ON THE NON-HOMOGENEOUS QUADRATIC BESSEL ZETA FUNCTION

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Abstract. This article studies the non-homogeneous quadratic Bessel zeta function $\zeta_{RB}(s, \nu, a)$, defined as the sum of the squares of the positive zeros of the Bessel function $J_\nu(z)$ plus a positive constant. In particular, explicit formulas for the main associated zeta invariants, namely, poles and residua $\zeta_{RB}(0, \nu, a)$ and $\zeta'_{RB}(0, \nu, a)$, are given.

From the point of view of differential geometry and mathematical physics, the Riemann zeta function appears as the operator zeta function associated to the Laplacian operator on the line segment [**5**, **6**, **14**, **17**, **18**]. A natural generalization of this setting is to consider a Sturm Liouville operator instead, *i.e.*, a singularity at one of the end points [**7**, **8**, **9**, **10**, **11**, **16**]. This leads again to a concrete zeta function, namely the Bessel zeta function, where the sum extends over the positive zeros of the Bessel function $J_\nu(z)$, and reduces for a suitable choice of ν to the classical Riemann case. Such a function was first considered and studied by Stolarsky in [**21**], where formulas for poles and residua are given, and more recently by other authors, who calculated the associated zeta invariants by different methods [**1**, **16**]. In these notes, we study the non-homogeneous version of this function. We determinate its poles and give formulas for the residua. In particular, we introduce two simple but quite general methods to calculate the value of the derivative at the origin, and therefore the regularized determinant of the associated Sturm-Liouville singular operator [**3**, **4**, **12**, **19**].

Consider the constant singular Sturm-Liouville operator

$$L_\nu + q^2 = -\frac{d^2}{dx^2} + \frac{4\nu^2 - 1}{4x^2} + q^2$$

on the line interval $(0, l]$, with positive real ν and q (the null cases can be easily obtained as limit cases). L_ν has the discrete resolution [**13**]

$$\left\{ \lambda_{\nu,n}^2 + q^2 = \frac{j_{\nu,n}^2}{l^2} + q^2, \quad \phi_{\nu,n}(x) = \frac{\sqrt{2x}J_\nu(\lambda_{\nu,n}x)}{lJ_{\nu+1}(j_{\nu,n})} \right\},$$

where the $j_{\nu,n}$ are the positive zeros of the Bessel function $J_\nu(z)$ in increasing order [**22**].

For analogy with the Riemann case, we consider the following non-homogeneous quadratic Bessel zeta function defined by

[MATHEMATIKA, **51** (2004), 123–130]

$$\zeta_{RB}(s, \nu, a) = \sum_{n=1}^{\infty} (n_{\nu,n}^2 + a^2)^{-s/2},$$

for $\mathrm{Re}(s) > 1$, where $\pi n_{\nu,n} = j_{\nu,n}$ and a is real and positive, and we study its analytical extension. We can state our main result for the Bessel zeta function; this comes as a corollary of the more general result stated in Proposition 3 below concerning the zeta function associated to the operator $L_\nu + q^2$.

PROPOSITION 1. *The function $\zeta_{RB}(s, \nu, a)$ has an analytic extension to the complex s-plane, smooth in ν, up to a discrete set of simple poles at $s = 1, -1, -3, \ldots$, whose residua can be computed using the known asymptotic expansions for the Bessel functions (more precisely, they are given by the residua of the function $z(s, \nu, a, \pi)$ in Proposition 3 multiplied by 2); in particular*

$$\mathrm{Res}_1(\zeta_{RB}(s, \nu, a), s = 1) = 1,$$

$$\mathrm{Res}_1(\zeta_{RB}(s, \nu, a), s = -1) = -\frac{1}{2\pi^2}\left(\nu^2 - \frac{1}{4} - \pi^2 a^2\right).$$

This extension is regular at $s = 0$, and

$$\mathrm{Res}_0(\zeta_{RB}(s, \nu, a), s = 0) = -\frac{1}{2}\left(\nu + \frac{1}{2}\right),$$

$$\mathrm{Res}_0(\zeta'_{RB}(s, \nu, a), s = 0) = -\frac{1}{2}\log \sqrt{2}\pi \frac{I_\nu(\pi a)}{a^\nu}.$$

Notice that the values for the homogeneous case follow immediately using the series expansion for the Bessel function $I_\nu(z)$ for small z, namely (see [16]),

$$\zeta'_{RB}(0, \nu, 0) = \frac{1}{2}\log \frac{2^{\nu - 1/2}\Gamma(\nu + 1)}{\pi^{\nu + 1}}.$$

We study the more general setting, *i.e.*, the function

$$z(s, \nu, q, l) = \zeta(s; L_\nu + q^2) = \sum_{n=1}^{\infty} (\lambda_{\nu,n}^2 + q^2)^{-s}.$$

Convergence of the series for $\mathrm{Re}(s) > \frac{1}{2}$ follows from classical estimates on the zeros of Bessel functions [22], and $\zeta_{RB}(2s, \nu, a) = z(s, \nu, a, \pi)$.

We present here two approaches to calculate the zeta invariants associated to $z(s, \nu, q, l)$. In the first, we produce an analytic representation that can be effectively used to get all the invariants; this will be particularly useful to generalize the method for the calculation of the zeta invariants associated to a disc or to a cone [20]. In the second approach, we give a very general lemma (Lemma 4) to deal with regularized products, and we apply it to calculate $z(0, \nu, q, l)$ and $z'(0, \nu, q, l)$.

We begin with the first approach. Using the Mellin transform [13, 15], we get the analytic representation

$$z(s, \nu, q, l) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t, \nu, q, l)\, dt,$$

where the trace of the heat operator is

$$f(t, \nu, q, l) = \mathrm{Tr}\, e^{-tL_\nu} = \sum_{n=1}^{\infty} e^{-(\lambda_{\nu,n}^2 + q^2)},$$

and from this the complex representation

$$f(t, \nu, q, l) = \frac{1}{2\pi i} \int_{\Lambda_c} e^{-\lambda t} R(\lambda, \nu, q, l) d\lambda,$$

where the contour is $\Lambda_c = \{\lambda \in \mathbf{C}: |\arg(\lambda - c)| = \pi/4\}$, for some $0 < c < q^2$, and the trace of the resolvent is

$$R(\lambda, \nu, q, l) = \sum_{n=1}^{\infty} \frac{1}{\lambda - (\lambda_{\nu,n}^2 + q^2)}.$$

We now observe that it is easy to express such a function in terms of special functions. In fact, taking the logarithmic derivative of the infinite product representation of the Bessel function $I_\nu(z)$, we get **[22]**

LEMMA 1.

$$R(\lambda, \nu, q, l) = \frac{\nu}{2z^2} - \frac{1}{2z} \frac{d}{dz} \log I_\nu(lz).$$

Here, $z = \sqrt{q^2 - \lambda}$, we set $\arg(q^2 - \lambda) = 0$ on the line $(-\infty, q^2)$ and we fix the sector $s_+ = \{z \in \mathbf{C}: |\arg z| < \pi/2\}$ for z.

At this point, it is worth observing that information about poles and residua of $z(s, \nu, q, l)$ can be obtained using the representation introduced, asymptotics expansions for Bessel functions and classical arguments **[13, 15]**. This is an easy way for producing the results relative to the so-called 'constant case' when studying regular singular operators **[7, 8, 9]**. More precisely, and for completeness, we can state the following results.

LEMMA 2. *For small t,*

$$f(t, \nu, q, l) = \sum_{i=0}^{l} a_i(\nu, q, l) t^{(i-1)/2} + O(t^{l/2})$$
$$= \frac{1}{2\sqrt{\pi}} t^{-1/2} - \frac{1}{2}\left(\nu + \frac{1}{2}\right) + \frac{1}{2l\sqrt{\pi}}\left(\nu^2 - \frac{1}{4} - l^2 q^2\right) t^{1/2} + O(t),$$

where

$$a_{2i}(\nu, q, l) = \frac{(-1)^i}{i!} \frac{l q^{2i}}{2\sqrt{\pi}} - \sum_{j=0}^{\infty} \sum_{k=1, k+2j=2i-1}^{\infty} \frac{(-1)^{j+k}}{2^k j! k!} \frac{q^{2j}}{l^k \Gamma(\frac{k}{2})} \frac{\Gamma(\nu + k + \frac{1}{2})}{\Gamma(\nu - k + \frac{1}{2})},$$

$$a_{2i+1}(\nu, q, l) = \frac{(-1)^{i+1}(\nu - \frac{1}{2}) q^{2i}}{2i!} - \sum_{j=0}^{\infty} \sum_{k=1, k+2j=2i}^{\infty} \frac{(-1)^{j+k}}{2^k j! k!} \frac{q^{2j}}{l^k \Gamma(\frac{k}{2})} \frac{\Gamma(\nu + k + \frac{1}{2})}{\Gamma(\nu - k + \frac{1}{2})}.$$

PROPOSITION 2. *The function $z(s, \nu, q, l)$ has an analytic extension to the whole complex s-plane, smooth in ν and q, up to a discrete set of simple poles at $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots,$ with residua*

$$\mathrm{Res}_1\left(z(s,\ \nu,\ q,\ l),\ s=\frac{1}{2}-k\right)=\frac{a_{2k}(\nu,\ q,\ l)}{\Gamma(\frac{1}{2}-k)},\qquad k=0,\ 1,\ 2,\dots$$

This extension is regular at $s=0$, and

$$\mathrm{Res}_0(z(s,\ \nu,\ q,\ l),\ s=0)=-\frac{1}{2}\left(\nu+\frac{1}{2}\right).$$

Notice that the contribution of the non-homogeneity term q^2 is shared equally among all the poles; in other words, the homogeneous Bessel zeta function has the same poles, but with (possibly) different residua (see [**21**]).

The information available is not enough to deal with the derivative, which is a harder point; however, we introduce the following quite general purpose result.

LEMMA 3. *Suppose that the zeta function $z(s,x)=\sum_{n=1}^{\infty}a_n(x)^{-s}$ has the following representation (with everything smooth in x):*

$$z(s,\ x)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}e^{-\lambda t}R(\lambda,\ x)d\lambda dt,$$

where the contour is as above, and there is a primitive function $-T(\lambda,\ x)$ for the function $R(\lambda,\ x)=-\frac{d}{d\lambda}T(\lambda,\ x)$, satisfying the following properties:

(*a*) *T is analytic near $\lambda=0$,*
(*b*) *for large λ and fixed x, there is an asymptotic expansion*

$$T(\lambda,\ x)=\cdots+A(x)\log(-\lambda)+B(x)+\cdots.$$

Then $z(s,\ x)$ can be analytically extended at $s=0$, and

$$z(0,\ x)=-A(x),$$
$$z'(0,\ x)=-B(x)+T(0,\ x).$$

Proof. Integrating the given complex representation for $z(s,x)$ by parts, first in λ and then in t, we get

$$z(s,\ x)=\frac{s}{\Gamma(s)}\int_0^{\infty}t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{e^{-\lambda t}}{-\lambda}T(\lambda,\ x)d\lambda dt.$$

At this point, it is convenient to isolate the constant part of T, writing

$$T(\lambda,\ x)=T_1(\lambda,\ x)+T_0(x);$$

in fact, it is clear that $T_0(x)$ gives no contribution, since

$$\int_{\Lambda_c}\frac{e^{-\lambda t}}{-\lambda}T_0(x)d\lambda=0.$$

Next, since by definition the zeta function is well defined for large s,

$$z(s, x) = \frac{s^2}{\Gamma(s+1)} \int_0^1 t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_c} \frac{e^{-\lambda t}}{-\lambda} T_1(\lambda, x) d\lambda dt + s^2 f(s),$$

where f is regular near $s = 0$. Because of the pole at $\lambda = 0$, we have to split the complex integral as follows to use the expansion for large λ (small t):

$$\int_{\Lambda_c} = \int_{\Lambda_{-c}} - \int_{C_c} = \int_{\Lambda_{-c}} + 2\pi i T_1(0, x),$$

where C_c is a circle around the origin of radius c. Moreover, by assumption (b),

$$\int_{\Lambda_{-c}} \frac{e^{-\lambda t}}{-\lambda} T_1(\lambda, x) d\lambda dt = \int_{\Lambda_{-c}} \frac{e^{-\lambda}}{-\lambda} T_1(\lambda/t, x) d\lambda dt$$
$$= \cdots + \gamma A(x) + A(x) \ln t + B(x) + \cdots,$$

where we have explicitly written only the relevant part. This means that we can write

$$z(s, x) = \frac{s}{\Gamma(s+1)} \left[\gamma A(x) - B(x) - \frac{1}{s} A(x) + T_1(0, x) \right] + s^2 g(s),$$

where again g is regular near $s = 0$, and, from that, the claim follows at once. □

Applying this argument to the function $z(s, \nu, q, l)$, we have

$$T(\lambda, \nu, q, l) = \nu \log lz - \log I_\nu(lz) - \log 2^\nu \Gamma(\nu + 1),$$

and hence we compute

$$A(\nu, q, l) = \frac{1}{2}\left(\nu + \frac{1}{2}\right),$$
$$B(\nu, q, l) = \frac{1}{2} \log 2\pi + \left(\nu + \frac{1}{2}\right) \log l - \log 2^\nu \Gamma(\nu + 1),$$
$$T(0, \nu, q, l) = \nu \log lq - \log I_\nu(lq) - \log 2^\nu \Gamma(\nu + 1).$$

This gives

$$\mathrm{Res}_0(z(s, \nu, q, l), s = 0) = -\frac{1}{2}\left(\nu + \frac{1}{2}\right),$$

and proves the following

PROPOSITION 3.

$$\mathrm{Res}_0(z'(s, \nu, q, l), s = 0) = -\log \sqrt{2\pi l} \frac{I_\nu(lq)}{q^\nu}.$$

We now turn to the second approach. First, we have the following

LEMMA 4. *Suppose that two sequences a_n and b_n, $n=1, 2, 3, \ldots$, of real positive numbers are given, which satisfy the following conditions*:

(*A*) *there are real s_a, s_b such that the two series*

$$\zeta_a(s)=\sum_{n=1}^{\infty} a_n^{-s}, \qquad \zeta_b(s)=\sum_{n=1}^{\infty} b_n^{-s},$$

converge for $\mathrm{Re}(s)>s_a$, s_b *respectively*;

(*B*) *the zeta function $\zeta_b(s)$ has an analytic extension at $s=0$*;

(*C*) $|a_n-b_n|<K$ *definitely for some constant K*;

(*D*) $s_b<1$.

Then the zeta function $\zeta_a(s)$ has an analytic extension at $s=0$ with $\zeta_a(0)=\zeta_b(0)$, the infinite product

$$\prod_{n=1}^{\infty}\frac{a_n}{b_n}=C$$

converges absolutely, and

$$\zeta_a'(0)=\zeta_b'(0)+\log C.$$

Proof. Let $c_n=a_n/b_n-1$. Then $|c_n|<Kb_n^{-1}$ by (B) and (C), and hence $\sum_{n=1}^{\infty}c_n$ converges absolutely and thus so does the infinite product. We can assume that $c_n<1$, and hence

$$\zeta_a(s)=\zeta_b(s)+\sum_{k=1}^{\infty}\binom{-s}{k}\sum_{n=1}^{\infty}b_n^{-s}c_n^k.$$

Now the series

$$\sum_{n=1}^{\infty}b_n^{-s}c_n^k$$

converges in a neighbourhood of $s=0$ for each k, by conditions (C) and (D). This means that ζ_a can be analytically extended to $s=0$ by using the extension of ζ_b. In particular, evaluating at $s=0$, we get $\zeta_a(0)=\zeta_b(0)$ and

$$\zeta_a'(0)=\zeta_b'(0)+\sum_{k=1}^{\infty}\frac{d}{ds}\binom{-s}{k}\sum_{n=1}^{\infty}b_n^{-s}c_n^k\Big|_{s=0}+\sum_{k=1}^{\infty}\binom{-s}{k}\sum_{n=1}^{\infty}\log b_n b_n^{-s}c_n^k\Big|_{s=0}$$

$$=\zeta_b'(0)+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=1}^{\infty}c_n^k=\zeta_b'(0)+\sum_{n=1}^{\infty}\log(1+c_n)=\zeta_b'(0)+\log C.$$

□

We can apply this lemma to the present case as follows. Let $a_n=\lambda_{\nu,n}^2+q^2$, $b_n=\pi^2/l^2\left[n+\frac{1}{2}\left(\nu-\frac{1}{2}\right)\right]^2$; then all the assumptions of Lemma 4 are satisfied, and $\zeta_b(s)$ is the Hurwitz zeta function $\zeta_H(2s,\frac{1}{2}(\nu+\frac{3}{2}))$. Thus,

$$\zeta_b'(0)=2\log\Gamma\left(\frac{1}{2}\left(\nu+\frac{3}{2}\right)\right)-\log 2\pi,$$

and we can compute (where $u = \frac{1}{2}(v - \frac{1}{2})$):

$$\begin{aligned}
\frac{1}{C} &= \lim_{z \to 0^+} \frac{\prod_{n=1}^{\infty} \left(1 + l^2/z^2(j_{v,n}^2 + q^2 l^2)\right)}{\prod_{n=1}^{\infty} \left(1 + l^2/\pi^2 z^2 (n+u)^2\right)} \\
&= \lim_{z \to 0^+} \frac{\prod_{n=1}^{\infty} \left(1 + l^2(1 + q^2 z^2)/z^2 j_{v,n}^2\right)}{\left(1 + l^2 q^2/j_{v,n}^2\right) \prod_{n=1}^{\infty} \left(1 + l^2/\pi^2 z^2 (n+u)^2\right)} \\
&= \lim_{z \to 0^+} \frac{z^v q^v I_v\left(l\sqrt{1 + q^2 z^2}/z\right) \Gamma(u + 1 + il/(\pi z))|^2}{I_v(lq)\Gamma^2(u+1)} \\
&= \frac{\sqrt{2}(lq)^v}{\pi^v I_v(lq) \Gamma^2\left(\frac{1}{2}\left(v + \frac{3}{2}\right)\right)}.
\end{aligned}$$

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