} - \frac{\pi}{4} \log(-\lambda) + \frac{\pi}{2} K_0(2a\sqrt{\lambda}) = 0. \quad (18)$$

If $n=1$, the possible eigenvalues are the solutions of the equation

$$\frac{1}{g_R} - \frac{\pi i}{4\sqrt{\lambda}} (1 - e^{2ia\sqrt{\lambda}}) = 0. \quad (19)$$

With $b = 4/g_R\pi$, Eq. (19) becomes

$$ib\sqrt{\lambda} + 1 - e^{2ia\sqrt{\lambda}} = 0,$$

which gives the system

$$\begin{cases} 1 - by - e^{-2ay} \cos(2ax) = 0 \\ bx - e^{-2ay} \sin(2ax) = 0. \end{cases} \quad (20)$$

With $x=0$, we obtain

$$y = \frac{1}{b} - \frac{e^{-2ay}}{b}.$$

Since $f_2(y) = \frac{1}{b} - e^{-2ay}/b$ has tangent with angular coefficient $f_2'(0) = 2a/b$, the system in Eq. (20) has one positive solution if and only if $a > b/2$. With $y=0$, the system in Eq. (20) becomes

$$\begin{cases} 1 - \cos(2ax) = 0 \\ bx - \sin(2ax) = 0. \end{cases}$$

This system has only the trivial solution $x=0$ if $b \neq 0$ and has infinitely many solutions $x = \pi k/a$, $k \in \mathbb{Z}$, if $b=0$.

We point out that the spectrum in the case $\mathbb{H}^1=[0, \infty)$, $g_R=\infty$, is as expected since in this case the operator reduces to the sum of the Laplacian on the positive half line with Dirichlet boundary conditions, plus the Laplacian on the interval $[0, a]$.

IV. DETERMINANT AND PARTITION FUNCTIONS

The aim of this section is to study the determinant of the operators described in Sec. III and, consequently to obtain explicit expression for the partition function of the associated models of the Casimir effect. We first recall how the zeta function regularization Ref. 19 (see also Ref. 10 and references therein) is used to define the infinite determinants of self-adjoint positive operators A . In fact, one defines

$$\det_{\zeta} A = e^{-\zeta'(0,A)},$$

where the zeta function of A is by definition

$$\zeta(s,A) = \sum_{\lambda \in \text{Sp } A} \lambda^{-s},$$

for $\text{Re}(s)$ sufficiently large and analytically continued elsewhere. Accordingly, we have for the the partition function $Z=(\det_{\zeta} \ell^2 A)^{-1/2}$,

$$\log Z = \frac{1}{2} \zeta'(0,A) - \frac{1}{2} \zeta(0,A) \log \ell^2,$$

where ℓ , a real nonvanishing number, is the usual renormalization parameter.

More precisely, we need relative zeta functions and relative zeta determinants. We recall first the main definitions and some properties of relative zeta determinants in Sec. IV A, and then we apply this method to the operators of interest in Sec. IV B.

A. Relative zeta determinant and relative partition function

We will use the notation introduced in Ref. 30 for the relative zeta functions and we refer to that work or to the original paper of Müller for more details.²⁴

Let \mathcal{H} be a separable Hilbert space, and let A and A_0 be two self-adjoint non-negative linear operators in \mathcal{H} . Suppose that $\text{Sp } A = \text{Sp}_c A$, namely, that A has a pure continuous spectrum. We recall that $R(\lambda, T) = (\lambda I - T)^{-1}$ denotes the resolvent of the operator T and $\rho(T)$ the resolvent set. Then, we introduce the following set of conditions on the pair (A, A_0) :

- (B1) the operator $R(\lambda, A) - R(\lambda, A_0)$ is of trace class for all $\lambda \in \rho(A) \cap \rho(A_0)$;
- (B2) as $\lambda \rightarrow \infty$ in $\rho(A) \cap \rho(A_0)$, there exists an asymptotic expansion of the form

$$\text{Tr}(R(\lambda, A) - R(\lambda, A_0)) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{K_j} a_{j,k} (-\lambda)^{\alpha_j} \log^k(-\lambda),$$

where $-\infty < \dots < \alpha_1 < \alpha_0$, $\alpha_j \rightarrow -\infty$, for large j , and $a_{j,k} = 0$ for $k > 0$; and

- (B3) as $\lambda \rightarrow 0$, there exists an asymptotic expansion of the form

$$\text{Tr}(R(\lambda, A) - R(\lambda, A_0)) \sim \sum_{j=0}^{\infty} b_j (-\lambda)^{\beta_j},$$

where $-1 \leq \beta_0 < \beta_1 < \dots$, and $\beta_j \rightarrow +\infty$, for large j .

We introduce the further consistency condition

- (C) $\alpha_0 < \beta_0$.

It was proved in Ref. 30 that if the pair of non-negative self-adjoint operators (A, A_0) satisfies conditions (B1)–(B3), then it satisfies conditions (1.1)–(1.3) of Ref. 24. In this situation, we define the relative zeta function for the pair (A, A_0) by the following equation;

$$\zeta(s; A, A_0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-tA} - e^{-tA_0}) dt, \tag{21}$$

when $\alpha_0 + 1 < \text{Re}(s) < \beta_0 + 1$, and by analytic continuation elsewhere, and we define the regularized relative determinant of the pair of operators (A, A_0) by

$$\det_\zeta(A, A_0) = e^{-(d/ds)\zeta(s; A, A_0)|_{s=0}}.$$

Introducing the relative spectral measure, we have the following useful representation of the relative zeta function.³⁰

Proposition 4.1. *Let A be a non-negative self-adjoint operator and assume that there exists an operator A_0 such that the pair (A, A_0) satisfies conditions (B1)–(B3) and (C). Then,*

$$\zeta(s; A, A_0) = \int_0^\infty v^{-2s} e(v; A, A_0) dv,$$

where the relative spectral measure is defined by

$$e(v; A, A_0) = \frac{v}{\pi i} \lim_{\epsilon \rightarrow 0^+} (r(v^2 e^{2i\pi - i\epsilon}; A, A_0) - r(v^2 e^{i\epsilon}; A, A_0)),$$

$$r(\lambda; A, A_0) = \text{Tr}(R(\lambda, A) - R(\lambda, A_0)).$$

The integral, the limit, and the trace exist.

Accordingly, we also define the zeta regularized partition function of a model described by the operator A , under the assumption that there exists a second operator A_0 such that the pair of operators (A, A_0) satisfies assumptions (B1)–(B3), by

$$\log Z = \frac{1}{2} \zeta'(0; A, A_0) - \frac{1}{2} \zeta(0; A, A_0) \log \ell^2, \tag{22}$$

where ℓ , a real nonvanishing number, is the usual renormalization parameter.

Next, we recall the main result in Ref. 30 about the decomposition of the relative partition function of a finite temperature quantum field theory on an ultrastatic space time. Let M be a smooth Riemannian manifold of dimension n , $n \in \mathbb{N}$, and consider the product $X = S^1_{\beta/2\pi} \times M$, where S^1_r is the circle of radius r . Let ξ be a complex line bundle over X , and L a self-adjoint non-negative linear operator in the Hilbert space $\mathcal{H}(M)$ of the L^2 sections of the restriction of ξ onto M with respect to some fixed metric g on M . Let H be the self-adjoint non-negative operator $H = -\partial_u^2 + L$ in the Hilbert space $\mathcal{H}(X)$ of the L^2 sections of ξ with respect to the product metric $du^2 \oplus g$ on X and with periodic boundary conditions on the circle. Assume that there exists a second operator L_0 defined on $\mathcal{H}(M)$ such that the pair (L, L_0) satisfies the previous assumptions (B1)–(B3). Then, by Lemma 2.2 of Ref. 30, it follows that there exists a second operator H_0 defined in $\mathcal{H}(X)$ such that the pair (H, H_0) satisfies those assumptions too. Under these requirements, we introduce the relative zeta regularized partition function of the model described by the pair of operators (H, H_0) using Eq. (22), and we have the following result (Ref. 30 Proposition 3.1.)

Proposition 4.2. *Let L be a non-negative self-adjoint operator on M , and $H = -\partial_u^2 + L$, on $S^1_r \times M$ as defined above. Assume that there exists an operator L_0 such that the pair (L, L_0) satisfies conditions (B1)–(B3). Then,*

$$\zeta(0; H, H_0) = -\beta \text{Res}_{s=-1/2} \zeta(s; L, L_0),$$

$$\zeta'(0; H, H_0) = -\beta \text{Res}_{s=-1/2} \zeta(s; L; L_0) - 2\beta(1 - \log 2) \text{Res}_{s=-1/2} \zeta(s; L, L_0) - 2 \log \eta(\beta; L, L_0),$$

where $H_0 = -\partial_u^2 + L_0$, and the relative Dedekind eta function is defined by

$$\log \eta(\tau; L, L_0) = \int_0^\infty \log(1 - e^{-\tau v}) e(v; L, L_0) dv.$$

The residues and the integral are finite.

B. Relative determinant for the Laplacian in the whole space

The aim of this section, and of the following one, is to investigate the determinant and the partition function of the operator $-\Delta^{g_R}$ described in Sec. III by means of the technique described in the previous section. More precisely, we consider in this section the operator acting in the space $L^2(\mathbb{R}^n)$, and in the following section the operator acting in the half space $L^2(\mathbb{H}^n)$. However, recalling the analysis of Sec. III, singular rank 1 perturbations of the Laplace operator on $L^2(\mathbb{R}^n)$ according to the definition given in Ref. 3 are well defined only for $n \neq 2$, $n \leq 3$. As a consequence, in the following, we shall restrict ourselves to the cases $n=1$ and $n=3$.

Remark 4.3. The operator $-\Delta^{g_R}$ in the three dimensional space was originally described in Ref. 5 and, more recently, in Ref. 2 (Sec. 1.1), and in Ref. 3 (Sec. 1.5). The operator $-\Delta^{g_R}$ in the one dimensional space was investigated in Ref. 1 (Sec. 1.3). In particular, it is worth to observe that the operators described by Albeverio and Kurasov are obtained as rank 1 singular perturbations of the maximal operator $-\Delta^0$ (in the language of Ref. 3 and Sec. III) fixing the value of the free parameter c by imposing the preservation of the symmetry under scaling transformations (as explained in Sec. III), while here we fix the value of the constant c by the condition described at the end of Sec. III, namely, $c = -b_n/a_n$. The two different prescriptions, however, define the same operator as follows by comparing the formulas for the difference of the resolvents given in Ref. 2 (Theorems I.1.1.2 and I.3.1.3) with the ones obtained here using Proposition 3.1 and Lemma 2.17.

The relative zeta function and the relative partition function for the operator $-\Delta^{g_R}$ in \mathbb{R}^3 have been evaluated in Sec. 4.1 of Ref. 30, and the following results were obtained:

$$\zeta(s; -\Delta^{g_R}, -\Delta^0) = \frac{1}{2} \frac{(2\pi^2 g_R)^{2s}}{\cos \pi s},$$

$$\log \eta(\tau; -\Delta^{g_R}, -\Delta^0) = \log \Gamma\left(\frac{\tau}{4\pi^3 g_R}\right) + \frac{1}{2} \log \frac{\tau}{4\pi^3 g_R} - \frac{\tau}{4\pi^3 g_R} \left(\log \frac{\tau}{4\pi^3 g_R} - 1\right) - \frac{1}{2} \log 2\pi,$$

$$\log Z = 2 \left(\log \frac{\ell}{2\pi^2 g_R} - 1\right) \frac{\beta}{8\pi^3 g_R} - \log \eta(\beta; -\Delta^{g_R}, -\Delta^0).$$

For completeness, we investigate here the one dimensional case. The trace of the difference of the resolvents is given in Corollary 3.2, and using the results of Sec. II D (see also Remark 3.3), we obtain

$$r(\lambda; -\Delta^{g_R}, -\Delta^0) = \text{Tr}(R(\lambda, -\Delta^{g_R}) - R(\lambda, -\Delta^0)) = -\frac{1}{2\lambda(ib\sqrt{\lambda} + 1)},$$

where $b = 1/\pi g_R$. The expansion for large λ is

$$r(\lambda) = \frac{1}{2ib\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right), \quad \text{Im } \sqrt{\lambda} > 0,$$

and for small λ

$$r(\lambda) = -\frac{1}{2\lambda} + O(1).$$

It follows that all the conditions (B1)–(B3) of Sec. IV A are satisfied with $\alpha_0 = -\frac{3}{2}$ or $\alpha_0 = -1$, and $\beta_0 = -1$. However, it should be noted that since $\alpha_0 = -\frac{3}{2}$ if $b \neq 0$, while $\alpha_0 = -1$, if $b = 0$, then condition (C) is satisfied when $b \neq 0$, but is not satisfied when $b = 0$. This is consistent with the fact that the limit case $b = 0$ gives $g_R = \infty$, which corresponds to the limit case of the Laplacian with Dirichlet boundary condition at $x = 0$, and a relative zeta function cannot be defined in this case.

Next, we evaluate the relative spectral measure

$$e(v; L, L_0) = \frac{v}{\pi i} \lim_{\epsilon \rightarrow 0^+} (r(v^2 e^{2i\pi - i\epsilon}; L, L_0) - r(v^2 e^{i\epsilon}; L, L_0)).$$

We obtain (note that the spectral measure vanishes in both limit cases $b = 0$ and $b = \infty$, corresponding to the Laplacian with Dirichlet boundary condition at $x = 0$ and to the free Laplacian)

$$e(v; -\Delta^{g_R}, -\Delta^0) = -\frac{b}{\pi(1 + b^2 v^2)},$$

and a simple calculation using the formula for the zeta function given in Proposition 4.1 gives

$$\zeta(s; -\Delta^{g_R}, -\Delta^0) = -\frac{(\pi g_R)^{-2s}}{2 \sin(\pi s)}.$$

Using the definition of the relative Dedekind eta function and Eq. (22) for the partition function, we also obtain

$$\log \eta(\tau; -\Delta^{g_R}, -\Delta^0) = -\log \Gamma\left(\frac{g_R}{2} \tau\right) - \frac{1}{2} \log \frac{g_R}{2} \tau + \frac{g_R}{2} \tau \left(\log \frac{g_R}{2} \tau - 1 \right) + \frac{1}{2} \log 2\pi,$$

$$\log Z = -\frac{\pi}{4} g_R \beta - \log \eta(\beta; -\Delta^{g_R}, -\Delta^0).$$

C. Relative determinant for the Laplacian in the half space

We pass now to study the case of main interest, namely, the operator $-\Delta^{g_R}$ acting in the space $L^2(\mathbb{H}^n)$, $n = 1$ and 3 . The case $n = 2$ presents nontrivial technical aspects that we are not able to tackle at the moment, and therefore, its investigation is postponed to a further occasion.

1. The case $n = 3$

The operator $-\Delta^{g_R}$ in \mathbb{H}^3 is the operator with resolvent given in Proposition 3.1 and corresponds to the Fourier transforms of the operator investigated in Sec. II E. In order to apply the results of Sec. IV A, we first need to check conditions (B1)–(B3). By Lemmas 2.15 and 3.4, the operator $-\Delta^{g_R}$ has pure continuous spectrum coinciding with the non-negative real axis for all $g_R \geq 0$. We will restrict ourselves to this case. Consider the pair $(-\Delta^{g_R}, -\Delta^0)$. By Corollary 3.2 and Lemma 2.23 (see also Remark 3.3), the difference of the resolvent is of trace class with trace

$$\begin{aligned} r(\lambda; -\Delta^{g_R}, -\Delta^0) &= \text{Tr}(R(\lambda, -\Delta^{g_R}) - R(\lambda, -\Delta^0)) = \frac{\pi^2(1 - e^{2ai\sqrt{\lambda}})}{2i\sqrt{\lambda} \left(\frac{1}{g_R} - \frac{\pi^2}{2} i\sqrt{\lambda} + \frac{\pi^2}{4a} e^{2ia\sqrt{\lambda}} \right)} \\ &= \frac{1 - e^{2ai\sqrt{\lambda}}}{i\sqrt{\lambda} \left(b - i\sqrt{\lambda} + \frac{e^{2ia\sqrt{\lambda}}}{2a} \right)}, \end{aligned} \tag{23}$$

where $b = 2/\pi^2 g_R$ and a is a real positive number that gives the position of the delta interaction. We obtain the expansions for large $|\lambda|$

$$r(\lambda) = \frac{1}{\lambda} - \frac{ib}{\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right), \quad \text{Im } \sqrt{\lambda} > 0,$$

and for small $|\lambda|$

$$r(\lambda) = -\frac{4a^2}{1+2ab} - \frac{4a^3i}{1+2ab}\sqrt{\lambda} - \frac{16a^4(1-ab)}{3(1+2ab)^2}\lambda + O(\lambda^{3/2}),$$

and this shows that all the conditions (B1)–(B3) of Sec. IV A are satisfied with $\alpha_0 = -1 < \beta_0 = 0$, and therefore, also condition (C) is satisfied.

Second, we evaluate the relative spectral measure

$$e(v; L, L_0) = \frac{v}{\pi i} \lim_{\epsilon \rightarrow 0^+} (r(v^2 e^{2i\pi - i\epsilon}; L, L_0) - r(v^2 e^{i\epsilon}; L, L_0)). \quad (24)$$

Substitution of the expression in Eq. (23) into Eq. (24) gives

$$e(v; -\Delta^{gR}, -\Delta^0) = -\frac{4a}{\pi} \frac{1 - 2ab + 2ab \cos(2av) - 2av \sin(2av) - \cos(2av)}{1 + 4a^2(b^2 + v^2) + 4ab \cos(2av) - 4av \sin(2av)}.$$

The function $e(v; -\Delta^{gR}, -\Delta^0)$ is a regular function of v for all $v \geq 0$. We show that there is no solution with $v \geq 0$ of the equation

$$1 + 4a^2(b^2 + v^2) + 4ab \cos(2av) - 4av \sin(2av) = 0.$$

Consider the two curves,

$$f_1(v) = 4av \sin(2av) - 4a^2v^2$$

and

$$f_2(v) = 1 + 4a^2b^2 + 4ab \cos(2av),$$

and assume that $a, b, v \geq 0$. Obviously, $f_1(v) \leq f_3(v)$, where $f_3(v) = 4av - 4a^2v^2$ is a parabola facing down with vertex $V = (1/2a, 1)$ that intercepts the horizontal axis in $v = 0$ and $v = 1/a$. On the other hand,

$$(1 + 2ab)^2 \leq f_2(v) \leq (1 - 2ab)^2,$$

and $f_2(v)$ oscillates around the value $1 + 4a^2b^2$, and $f_2(\pi/4a) = 1 + 4a^2b^2$. This suggests to split the problem into three intervals $[0, \pi/4a]$, $[\pi/4a, 1/a]$, and $[1/a, \infty)$. In the interval $[0, \pi/4a]$, f_2 is decreasing, and therefore,

$$f_2(v) \geq f_2\left(\frac{\pi}{4a}\right) = 1 + 4a^2b^2.$$

On the other hand, in the same interval we have $f_3(v) \leq 1$, and therefore $f_3 < f_2$ in this interval. In the interval $[1/a, \infty)$, $f_3(v) \leq 0$, while $f_2(v) \geq (1 - 2ab)^2$, and the value is zero if and only if $v = \pi(1 + 2k)/2a$, $k \in \mathbb{Z}$. However, we have $f_3(v) = 0$ if and only if $v = 0$ or $v = 1/a$, and $1/a < \pi/2a$. Therefore, $f_3 < f_2$ in this interval. Eventually, consider the interval $[\pi/4a, 1/a]$. In this interval, f_2 is decreasing and hence

$$f_2(v) \geq f_2\left(\frac{1}{a}\right) = 1 + 4a^2b^2 + 4ab \cos 2.$$

Also f_3 is decreasing, and hence

$$f_3(v) \leq f_2\left(\frac{\pi}{4a}\right) = \pi - \frac{\pi^2}{4} > 0.$$

Now, we can check that $1 + 4a^2b^2 + 4ab \cos 2 > \pi - \pi^2/4$, for all a and b , hence $f_3 < f_2$ on this interval, and this concludes the proof that $e(v; -\Delta^{gR}, -\Delta^0)$ is a regular function of v for all $v \geq 0$.

Third, we use Proposition II.1 in order to obtain a suitable analytic extension of the relative zeta function. For we need the behavior for small and large v of the function $e(v; -\Delta^{gR}, -\Delta^0)$. We have

$$e(v; -\Delta^{gR}, -\Delta^0) = O(v^2)$$

for $v \rightarrow 0^+$, and

$$e(v; -\Delta^{gR}, -\Delta^0) = \frac{2 \sin(2av)}{\pi v} + \frac{2 \sin^2(2av)}{a \pi v^2} - \frac{2(1-2ab)\sin^2(av)}{a \pi v^2} + O(v^{-3})$$

for $v \rightarrow +\infty$. So we decompose

$$e(v; -\Delta^{gR}, -\Delta^0) = e_0(v; -\Delta^{gR}, -\Delta^0) + e_\infty(v; -\Delta^{gR}, -\Delta^0),$$

where

$$e_\infty(v; -\Delta^{gR}, -\Delta^0) = \frac{2 \sin(2av)}{\pi v} + \frac{2 \sin^2(2av)}{a \pi v^2} - \frac{2(1-2ab)\sin^2(av)}{a \pi v^2},$$

$$e_0(v; -\Delta^{gR}, -\Delta^0) = e(v; -\Delta^{gR}, -\Delta^0) - e_\infty(v; -\Delta^{gR}, -\Delta^0),$$

and

$$\begin{aligned} \zeta(s; -\Delta^{gR}, -\Delta^0) &= \zeta_0(s; -\Delta^{gR}, -\Delta^0) + \zeta_\infty(s; -\Delta^{gR}, -\Delta^0) \\ &= \int_0^\infty v^{-2s} e_0(v; -\Delta^{gR}, -\Delta^0) dv + \int_0^\infty v^{-2s} e_\infty(v; -\Delta^{gR}, -\Delta^0) dv. \end{aligned}$$

Now $e_0(v; -\Delta^{gR}, -\Delta^0)$ goes to a constant for $v \rightarrow 0$ and vanishes as v^{-3} for $v \rightarrow \infty$, and so the function $\zeta_0(s; -\Delta^{gR}, -\Delta^0)$ is a regular function of s in the interval $-1 < \operatorname{Re}(s) < \frac{1}{2}$. The function $\zeta_\infty(s; -\Delta^{gR}, -\Delta^0)$ can be studied explicitly. We evaluate the integrals

$$\int_0^\infty v^{-2s-1} \sin(2av) dv = -(2a)^{2s} \sin(\pi s) \Gamma(-2s)$$

for $-\frac{1}{2} < \operatorname{Re}(s) < \frac{1}{2}$ (Ref. 17, Sec. 3.761.4),

$$\int_0^\infty v^{-2s-2} \sin^2(av) dv = (2a)^{2s+1} \sin(\pi s) \Gamma(-2s-1),$$

for $-\frac{1}{2} < \operatorname{Re}(s) < \frac{1}{2}$ (Ref. 17, Sec. 3.823). Collecting, we have

$$\zeta_\infty(s; -\Delta^{gR}, -\Delta^0) = \frac{4}{\pi} (2a)^{2s} \sin(\pi s) \Gamma(-2s-1) (ab + s + 2^{2s}).$$

Thus, we have the following representation for the relative zeta function when $-\frac{1}{2} < \operatorname{Re}(s) < 1/2$:

$$\zeta(s; -\Delta^{gR}, -\Delta^0) = \frac{4}{\pi} (2a)^{2s} \sin(\pi s) \Gamma(-2s-1) (ab + s + 2^{2s}) + \int_0^\infty v^{-2s} e_0(v; -\Delta^{gR}, -\Delta^0) dv.$$

This representation can be used in order to study the analytic continuation and, in particular, evaluate the residue and the finite part at $s=-\frac{1}{2}$. We obtain

$$\operatorname{Res}_{s=-1/2} \zeta(s; -\Delta^{g_R}, -\Delta^0) = \frac{2}{\pi^2 g_R},$$

$$\begin{aligned} \operatorname{Res}_{s=-1/2} \zeta(s; -\Delta^{g_R}, -\Delta^0) &= \frac{1 + \log 2}{\pi a} + \frac{2b(\gamma + \log(2a))}{\pi} + \zeta_0\left(-\frac{1}{2}; -\Delta^{g_R}, -\Delta^0\right) = \frac{1 + \log 2}{\pi a} \\ &+ \frac{4(\gamma + \log(2a))}{\pi^2 g_R} + \int_0^\infty v e_0(v; -\Delta^{g_R}, -\Delta^0) dv. \end{aligned}$$

Using Proposition 4.2 and the formula in Eq. (22), we obtain the formula for the relative partition function

$$\begin{aligned} \log Z &= -\frac{\beta}{2} \operatorname{Res}_{s=-1/2} \zeta(s; -\Delta^{g_R}, -\Delta^0) - \beta(1 - \log(2\ell)) \operatorname{Res}_{s=-1/2} \zeta(s; -\Delta^{g_R}, -\Delta^0) - \log \eta(\beta; -\Delta^{g_R}, -\Delta^0) \\ &= \frac{2\beta(\log(2\ell) - 1)}{\pi^2 g_R} + \frac{\beta}{2} \left(\frac{1 + \log 2}{\pi a} + \frac{4(\gamma + \log(2a))}{\pi^2 g_R} - \int_0^\infty v e_0(v; -\Delta^{g_R}, -\Delta^0) dv \right) \\ &\quad - \log \eta(\beta; -\Delta^{g_R}, -\Delta^0). \end{aligned}$$

As a consequence, the vacuum energy of the system reads as

$$E_c = -\lim_{\beta \rightarrow \infty} \frac{\partial}{\partial \beta} \log Z = \frac{1 + \log 2}{2\pi a} + \frac{2}{\pi^2 g_R} \left(\gamma + 1 + \log \frac{a}{\ell} \right) + \frac{1}{2} \int_0^\infty v e_0(v; -\Delta^{g_R}, -\Delta^0) dv,$$

since for large β , the exponential in the integral dominates in the definition of $\log \eta(\beta; -\Delta^{g_R}, -\Delta^0)$.

We are interested in the behavior of the force $p = -(\partial/\partial a)E_c$ of the vacuum for small values of a . We need the expansion for small a of the integral

$$\zeta_0\left(-\frac{1}{2}; -\Delta^{g_R}, -\Delta^0\right) = \int_0^\infty v^{-2s} e_0(v; -\Delta^{g_R}, -\Delta^0) dv = \frac{1}{a} \int_0^\infty \frac{x}{a} e_0\left(\frac{x}{a}; -\Delta^{g_R}, -\Delta^0\right) dx.$$

Therefore, we study the function

$$f(x, a) = \frac{x}{a} e_0\left(\frac{x}{a}; -\Delta^{g_R}, -\Delta^0\right) = \frac{N(x, a)}{D(x, a)},$$

where

$$\begin{aligned} N(x, a) &= -4x \sin x (\cos(5x) + 4ab \cos(3x) + 4a^2 b^2 \cos x) - 2 \sin x (2ab \sin(5x) \\ &\quad + (1 + 8a^2 b^2) \sin(3x) + 4ab(1 - ab + 2a^2 b^2) \sin x) \end{aligned}$$

and

$$D(x, a) = \pi x (1 + 4a^2 b^2 + 4x^2 + 4ab \cos(2x) - 4x \sin(2x)) = \pi x g(x, a).$$

This shows that the integral

$$\int_0^\infty f(x,a)dx = \sum_i \int_0^\infty \frac{N_i(x,a)}{D(x,a)} dx$$

decomposes as a finite sum of terms, and in each term the numerator factors as

$$N_i(x,a) = a^{p_i} h_i(x),$$

where p_i is 0, 1, 2, or 3, and the functions $h_i(x)$ are bounded. Thus, it remains to deal with the denominator. As a function of a , $g(x,a)$ is a parabola “facing up” so $g(x,a) \geq g(x,a_0)$, where a_0 is the vertex: so the solution of $\partial_a g(x,a) = 8b^2 a + 4b \cos(2x) = 0$, i.e., $a_0 = -\cos(2x)/2b$. Therefore,

$$D(x,a) \geq D(x,a_0) = \pi x \left(1 + \frac{\cos^2(2x)}{2} + 4x^2 - 4x \sin(2x) \right) > 0,$$

where it is easy to see that the function $D(x,a_0)$ is positive for all x . Thus, $|D(x,a)| > |D(x,a_0)|$, and

$$\int_0^\infty |f(x,a)| dx \leq \sum_i \int_0^\infty \frac{a^{p_i} |h_i(x)|}{|D(x,a_0)|}.$$

Now, it is also easy to see that

$$\int_0^\infty \frac{a^{p_i} |h_i(x)|}{|D(x,a_0)|} < \infty$$

for all i since $a^{p_i} |h_i(x)| / |D(x,a_0)| \sim 1/x^2$ for each i . This proves that the integral

$$\int_0^\infty \frac{x}{a} e_0 \left(\frac{x}{a}; -\Delta^{g_R}, -\Delta^0 \right) dx$$

converges uniformly for a in compact sets, and therefore, we can evaluate the behavior for small a taking the expansion of the integrand for small values of a . We obtain

$$\zeta_0 \left(-\frac{1}{2}; -\Delta^{g_R}, -\Delta^0 \right) = \frac{1}{a} (I_0 + abI_1 + a^2 b^2 I_2) + O(a^2).$$

The integrals I_n can be performed numerically. In particular, we have

$$I_0 = -\frac{2}{\pi} \int_0^\infty \frac{\sin u (\sin(3u) + 2u \cos(5u))}{u(1 + 4u^2 - 4u \sin(2u))} du \sim -0.12,$$

$$I_1 = -\frac{4}{\pi} \int_0^\infty \frac{\sin u (4u(1 + 4u^2) \cos(3u) - 4u \cos u - 4u^2 \sin(5u) + (1 + 16u^2) \sin u)}{u(1 + 4u^2 - 4u \sin(2u))^2} du \sim -0.51,$$

$$I_2 = -\frac{32}{\pi} \int_0^\infty \frac{u \sin u (8u^3 \cos u - 6u \cos(3u) - 12u^2 \sin u + \sin(5u))}{(1 + 4u^2 - 4u \sin(2u))^3} du \sim -1.04.$$

This gives the behavior of the force for small a ,

$$p = -\frac{\partial}{\partial a} E_c = \frac{1 + \log 2 + 2\pi I_0}{2\pi a^2} - \frac{2}{\pi^2 g_R a} - \frac{2I_2}{\pi^4 g_R^2} + O(a).$$

Using the numerical results given above, we see that for small values of a the force is positive ($p \sim 0.15/a^2$).

2. The case $n=1$

The operator $-\Delta^{g_R}$ in $\mathbb{H}^1=[0, \infty)$ is the operator with resolvent given in Proposition 3.1 and corresponds to the Fourier transform of the operator investigated in Sec. II E. By Lemmas 2.15 and 3.4, the operator $-\Delta^{g_R}$ has pure continuous spectrum coinciding with the non-negative real axis if $a \leq 2g_R/\pi < \infty$. We will restrict our considerations to this case. Consider the pair $(-\Delta^{g_R}, -\Delta^0)$. By Corollary 3.2 and Lemma 2.23 (see also Remark 3.3), the difference of the resolvent is of trace class with trace

$$r(\lambda; -\Delta^{g_R}, -\Delta^0) = \text{Tr}(R(\lambda, -\Delta^{g_R}) - R(\lambda, -\Delta^0)) = -\frac{\pi}{4i\lambda^{3/2}} \frac{1 + e^{2ai\sqrt{\lambda}}(2ia\sqrt{\lambda} - 1)}{\frac{1}{g_R} + \frac{\pi}{4i\sqrt{\lambda}}(1 - e^{2ia\sqrt{\lambda}})}.$$

We obtain the expansions for large values of $|\lambda|$

$$r(\lambda; -\Delta^{g_R}, -\Delta^0) = \frac{ig_R\pi}{4\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right), \quad \text{Im } \sqrt{\lambda} > 0,$$

and for small values of $|\lambda|$

$$r(\lambda; -\Delta^{g_R}, -\Delta^0) = \frac{ia^2g_R\pi}{(ag_R\pi - 2)\sqrt{\lambda}} + \frac{a^3g_R\pi(ag_R\pi - 8)}{3(ag_R\pi - 2)^2} + O(\sqrt{\lambda}).$$

This shows that all the conditions (B1)–(B3) of Sec. IV A are satisfied with $\alpha_0 = -3/2 < \beta_0 = -1/2$, and therefore, condition (C) is also satisfied. Second, we evaluate the relative spectral measure (see definition in Proposition 4.1). We obtain

$$e(v; -\Delta^{g_R}, -\Delta^0) = \frac{4b \sin(av)}{\pi} \frac{(ab + 1)\sin(av) - 2av \cos(av)}{b^2 + 2v^2 - b^2 \cos(2av) - 2bv \sin(2av)},$$

where we have set $b = \pi g_R/2$. Proceeding as in the previous section, we show that the function $e(v; -\Delta^{g_R}, -\Delta^0)$ is a regular function of v for all $v \geq 0$. Third, we use Proposition 4.1 in order to obtain a suitable analytic extension of the relative zeta function. We need the behavior for small and large values of v of the function $e(v; -\Delta^{g_R}, -\Delta^0)$. We have

$$e(v; -\Delta^{g_R}, -\Delta^0) = \frac{2a^2b}{\pi(1 - ab)} + O(v^2)$$

for $v \rightarrow 0^+$, and

$$e(v; -\Delta^{g_R}, -\Delta^0) = \frac{2ab \sin(2av)}{\pi v} + \frac{2b(ab \sin^2(2av) - (1 + ab)\sin^2(av))}{\pi v^2} + O(v^{-3})$$

for $v \rightarrow +\infty$. So we decompose

$$e(v; -\Delta^{g_R}, -\Delta^0) = e_0(v; -\Delta^{g_R}, -\Delta^0) + e_\infty(v; -\Delta^{g_R}, -\Delta^0),$$

where

$$e_\infty(v; -\Delta^{g_R}, -\Delta^0) = \frac{2ab \sin(2av)}{\pi v} + \frac{2b(ab \sin^2(2av) - (1 + ab)\sin^2(av))}{\pi v^2},$$

$$e_0(v; -\Delta^{g_R}, -\Delta^0) = e(v; -\Delta^{g_R}, -\Delta^0) - e_\infty(v; -\Delta^{g_R}, -\Delta^0),$$

and

$$\begin{aligned} \zeta(s; -\Delta^{g_R}, -\Delta^0) &= \zeta_0(s; -\Delta^{g_R}, -\Delta^0) + \zeta_\infty(s; -\Delta^{g_R}, -\Delta^0) \\ &= \int_0^\infty v^{-2s} e_0(v; -\Delta^{g_R}, -\Delta^0) dv + \int_0^\infty v^{-2s} e_\infty(v; -\Delta^{g_R}, -\Delta^0) dv. \end{aligned}$$

As for the three dimensional case, $e_0(v; -\Delta^{g_R}, -\Delta^0)$ is a constant at $v=0$ and goes to zero as v^{-3} at infinity. Then $\zeta_0(s; -\Delta^{g_R}, -\Delta^0)$ is a regular function of s in the interval $-1 < \text{Re}(s) < \frac{1}{2}$.

The function $\zeta_\infty(s; -\Delta^{g_R}, -\Delta^0)$ can be studied explicitly. The integrals involved are of the same type of the ones evaluated in the previous section. For $-\frac{1}{2} < \text{Re}(s) < \frac{1}{2}$, we obtain

$$\zeta_\infty(s; -\Delta^{g_R}, -\Delta^0) = \frac{(2a)^{2s+1} b(2s + (2^{2s+1} - 1)ab)}{\pi} \Gamma(-2s - 1) \sin(\pi s).$$

Thus, we have the following representation for the relative zeta function when $-\frac{1}{2} < \text{Re}(s) < \frac{1}{2}$:

$$\zeta(s; -\Delta^{g_R}, -\Delta^0) = \frac{(2a)^{2s+1} b(2s + (2^{2s+1} - 1)ab)}{\pi} \Gamma(-2s - 1) \sin(\pi s) + \int_0^\infty v^{-2s} e_0(v; -\Delta^{g_R}, -\Delta^0) dv.$$

This representation can be used in order to study the analytic continuation and, in particular, evaluate the residue and the finite part at $s = -\frac{1}{2}$. We obtain

$$\text{Res}_{s=-1/2} \zeta(s; -\Delta^{g_R}, -\Delta^0) = -\frac{b}{2\pi} = -\frac{g_R}{4},$$

$$\begin{aligned} \text{Res}_{s=-1/2} \zeta(s; -\Delta^{g_R}, -\Delta^0) &= \frac{b(1 - \gamma - \log(2a) + ab \log 2)}{\pi} + \zeta_0\left(-\frac{1}{2}; -\Delta^{g_R}, -\Delta^0\right) \\ &= \frac{g_R(2 - \gamma - 2 \log(2a) + a g_R \pi \log 2)}{4} + \int_0^\infty v e_0(v; -\Delta^{g_R}, -\Delta^0) dv. \end{aligned}$$

Using Proposition 4.2 and the formula in Eq. (22), we obtain the formula for the relative partition function

$$\begin{aligned} \log Z &= -\frac{\beta}{2} \text{Res}_{s=-1/2} \zeta(s; -\Delta^{g_R}, -\Delta^0) - \beta(1 - \log(2\ell)) \text{Res}_{s=-1/2} \zeta(s; -\Delta^{g_R}, -\Delta^0) - \log \eta(\beta; -\Delta^{g_R}, -\Delta^0) \\ &= \frac{\beta g_R}{8} \left(2\gamma - a g_R \pi \log 2 + 2 \log \frac{a}{\ell} \right) - \frac{\beta}{2} \int_0^\infty v e_0(v; -\Delta^{g_R}, -\Delta^0) dv - \log \eta(\beta; -\Delta^{g_R}, -\Delta^0). \end{aligned}$$

As a consequence, the vacuum energy of the system reads as

$$E_c = -\lim_{\beta \rightarrow \infty} \frac{\partial}{\partial \beta} \log Z = -\frac{g_R}{8} \left(2\gamma - a g_R \pi \log 2 + 2 \log \frac{a}{\ell} \right) + \frac{1}{2} \int_0^\infty v e_0(v; -\Delta^{g_R}, -\Delta^0) dv,$$

since for large β , the exponential function dominates in the function $\log \eta(\beta; -\Delta^{g_R}, -\Delta^0)$.

We are interested in the behavior of the force $p = -(\partial/\partial a)E_c$ of the vacuum for small a . Therefore, we need an expansion for small values of a of the integral

$$\zeta_0\left(-\frac{1}{2}; -\Delta^{g_R}, -\Delta^0\right) = \int_0^\infty v e_0(v; -\Delta^{g_R}, -\Delta^0) dv.$$

Proceeding as in the previous section, we show that the integral converges uniformly on compact subsets. Expanding for small values of a , we obtain

$$\zeta_0\left(-\frac{1}{2}; -\Delta^{g_R}, -\Delta^0\right) = -\frac{2 \log 2}{\pi} ab^2 + O(a^2).$$

This gives the behavior of the force for small values of a ,

$$p = -\frac{\partial}{\partial a} E_c = \frac{g_R}{4a} + \frac{g_R^2 \pi \log 2}{8} + O(a).$$

In this case, the force is positive for small values of a .

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APPENDIX

Lemma A.1: Let D_N^n be the closed disk of radius N centered at the origin in \mathbb{R}^n , $n=1, 2, 3$. Then, for all $\lambda \in \mathbb{C} - (-\infty, 0)$ and for large N

$$\int_{D_N^3} \frac{1}{|x|^2 - \lambda} d^3x = 4\pi N + 2\pi^2 i \sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda}) + o(1),$$

$$\int_{D_N^2} \frac{1}{|x|^2 - \lambda} d^2x = 2\pi \log N - \pi \log(-\lambda) + o(1),$$

$$\int_{D_N^1} \frac{1}{|x|^2 - \lambda} dx = \frac{i\pi}{\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})} - \frac{2}{N} + o\left(\frac{1}{N}\right).$$

Proof: Using polar coordinates, we have

$$\int_{D_N^3} \frac{1}{|x|^2 - \lambda} d^3x = 4\pi \int_0^N \frac{r^2}{r^2 - \lambda} dr = 4\pi N + 4\pi \lambda \int_0^N \frac{dr}{r^2 - \lambda} = 4\pi N + 2\pi^2 i \sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda}) + o(1).$$

In a similar way, we get corresponding results for $n=2$ and 1.

Lemma A.2: For $n=1, 2, 3$, we have

$$\int_{\mathbb{R}^n} \frac{d^n x}{|x|^2 \pm i} = \begin{cases} \sqrt{2} \pi^2, & n=3 \\ \frac{\pi^2}{2}, & n=2 \\ \frac{\pi^2}{\sqrt{2}}, & n=1. \end{cases}$$

Proof: We observe that

$$\frac{1}{|x|^2 \pm i} = \frac{1}{|x|^4 + 1} = \frac{1}{2i} \left(\frac{1}{|x|^2 + i} - \frac{1}{|x|^2 - i} \right).$$

Then the results follow as a consequence of Lemma A.1.

Lemma A.3: Let B_N^n be the closed half disk of radius N centered at the origin in \mathbb{R}^n , $n=1, 2, 3$. Then, for all $\lambda \in \mathbb{C} - (-\infty, 0)$, $a \in \mathbb{R}^n$, and large N

$$\int_{B_N^3} \frac{\sin^2(ax)}{|x|^2 - \lambda} d^3x = \pi N + \frac{i\pi^2}{2} \sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda}) - \frac{\pi^2}{4a} e^{2ia\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})} + o(1),$$

$$\int_{B_N^2} \frac{\sin^2(ax)}{|x|^2 - \lambda} d^2x = \frac{\pi}{2} \log N - \frac{\pi}{4} \log(-\lambda) + \frac{\pi}{2} K_0(2ai\sqrt{\lambda}) + o(1),$$

$$\int_{B_N^1} \frac{\sin^2(ax)}{|x|^2 - \lambda} dx = \frac{i\pi}{4\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})} (1 - e^{2ia\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})}) - \frac{1}{2N} + o\left(\frac{1}{N}\right),$$

where $K_0(z)$ is a Bessel function.

Proof: (See also Ref. 17 Secs. 3.723.3 and 3.723.10). First of all we observe that

$$\int_{B_N^n} \frac{\sin^2(ax)}{|x|^2 - \lambda} d^n x = \frac{1}{2} \int_{D_N^n} \frac{\sin^2(ax)}{|x|^2 - \lambda} d^n x = \frac{1}{4} \int_{D_N^n} \frac{1}{|x|^2 - \lambda} d^n x - \frac{1}{4} \int_{D_N^n} \frac{\cos(2ax)}{|x|^2 - \lambda} d^n x.$$

Now we see that for $n=1, 2, 3$, the first integral in the latter line has been already computed in Lemma 5.1, while for the second one, by taking polar coordinates and putting a on the positive z -axis, we get ($n=3$)

$$\begin{aligned} \int_{D_N^3} \frac{\cos(2ax)}{|x|^2 - \lambda} d^2x &= 2\pi \int_{r=0}^N \int_{u=-1}^1 \frac{r^2 \cos(2aru)}{r^2 - \lambda} du dr = \frac{2\pi}{a} \int_{r=0}^N \frac{r \sin(2ar)}{r^2 - \lambda} dr \\ &= \frac{\pi^2}{a} e^{2ia\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})} + o(1), \end{aligned}$$

while for $n=2, 1$, we obtain, respectively,

$$\int_{D_N^2} \frac{\cos(2ax)}{|x|^2 - \lambda} d^2x = -2\pi K_0(2ai\sqrt{\lambda}) + o(1),$$

$$\int_{-N}^N \frac{\cos(2ax)}{x^2 - \lambda} dx = \frac{\pi i e^{2ia\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})}}{\sqrt{\lambda} \operatorname{sgn}(\operatorname{Im} \sqrt{\lambda})} + o(1/N),$$

with $K_0(z)$ being a Bessel function. The thesis follows from these results and Lemma A.1.

By an easy computation, we get the following:

Lemma A.4: For $n=1, 2, 3$, $a \in \mathbb{R}^n$, and $\operatorname{Im}(\sqrt{\lambda}) > 0$, we have

$$\int_{\mathbb{R}^n} \frac{d^n x}{(|x|^2 - \lambda)^2} = \begin{cases} \frac{i\pi^2}{\sqrt{\lambda}}, & n = 3 \\ -\frac{\pi}{\lambda}, & n = 2 \\ -\frac{i\pi}{2\lambda^{3/2}}, & n = 1. \end{cases}$$

Lemma A.5: For $n=1, 2, 3$, $a \in \mathbb{R}^n$, and $\operatorname{Im}(\sqrt{\lambda}) > 0$, we have

$$\int_{\mathbb{R}^n} \frac{\sin^2(ax)}{(|x|^2 - \lambda)^2} d^n x = \begin{cases} \frac{i\pi^2}{2\sqrt{\lambda}} (1 - e^{2ia\sqrt{\lambda}}), & n = 3 \\ -\frac{\pi}{2\lambda} - \frac{\pi ia}{\sqrt{\lambda}} K_1(-2ia\sqrt{\lambda}), & n = 2 \\ -\frac{i\pi}{4\lambda^{3/2}} (1 - e^{2ia\sqrt{\lambda}}) + \frac{\pi a}{2\lambda} e^{2ia\sqrt{\lambda}}, & n = 1, \end{cases}$$

where $K_1(z)$ is a Bessel function.

Proof: As in Lemma A.3, we write

$$\int_{\mathbb{R}^n} \frac{\sin^2(ax)}{(|x|^2 - \lambda)^2} d^n x = \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{(|x|^2 - \lambda)^2} d^n x - \frac{1}{2} \int_{\mathbb{R}^n} \frac{\cos(2ax)}{(|x|^2 - \lambda)^2} d^n x.$$

Then, choosing polar coordinates and putting a along the positive z -axis, for $n=3$, we get

$$\int_{\mathbb{R}^3} \frac{\cos(2ax)}{(|x|^2 - \lambda)^2} d^3 x = 2\pi \int_{r=0}^{\infty} \int_{u=-1}^1 \frac{r^2 \cos(2aru)}{(r^2 - \lambda)^2} du dr = \frac{2\pi}{a} \int_{r=0}^{\infty} \frac{r \sin(2ar)}{(r^2 - \lambda)^2} dr = \frac{i\pi^2}{\sqrt{\lambda}} e^{2ia\sqrt{\lambda}}.$$

In a similar way, we obtain

$$\int_{\mathbb{R}^2} \frac{\cos(2ax)}{(|x|^2 - \lambda)^2} d^2 x = \frac{2\pi ia}{\sqrt{\lambda}} K_1(-2ia\sqrt{\lambda}),$$

$$\int_{\mathbb{R}} \frac{\cos(2ax)}{(x^2 - \lambda)^2} dx = -\left(\frac{\pi a}{\lambda} + \frac{i\pi}{2\lambda^{3/2}}\right) e^{2ia\sqrt{\lambda}}.$$

The thesis follows from these results and Lemma A.4.

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