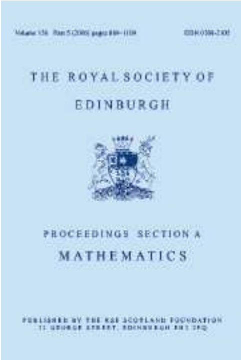

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Zeta invariants for sequences of spectral type, special functions and the Lerch formula

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We discuss the spectral properties of a class of sequences of what we call 'spectral' type. We introduce an effective method to calculate the zeta invariants for this type of sequence. Such invariants are given in terms of some new and old special functions, and we consider a number of examples in which we study the properties of these special functions.

1. Introduction

Since it was introduced in the early 1970s [2,25] as a tool to deal with determinants of elliptic operators and analytic torsion, the zeta function has played a fundamental role in the new discipline of geometric analysis, where analytic methods are used to study geometrical objects. The zeta function technique in this context is a new tool with which to obtain information. Such information is given in terms of particular values of the zeta function, which consequently are called zeta invariants. The first example is the Euler characteristic of a closed surface, given by the value of the associated zeta function at the origin. In fact, it emerges that the more interesting zeta invariants are precisely the values of the zeta function and that of its first derivative at the origin, the latter being the so-called regularized determinant. An enormous amount of work has been produced in this area since then, and very important results have been achieved (just two examples are [11,18]). In particular, the zeta function technique of regularizing the determinant of an elliptic operator became of primary importance in mathematical physics [16]. In this direction, many works [8,21,28,33,35] appeared in the literature, where the aim was to give complete results for a number of cases where it was possible to find out a relationship with some known special functions. These works are all related to the case of compact connected Riemannian manifold, where the zeta function can always be defined in a subset of the complex plane, by the sum

$$\zeta_M(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

of the negative complex powers of the eigenvalues λ_n of the Laplace operator induced by the Riemannian metric [26]. In particular, when a constant term is added to the Laplace operator, the zeta invariants of the non-homogeneous resulting zeta function are related to those of the homogeneous zeta function by a generalization (for the derivative) of the classical Lerch formula relating the derivatives of the Riemann and Hurwitz zeta functions:

$$\zeta'_{\mathrm{H}}(0,q) = \zeta'_{\mathrm{R}}(0) + \ln \Gamma(q).$$

From a more geometrical point of view, this is a relation between the regularized determinants of the non-homogeneous and the homogeneous Laplacian operators on the circle (see § 3.1). This is the point of view that we will take here, and in fact one of our results is a series of generalizations of the above Lerch formula for some interesting cases (see § 3). With this purpose in mind, the approach adopted here consists in providing a general abstract setting for the problem: namely, in § 2 we introduce the zeta function associated with a particular kind of sequence of complex numbers that we call a *sequence of spectral type*, which covers all the cases of interest. Besides the zeta function, we introduce, for such sequences, other spectral functions, in terms of which we give a general Lerch formula (proposition 2.9). Under this geometrical point of view, the main results of our analysis are an effective tool (generalizing a technique introduced in [32]) with which to calculate the zeta invariants that apply for any compact connected manifolds (corollary 2.4), an explanation of the appearance of the spectral functions in the generalized Lerch formula, and a generalization of the latter on the 2-sphere (cf. the last remark in [30, § 5]).

Taking another point of view, the Lerch formula is a good tool for providing information (and, in particular, the constant term of the asymptotic expansion) for some classes of special function. In the Lerch formula for the circle (see equation (3.1), below), the derivatives of the homogeneous and non-homogeneous Riemann zeta functions evaluated at $s = 0$ are related by means of the logarithm of the Gamma function. When considering different types of zeta function, different types of special functions will appear in the Lerch formula. From this point of view, the zeta function technique is a very useful device with which to study the properties of these special functions [5,19,29,31,33]. A class of zeta function that has been investigated for a long time is that of the multiple zeta functions [4]. As an application of our methods, we analyse this class of zeta functions and the associated special functions, obtaining in § 3.2 some new interesting interpretations and results. We conclude by observing that this analysis is also very important in physics. In fact, adding a constant to the Laplacian on a compact manifold means adding a constant potential term m, usually a mass term, to the associated physical theory, and it is of great importance in physics to know the behaviour for large and small m [9].

2. Sequences of spectral type

2.1. Definition and spectral functions

Let S be a sequence[1] of complex numbers whose only accumulation point is the point at infinity. For our purposes, we can assume without loss of generality that $0 \notin S$. We can order the elements of S using the natural numbers, and we write

[1] By a sequence of elements of a set X we mean a mapping from the natural numbers to X.

$S = \{a_n\}_{n\in\mathbb{N}_0}$, with $0 < |a_1| \leqslant |a_2| \leqslant \cdots$. The number

$$\alpha = \limsup_{n\to\infty} \frac{\ln n}{\ln |a_n|},$$

is the *exponent of convergence* of the sequence S. If α is finite, the sequence is of *finite exponent*. For a sequence S of finite exponent α, the series $\sum_{n=1}^{\infty} a_n^{-s}$ converges (locally) uniformly and absolutely for $\mathrm{Re}(s) > \alpha$, and if we assume $\mathrm{Re}(a_n)$ is bounded below, the series $\sum_{n=1}^{\infty} \mathrm{e}^{-a_n t}$ also converges (locally) uniformly and absolutely for $t > 0$. In geometry, the most natural situation where the above setting applies is when $S = \sigma_p P - \ker P = \{\lambda_n\}_{n=1}^{\infty}$ is the sequence of the positive eigenvalues of a linear operator P. The operator P can be an elliptic self-adjoint operator on a compact manifold[2], but more general settings are important. In particular, for studying topology and geometry of manifolds, P is the Laplace operator Δ_M on a compact manifold M, and the associated zeta function $\zeta(s, \Delta_M) = \sum_{n=1}^{\infty} \lambda_n^{-s}$ detects much information on the geometry and the topology of the space [26]. When P is self-adjoint, positive definite and elliptic, there exists an asymptotic expansion for the eigenvalues $\lambda_n = Kn^{\epsilon} + o(n^{\epsilon})$, for large n, with some $K, \epsilon > 0$, and the function $\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n t}$ is the trace of the heat operator and $\det P = \mathrm{e}^{-\zeta'(s,P)}$ is the *functional determinant* of P [2]. A full asymptotic expansion [20, 24] for the trace of the heat kernel

$$\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n t} \sim t^{-m/2} \sum_{j=0}^{\infty} e_j t^{j/2}$$

can be obtained by using classical methods in pseudodifferential operator theory (see, for example, [14]). As a consequence, the analytic continuation of the associated zeta function is regular at $s = 0$ with $\zeta(0, \Delta_M) = e_m - \dim\ker \Delta_M$, and has possible poles at $s = \frac{1}{2}(m - j)$, $j = 0, 1, 2, \ldots$, which are non-negative integers, with residues

$$\mathrm{res}_1\left(\zeta(s, \Delta_M),\ s = \frac{m-j}{2}\right) = \frac{e_j}{\Gamma(\frac{1}{2}(m-j))}.$$

This can be obtained using the Mellin transform and the expansion for the trace of the heat kernel. In particular, all the coefficients of odd index vanish if the manifold has no boundary [6]. The growth estimate for the eigenvalues can be deduced applying Ikehara's Tauberian theorem [36] from the residues at the first pole of the zeta function. Furthermore, in the case of the Laplace operator Δ_M on a manifold of dimension m, a better estimate for the growth of the eigenvalues is available. That is to say, we have the Weyl formula [17,22,27] (where the coefficients are known)

$$\lambda_n = K_0 n^{2/m} + O(n^{1/m}),$$

or even

$$\lambda_n = K_0 n^{2/m} + K_1 n^{1/m} + o(n^{1/m}),$$

if some suitable geometric conditions are satisfied. The above estimates also hold if a constant (potential) term is added to the Laplacian. This suggests that we

[2]When M has a boundary, some suitable boundary conditions must be imposed [14, § 1.11].

start the analysis from the properties of the sequence itself, with the final aim of obtaining properties and relationships among the zeta invariants directly from some information about the sequence. This is our point of view here. Our main result is to show that the main zeta invariants are determined by knowledge of the asymptotic expansion of a spectral function associated with S (proposition 2.6). We can also introduce a sufficient condition on the sequence to ensure the existence of such an expansion (lemma 2.5). To obtain these results we need to restrict the class of allowed sequences. To do this we require two further conditions. First, we need to introduce three spectral functions associated with S.

Given a sequence $S = \{a_n\}_{n=1}^{\infty}$ with finite exponent α, the *zeta function* associated with S is the function of the complex variable s defined by the series

$$\zeta(s, S) = \sum_{n=1}^{\infty} a_n^{-s},$$

when $\mathrm{Re}(s) > \alpha$, and by analytic continuation elsewhere; if $\mathrm{Re}(a_n)$ is bounded below, the *heat function* associated with S is the function of the real positive variable t defined by the series

$$f(t, S) = \sum_{n=0}^{\infty} \mathrm{e}^{-a_n t},$$

with $a_0 = 0$. For a sequence S as above, there exists a least integer p such that the series $\sum_{n=1}^{\infty} a_n^{-1-p}$ is convergent. If α is not an integer, p is the greatest integer less than α; if α is an integer, p may be either α or $\alpha - 1$. In any cases $\alpha - 1 \leqslant p \leqslant \alpha$. The Weierstrass canonical product,

$$F(z, S) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right) \exp\left(\sum_{j=1}^{p} \frac{(-1)^j}{j} \frac{z^j}{a_n^j}\right),$$

converges uniformly and absolutely in any bounded closed region of the plane, and $F(z, S)$ is an integral function of finite order[3] $\rho = \alpha$ (this is the first Borel theorem), which vanishes if and only if $z = -a_n$ for some n. The integer p is called the *genus* of the canonical product, and F the *Fredholm determinant* associated with S.

We can now introduce the definition of the type of sequence we will work with.

DEFINITION 2.1. First, we use the notation $\Sigma_{\theta,c}$ to denote the closed sector of the positive complex plane of angle θ translated by the real constant c, namely

$$\Sigma_{\theta,c} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - c)| \leqslant \tfrac{1}{2}\theta\}.$$

We call a sequence $S = \{a_n\}_{n=1}^{\infty}$ of finite exponent α a *sequence of spectral type* if the following two properties hold:

(i) the points a_n are all contained in a translated closed sector $\Sigma_{\theta,c}$ of angle $\theta < \pi$ and shift $c > 0$ of the complex plane;

[3]Recall that the *order* of an integral function of finite order f is

$$\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r},$$

where $M(r)$ is the maximum modulus of $f(z)$ on the circle $|z| = r$.

(ii) the logarithm of the Fredholm determinant $\ln F(z,S)$ associated with S has an asymptotic expansion in terms of powers and logarithms for large z along any radius in the complex plane disjoint from $-\Sigma_{\theta,c}$ at least up to the constant term, namely

$$\ln F(z,S) = \sum_{k=0}^{K}\sum_{j=0}^{J} a_{j,k} z^{\alpha_j} \ln^k z + o(1),$$

where the α_j are real and $\alpha_0 > \alpha_1 > \cdots > \alpha_J \geqslant 0$.

The choice of characterizing a sequence of spectral type by a property of one of the associated spectral functions is suggested by the main application, i.e. the Laplacian over compact manifolds, where this property is always satisfied as we will show in corollary 2.4. First, we show that the existence of an asymptotic expansion for the Fredholm determinant and for the heat function imply each other. For, consider the asymptotic sequences of functions (see [20] for terminology) $f_{\delta,j}(t) = t^{-\delta}\log^j t$ and $g_{\alpha,k}(z) = z^\alpha \log^k z$, where $\{\delta\}$ and $\{\alpha\}$ are any two sets of complex numbers with $\mathrm{Re}(\delta), \mathrm{Re}(\alpha) \to -\infty$, and $j/k = 0, 1, \ldots, J_\delta/K_\alpha \in \mathbb{N}$, for each δ and α, respectively (see also [7, p. 372]). We can then prove the following lemma.

Lemma 2.2. *Let $S = \{a_n\}_{n=1}^\infty$ be a sequence with finite exponent and suppose the points a_n are all contained in a sector $\Sigma_{\theta,c}$, $\theta < \pi$, $c > 0$. Then, the heat function associated with S has a full*[4] *asymptotic expansion with respect to the asymptotic sequence $\{f_{\delta,j}(t) = t^{-\delta}\log^j t\}$ as $t \to 0^+$ if and only if the logarithm of the Fredholm determinant associated with S has a full asymptotic expansion with respect to the asymptotic sequence $\{g_{\alpha,k}(z) = z^\alpha \log^k z\}$ as $z \to \infty$ along any radius disjoint from $-\Sigma_{\theta,c}$.*

Remark 2.3 (notation). We will often use the complex variable $-\lambda = z$, defined by cutting the complex plane along the real positive axis and setting $\arg(-\lambda) = 0$ on the line $(-\infty, 0]$, as usual, to deal with contour integrals of Hankel type (see, for example, [37] for more details).

Proof. First, suppose the heat function has a full expansion. By classical asymptotic analysis [20, 24], in order to prove that $F(z,S)$ has an expansion it is sufficient to prove that one of its derivatives has such an expansion. Thus, we introduce the function

$$R(\lambda,S) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\ln F(-\lambda,S) = \sum_{n=1}^{\infty}\left[\frac{1}{\lambda - a_n} + \sum_{j=0}^{p-1}\frac{\lambda^j}{a_n^{j+1}}\right],$$

which we call the resolvent function associated with S, together with its derivatives

$$R^{(k)}(\lambda,S) = \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}R(\lambda,S).$$

In particular, note that

$$R^{(p)}(\lambda,S) = \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}R(\lambda,S) = -p!\zeta(p+1, S-\lambda) = -p!\sum_{n=1}^{\infty}(a_n - \lambda)^{-p-1},$$

[4] The equivalence of expansions of finite order can also be shown by a similar proof.

since the last series is uniformly convergent and, by applying the Mellin transform,

$$R^{(p)}(\lambda, S) = -\int_0^\infty t^p e^{\lambda t}(f(t,S) - 1)\,dt.$$

Thus, we have expressed $R^{(p)}(\lambda, S)$ as the Laplace transform of the heat function and we use the Watson lemma [24] to get the expansion for $R^{(p)}$ from that of f.

For the other implication, suppose that $\ln F(z, S)$ has a full expansion. Let $\Lambda_{\theta,c} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - c)| = \frac{1}{2}\theta\}$ be the boundary of the sector $\Sigma_{\theta,c}$ appearing in the definition of the sequence (where $c < |a_1|$). Then, we can write

$$f(t,S) - 1 = \frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} e^{-\lambda t} R(\lambda, S)\,d\lambda.$$

In fact, both the series are uniformly and absolutely convergent, and

$$\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} e^{-\lambda t} R(\lambda, S)\,d\lambda = \sum_{n=1}^\infty e^{-a_n t},$$

since the integrals of all the non-negative integer powers of $(-\lambda)$ vanish, as may be seen immediately by modifying the contour to the usual Hankel contour and using the Hankel formula for the Gamma function. Next, we integrate by parts to get

$$f(t,S) - 1 = \frac{t}{2\pi i}\int_{\Lambda_{\theta,c}} e^{-\lambda t} \ln F(-\lambda, S)\,d\lambda,$$

and we can use the expansion of the logarithm of the Fredholm determinant in the above integral when t is small, changing the variable in λ/t. In fact, by modifying the contour in a suitable way, we will get integrals of Hankel type in each term. □

COROLLARY 2.4. *The sequence of the eigenvalues of the Laplace operator with a regular potential on a compact connected manifold is a sequence of spectral type.*

Next, we give a sufficient condition for a sequence to be of spectral type. This is quite a well-known result, but we present a direct simple proof for completeness.

LEMMA 2.5. *Let $S = \{a_n\}_{n=1}^\infty$ be a sequence of finite exponent α, and suppose the following expansion for the general term of S holds for large n:*

$$a_n = \sum_{j=0}^J K_j n^{\beta_j} + o\left(\frac{1}{n}\right),$$

with real K_j and β_j, and $1/\alpha = \beta_0 > \beta_1 > \cdots > \beta_J \geqslant 0$. Then, S is of spectral type.

Proof. It is clear that the points of S are all contained in the real positive axis, up to a finite number of cases. Then, by a similar argument to that in lemma 2.2, we just have to prove that the associated heat function has an asymptotic expansion for small t. We do not lose generality by assuming $J = K_0 = K_1 = 1$. Then

$$\sum_{n=1}^\infty e^{-(n^{\beta_0}+n^{\beta_1})t}[e^{-o(n^{-1})t} - 1]$$

is uniformly convergent to 0 for $t \to 0^+$. Therefore, to prove that the heat function has an asymptotic expansion for small t, it is sufficient to prove that the function

$$f_0(t) = \sum_{n=0}^{\infty} \mathrm{e}^{-(n^{\beta_0}+n^{\beta_1})t},$$

has such an expansion. To do that, we apply the Plana theorem [37] to the sum $f_0(t) = \sum_{n=0}^{\infty} \phi(n,t)$, where $\phi(z,t) = \mathrm{e}^{-(z^{\beta_0}+z^{\beta_1})t}$ is a function of the complex variable z and of the real non-negative variable t. To give a precise definition to the complex powers, we cut the complex plane along the negative axis, and set $z^{\beta} = \mathrm{e}^{\beta \ln x + \mathrm{i}\beta\theta} = x^{\beta}\mathrm{e}^{\mathrm{i}\beta\theta}$. If $-\frac{1}{2}\pi \leqslant \beta_0\theta \leqslant \frac{1}{2}\pi$, ϕ is analytic and bounded uniformly for all $t \geqslant 0$. Let first assume $\beta_0 \leqslant 1$, then $(\beta_0) = \beta_0$ and $[\beta_0] = 0$, ϕ is analytic and bounded in the strip $0 \leqslant \mathrm{Re}(z) \leqslant M$, for all $M > 0$, and the Plana theorem gives

$$\sum_{n=0}^{M} \phi(n,t) = \tfrac{1}{2}[\phi(0,t) - \phi(M,t)] + \int_0^M \phi(x,t)\,\mathrm{d}x \\ + \mathrm{i}\int_0^{\infty} \frac{\phi(\mathrm{i}y,t) - \phi(-\mathrm{i}y,t) + \phi(M+\mathrm{i}y,t) - \phi(M-\mathrm{i}y,t)}{\mathrm{e}^{2\pi y}-1}\,\mathrm{d}y, \quad (2.1)$$

where the last integral in (2.1) is uniformly convergent. The limit for $M \to +\infty$ gives

$$f_0(t) = \frac{1}{2} + \int_0^{\infty} \mathrm{e}^{-(x^{\beta_0}+x^{\beta_1})t}\,\mathrm{d}x \\ + 2\int_0^{\infty} \frac{\mathrm{e}^{-(y^{\beta_0}\cos\frac{1}{2}\pi\beta_0 + y^{\beta_1}\cos\frac{1}{2}\pi\beta_1)t}\sin[(y^{\beta_0}\sin\frac{1}{2}\pi\beta_0 + y^{\beta_1}\sin\frac{1}{2}\pi\beta_1)t]}{\mathrm{e}^{2\pi y}-1}\,\mathrm{d}y. \quad (2.2)$$

The expansion of the integral in the middle term of (2.2) can be calculated by expanding one of the exponentials and making the substitution $u = x^{\beta_0}$:

$$\int_0^{\infty} \mathrm{e}^{-x^{\beta_0}t}\mathrm{e}^{-x^{\beta_1}t}\,\mathrm{d}x = \int_0^{\infty} \mathrm{e}^{-x_0^{\beta}t} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{\beta_1 k} t^k\,\mathrm{d}x \\ = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k \int_0^{\infty} \mathrm{e}^{-ut} u^{k(\beta_1/\beta_0)+(1/\beta_0)-1}\,\mathrm{d}u \\ \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma\left(\frac{\beta_1}{\beta_0}k + \frac{1}{\beta_0}\right) t^{k-k(\beta_1/\beta_0)-(1/\beta_0)}.$$

In the integral in the last term of (2.2) we can expand the factors, due to uniform convergence, and after rearranging the terms in the product of the series we get the thesis when $\beta \leqslant 1$. When $\beta > 1$, $\beta = [\beta] + (\beta)$ and $z^{\beta} = x^{\beta}\mathrm{e}^{\mathrm{i}(\beta)\theta}$. □

Note that the proof of the existence of the asymptotic expansion given for lemma 2.5 also provides a method of determining the coefficients in the expansion when the terms in the expansion of the sequence are all known. An application of this method is given in [31]. Notice also that, by applying lemma 2.5 to the first term in the growth of the eigenvalues, we obtain the first term in the expansion of the heat function explicitly.

2.2. Zeta invariants for homogeneous sequences

Next, we generalize a technique introduced in [32] and we give an effective method of determining the main zeta invariants for a sequence of spectral type when we have some knowledge of the asymptotic expansion of the associated Fredholm determinant. To deal with the applications we have in mind, we need only a particular kind of spectral-type sequence, namely the regular sequences, as defined below.

A sequence of spectral type S is called *regular* if the coefficients $a_{j,k}$ in the expansion of the logarithm of the associated Fredholm determinant $\ln F(z,S)$ vanish for all $k \neq 0,1$. A result similar to proposition 2.6 for a larger class of sequences of spectral type can be proved by precisely the same means, but is more complicated. In particular, note that, for regular sequences of spectral type, the point $s=0$ is a regular point for the associated zeta function $\zeta(s,S)$. In the more general situation, a generalization of proposition 2.6 does not state that $s=0$ is a regular point, but rather gives some coefficients of the Laurant expansion of the zeta function $\zeta(s,S)$ near $s=0$. In fact, possible terms of the type $\ln^{k+1}(-\lambda)$, with $k>0$ in the asymptotic expansion of $\ln F(\lambda,S)$, should imply a pole of order k at $s=0$ for the zeta function. In particular, such terms appear when studying the spectrum of the Laplace operator on manifolds with singularities of conical type [7, 12, 32].

PROPOSITION 2.6. *If S is a regular sequence of spectral type, then the associated zeta function $\zeta(s,S)$ is regular at $s=0$ and*

$$\operatorname{res}_0(\zeta(s,S), s=0) = F_1, \qquad \operatorname{res}_0(\zeta'(s,S), s=0) = F_0,$$

where F_0 and F_1 are the coefficients of the constant term and of the logarithmic term, respectively, in the asymptotic expansion of the logarithm of the Fredholm determinant associated with S, namely, for large z:

$$\ln F(z,S) \sim \cdots + F_1 \ln z + F_0 + \cdots .$$

Moreover, $\zeta(s,S)$ is regular in the positive complex half-plane $\mathrm{Re}(s) > -\epsilon$, up to a finite set of poles. The poles are at most $J+1$, are located at $s = \alpha_J, \alpha_{J-1}, \dots, \alpha_0 = \alpha$, and are of order at most 2. The residues can be calculated explicitly from the coefficients $a_{j,k}$ in the asymptotic expansion of $\ln F(z,S)$.

Proof. Using the Mellin transform, we obtain the following analytic representation of the zeta function:

$$\zeta(s,S) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}[f(t,S)-1]\,\mathrm{d}t = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{2\pi \mathrm{i}} \int_{\Lambda_{\theta,c}} \mathrm{e}^{-\lambda t} R(\lambda,S)\,\mathrm{d}\lambda\,\mathrm{d}t,$$

where the contour and the resolvent function $R(\lambda,S)$ were introduced in the proof of lemma 2.2. The behaviour of $R(\lambda,S)$ for large λ along some radius in the complex plane disjoint from $\Sigma_{\theta,c}$ is at most $o((-\lambda)^p)$. In fact, taking the pth derivative or $R(\lambda,S)$ with respect to λ, we get a uniformly convergent series of functions of λ (outside $\Sigma_{\theta,c}$), vanishing for large λ. Next, observe that

$$R(\lambda,S) = -\frac{\mathrm{d}}{\mathrm{d}\lambda} T(\lambda,S),$$

where

$$T(\lambda, S) = -\ln \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{a_n}\right) \exp\left(\sum_{j=1}^{p} \frac{(-1)^j}{j} \frac{(-\lambda)^j}{(a_n)^j}\right) + T_0.$$

Actually, since we have seen that the possible presence of any further term that is a polynomial in $-\lambda$ in $T(\lambda, S)$ does not affect the analytic representation of the zeta function, we can take the more general possible form for $T(\lambda, S)$:

$$T(\lambda, S) = T(\lambda, S) + P_k(\lambda),$$

where $P_k(\lambda)$ is a polynomial of degree k in $-\lambda$ with $P_k(0) = T_0$. The introduction of this more general form for T is due to the fact that, in applications, we often find that T can be expressed in terms of some known special functions. Since $\mathrm{e}^{-T(\lambda,S)}$ is an integral function of $-\lambda$ of finite order α with zeros at a_n, which does not vanish at the origin, by the Hadamard factorization theorem we have

$$\mathrm{e}^{-T(\lambda,S)} = \mathrm{e}^{Q_p(-\lambda)} F(-\lambda, S),$$

where Q_p is a polynomial of degree p. Also, due to the known behaviour of $R(\lambda, S)$, we have, for large λ outside $\Sigma_{\theta,c}$, $T(\lambda, S) = o((-\lambda)^{p+1})$. In particular, $T(\lambda, S)$ has an asymptotic expansion, since S is of spectral type, and we let

$$\begin{aligned} T(\lambda, S) &= -\ln F(-\lambda, S) - Q_p(\lambda) \\ &= \sum_{k=0,1} \sum_{j=0}^{J} a_{j,k} (-\lambda)^{\alpha_j} \ln^k(-\lambda) + o(1) + Q_p(\lambda) \\ &= \cdots + B + A \ln(-\lambda) + \cdots . \end{aligned}$$

The introduction of the function $T(\lambda, S)$ allows us to integrate by parts first in the λ integrals and then in the t integrals, in the analytic representation of the zeta function

$$\zeta(s, S) = \frac{s}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{1}{2\pi \mathrm{i}} \int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} T(\lambda, S)\, \mathrm{d}\lambda\, \mathrm{d}t.$$

The presence of a factor with a zero of second order at $s = 0$, allows us to use the standard technique (see, for example, [14]) employed to calculate the residues to also treat the derivative at $s = 0$. In fact, splitting the t integral at $t = 1$, the integral $\int_1^{\infty}$ represents a regular function of s near $s = 0$, and, hence, makes no contribution to the derivative. We can write

$$\zeta(s, S) = \frac{s}{\Gamma(s)} \int_0^{1} t^{s-1} \frac{1}{2\pi \mathrm{i}} \int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} T(\lambda, S)\, \mathrm{d}\lambda\, \mathrm{d}t + O(s^2).$$

Concerning the integral $\int_0^1$, we can change the variable λ to λ/t, and use the expansion of $T(\lambda, S)$ for large λ, provided that we can modify the contour without crossing the pole at $\lambda = 0$. A way to do that is to split the contour as $\Lambda_{\theta,c} = \Lambda_{\theta,-c} - C_c$, where C_c is a circle of radius c and centre at the origin. Thus,

$$\zeta(s, S) = \frac{1}{\Gamma(s)} \left[\cdots - \frac{A}{s} + \gamma A - B + T(0, S) + \cdots\right] + O(s^2),$$

$$\mathrm{res}_0(\zeta(s, S), s = 0) = -A, \qquad \mathrm{res}_0(\zeta'(s, S), s = 0) = -B + T(0, S),$$

and it is clear that $-A = F_1$ and $-B + T(0, S) = F_0$. The same method can be applied to prove the second part of the proposition, using the complete expansion given above and recalling that

$$\int_0^1 t^x \ln t \, \mathrm{d}t = -\frac{1}{(x+1)^2}.$$

□

Proposition 2.6 suggests calling a regular sequence of spectral type *simply regular* if the associated zeta function has only simple poles. The conditions for this follow.

LEMMA 2.7. *A regular sequence S of spectral type with genus p is simply regular if one of the following (non-equivalent) conditions is satisfied:*

(i) *the unique logarithmic terms appearing in the expansion of $\ln F(-\lambda, S)$ are of the form $(-\lambda)^k \ln(-\lambda)$, with integer $k \leqslant p$;*

(ii) *there are no logarithmic terms in the expansion of the heat function $f(t, S)$.*

2.3. Non-homogeneous sequences

In this section we analyse the problem of the contribution to the zeta invariants of the addition of a constant term to the general term of the sequence S. Namely, we consider a shift of the sequence S to the sequence $S_a = \{a_n + a\}_{n=1}^{\infty}$, with real $a \geqslant 0$, and $S_0 = S$, and, consequently, the non-homogeneous zeta function

$$\zeta(s, S_a) = \sum_{n=1}^{\infty} (a_n + a)^{-s}.$$

We assume the sequences are all regular and simply regular. It emerges that a general form of the Lerch formula still holds (see also [35]), and this can be proved as follows. First, expand the power of the binomial in the definition of the zeta function

$$\zeta(s, S_a) = \sum_{j=0}^{\infty} \binom{-s}{j} \zeta(s + j, S_0) a^j.$$

Now, if we write the first p terms explicitly, we obtain

$$\zeta(s, S_a) = \sum_{j=0}^{p} \binom{-s}{j} \zeta(s + j, S_0) a^j + \sum_{j=p+1}^{\infty} \binom{-s}{j} \zeta(s + j, S_0) a^j.$$

We can calculate the s-derivative of the first term at $s = 0$ using the expansion

$$\binom{-s}{j} \zeta(s + j, S_0) = \frac{(-1)^j}{j} \{R_1(j) + [R_0(j) + (\gamma + \psi(j)) R_1(j)] s\} + O(s^2),$$

where $R_k(j) = \mathrm{res}_k(\zeta(s, S_0), s = j)$, for $j > 0$. For the second term, note that

$$\frac{\mathrm{d}}{\mathrm{d}s} \binom{-s}{j} \bigg|_{s=0} = \frac{(-1)^j}{j},$$

and, hence, taking the s-derivative at $s = 0$, we get

$$\sum_{j=p+1}^{\infty} \frac{(-1)^j}{j} \zeta(j, S_0) a^j = -\ln \prod_{n=1}^{\infty} \left(1 + \frac{a}{a_n}\right) \exp\left\{ \sum_{j=1}^{p} \frac{(-1)^j}{j} \frac{a^j}{a_n^j} \right\}.$$

We have then proved the following facts.

Lemma 2.8. *For* $|a| < |a_1|$,

$$\ln F(a, S_0) = \sum_{j=p+1}^{\infty} \frac{(-1)^{j+1}}{j} \zeta(j, S_0) a^j.$$

Proposition 2.9 (Lerch formula). *For all* a,

$$\zeta(0, S_a) = \zeta(0, S_0) + \sum_{j=1}^{p} \frac{(-1)^j}{j} R_1(j) a^j,$$

$$\zeta'(0, S_a) = \zeta'(0, S_0) + \sum_{j=1}^{p} \frac{(-1)^j}{j} [R_0(j) + (\gamma + \psi(j)) R_1(j)] a^j - \ln F(a, S_0).$$

This gives a relation between the functional determinant and the Fredholm determinant [29,35]. Notice that, when considering the zeta function associated with the Laplace operator Δ_M, the residues at the poles are expressible in terms of the coefficients of the heat kernel expansion, namely

$$R_1(j) = \frac{e_{m-2j}}{\Gamma(j)}.$$

We will apply the formula in proposition 2.9 to some interesting situations in the next section. As a first example, we show here how it can be used to prove the following factorization lemma of Choi and Quine [13].

Corollary 2.10. *Let* $S = \{a_n\}_{n=1}^{\infty}$ *be a simply regular sequence of spectral type with finite exponent* α *and genus* p. *Let* $L = \{a_n^2\}_{n=1}^{\infty}$, *and* a *be any real number. Then*

$$\begin{aligned} \zeta(0, L_{a^2}) &= \tfrac{1}{2}[\zeta(0, S_{\mathrm{i}a}) + \zeta(0, S_{-\mathrm{i}a})], \\ \zeta'(0, L_{a^2}) &= \zeta'(0, S_{\mathrm{i}a}) + \zeta'(0, S_{-\mathrm{i}a}) \\ &\quad - \sum_{j=1}^{[p/2]} \frac{(-1)^j}{j} \sum_{k=1}^{j} \frac{1}{2k-1} \operatorname{res}_1(\zeta(s, S_0), s = 2j) a^{2j}. \end{aligned}$$

Proof. It is clear that L_a has exponent $\frac{1}{2}\alpha$ and genus $[\frac{1}{2}p]$. Since $\zeta(s, L_0) = \zeta(2s, S_0)$,

$$\operatorname{res}_k(\zeta(s, L_0), s = j) = 2^{-k} \operatorname{res}_k(\zeta(s, S_0), s = 2j),$$

and the thesis follows now from proposition 2.9, using the relation

$$\psi(2k) = \tfrac{1}{2}\psi(k) - \tfrac{1}{2}\gamma + \sum_{j=1}^{k} \frac{1}{2j-1}.$$

□

3. Applications and examples

In this section, we apply the devices provided in § 2 to different examples. In each case, we put in evidence the relative Lerch formula, equations (3.1)–(3.6). In particular, in § 3.1 we review the classical case of the circle, while in § 3.3 we deal with the 2-sphere.

3.1. The zeta function on the circle

The aim of this section is to show how the method outlined in § 2 works on the simplest case: this is the one-dimensional case with $S = \{a_n = n\}$ and $L = \{a_n = n^2\}$, the former corresponding to the positive part of the operator $\mathrm{i}(\mathrm{d}/\mathrm{d}x)$ and the latter relating to the Laplace operator on the circle. In the first case, $\alpha = p = 1$, the zeta function has a unique simple pole at $s = 1$ with residue 1, and regular part γ,

$$f(t, S) - 1 = \sum_{n=1}^{\infty} \mathrm{e}^{-nt} = \frac{1}{\mathrm{e}^t - 1} = \frac{1}{t} + \sum_{j=1}^{\infty} \frac{1}{j!} B_j t^{j-1},$$

where the B_j are the Bernoulli numbers (see [15, 9.71, $B_2 = \frac{1}{6}$]),

$$R(\lambda, S) = \gamma + \psi(-\lambda + 1),$$

and a possible choice for $T(\lambda, S) = -\ln F(-\lambda, S)$ is $\gamma(-\lambda) + \ln \Gamma(-\lambda + 1)$, so

$$\begin{aligned} T(\lambda, S) &= -\ln F(-\lambda, S) \\ &= -\ln \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{n}\right) \mathrm{e}^{-(-\lambda)/n} \\ &= (-\lambda)\ln(-\lambda) + (\gamma - 1)(-\lambda) + \tfrac{1}{2}\ln(-\lambda) + \tfrac{1}{2}\ln 2\pi \\ &\quad + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)} (-\lambda)^{1-2j}, \end{aligned}$$

and $T(0, S) = 0$. Applying proposition 2.6, we obtain

$$\zeta(0, S) = -\tfrac{1}{2}, \qquad \zeta'(0, S) = -\tfrac{1}{2}\ln 2\pi.$$

In the second case, $\alpha = \frac{1}{2}$, $p = 0$, the zeta function has a unique simple pole at $s = \frac{1}{2}$ with residue $\frac{1}{2}$, and regular part γ,

$$f(t, L) - 1 = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t} = \frac{1}{2}\sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\pi^2 n^2/t} - \frac{1}{2} = \frac{1}{2}\sqrt{\frac{\pi}{t}} - \tfrac{1}{2} + O(\mathrm{e}^{-1/t})$$

by the Poisson summation formula (the Fourier expansion for the theta function),

$$\begin{aligned} R(\lambda, L) &= \frac{1}{2(-\lambda)} - \frac{\pi}{2} \frac{\coth \pi\sqrt{-\lambda}}{\sqrt{-\lambda}}, \\ T(\lambda, L) &= -\ln F(-\lambda, L) = -\ln \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{n^2}\right) = -\ln \frac{\sinh \pi\sqrt{-\lambda}}{\pi\sqrt{-\lambda}} \\ &= -\pi\sqrt{-\lambda} + \tfrac{1}{2}\ln(-\lambda) + \ln 2\pi + O(\mathrm{e}^{-2\pi\sqrt{-\lambda}}), \end{aligned}$$

and $T(0,L)=0$. Applying proposition 2.6,

$$\zeta_{S^1}(0,0)=2\zeta(0,L)=-1,\qquad \zeta'_{S^1}(0,0)=2\zeta'(0,L_0)=-2\ln 2\pi,$$

where we introduce the notation $\zeta_{S^n}(s,q)$ for the zeta function associated with the non-homogeneous Laplace operator on S^n (thus, $\zeta_{S^1}(s,a^2)=2\zeta(s,L_{a^2})$). Applying proposition 2.9, we can also deal with the non-homogeneous associated problems, namely the zeta functions

$$\zeta(s,S_a)=\sum_{n=1}^{\infty}(n+a)^{-s},\qquad \zeta(s,L_{a^2})=\sum_{n=1}^{\infty}(n^2+a^2)^{-s},$$

to obtain

$$\zeta(0,S_a)=-\tfrac{1}{2}-a,\qquad \zeta(0,L_{a^2})=-\tfrac{1}{2},$$
$$\zeta'(0,S_a)=\zeta'(0,S_0)-\gamma a-\ln F(a,S)=\zeta'(0,S_0)+\ln\Gamma(a+1),$$

where

$$\zeta'(0,S_a)=\zeta'(0,S_0)+\ln\Gamma(a+1) \tag{3.1}$$

is the classical Lerch formula, and

$$\zeta'(0,L_{a^2})=\zeta'(0,L_0)-\ln F(a^2,S)=-\ln 2\pi-\ln\frac{\sinh\pi a}{\pi a},$$

where

$$\zeta'_{S^1}(0,a^2)=\zeta'_{S^1}(0,0)-2\ln\frac{\sinh\pi a}{\pi a} \tag{3.2}$$

is a first generalization of such a Lerch formula. In particular, note also that

$$\zeta'(0,L_{a^2})=\zeta'(0,S_{\mathrm{i}a})+\zeta'(0,S_{-\mathrm{i}a}),$$

as expected from corollary 2.10.

3.2. Multiple zeta and Gamma functions

Multiple Gamma functions, and in particular the double Gamma or G function, were introduced long time ago by Barnes [4] as natural generalizations of the Euler Gamma function, and multiple zeta functions appeared in the same context as generalizations of the Riemann zeta function. Consequently, different approaches and various generalizations appeared with different aims and applications: in [33], multiple Gamma functions are introduced in order to study the functional determinant on the spheres, while a generalization of the Barnes G function [3] is considered in [29]. We study the two classes of zeta function associated with these two classes of special function, and we show how they are related in a very natural algebraic way. We introduce a slightly different definition for the multiple Gamma function, which will turn out to be a more natural generalization of the Euler function.

Let $n=(n_0,\dots,n_m)$ be a positive integer vector in $\mathbb{N}_0^{m+1}$, where $m=0,1,\dots$ is fixed, and a is a non-negative real constant. We prefer to avoid the zero vectors in the definition, since this allows more compact formulae in the results. We then consider the sequence $S_{1,a}$, where the general term $n+a$, with $n \in \mathbb{N}_0$, appears n^m times and $S_{2,a} = \{n_0 + \cdots + n_m + a\}_{n\in\mathbb{N}_0^{m+1}}$. The associated multiple zeta functions are

$$z_m(s,a) = \zeta(s, S_{1,a}) = \sum_{n=1}^{\infty} n^m (n+a)^{-s},$$
$$\zeta_m(s,a) = \zeta(s, S_{2,a}) = \sum_{n\in\mathbb{N}_0^{m+1}} (n_0 + n_1 + \cdots + n_m + a)^{-s},$$

for $\mathrm{Re}(s) > m+1$, while the associated special functions are[5]

$$\begin{aligned} G_m(z+1) = \exp\Bigg\{ &\frac{(-1)^m}{m+1}(2\gamma + \psi(m+1))z^{m+1} \\ &- \sum_{j=1}^{m} \frac{(-1)^j}{j}\zeta_{\mathrm{R}}(j-m)z^j + \sum_{j=1}^{m}(-1)^j\binom{m}{j}\zeta_{\mathrm{R}}'(j-m)z^j \Bigg\} \\ &\times \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{n^m} \exp\left\{ n^m \sum_{j=1}^{m+1} \frac{(-1)^j}{j}\frac{z^j}{n^j}\right\}, \end{aligned}$$

$$\begin{aligned} \frac{1}{\Gamma_m(z+1)} = \exp\Bigg\{ &\frac{(-1)^m}{m!}\sum_{j=0}^{m}\Bigg[(2\gamma + \psi(j+1))s_{m+1,j+1} \\ &+ \sum_{k=0,k\neq j}^{m} (-1)^{k+j} s_{m+1,k+1}\zeta_{\mathrm{R}}(j+1-k)\Bigg]\frac{z^{j+1}}{j+1}\Bigg\} \\ &\times \prod_{n\in\mathbb{N}_0^{m+1}}\left(1+\frac{z}{n_0+\cdots+n_m}\right)\exp\left\{\sum_{j=1}^{m+1}\frac{(-1)^j}{j}\frac{z^j}{(n_0+\cdots+n_m)^j}\right\}, \end{aligned}$$

where the constants $s_{j,k}$ are the Stirling numbers; their appearance will be clarified in the following. Some remarks on these definitions are in order. The definition of the function G_m is that given by Shuster [29], and is in fact the natural generalization of the definition given by Barnes [3] for the G function. Actually, it is the most natural generalization of the G function for at least the following reasons: it reduces to the Barnes G function for $m = 1$; it is normalized by $G_m(1) = 1$; it satisfies a functional equation similar to the one characterizing the G function. The definition of the function Γ_m is essentially that given by Barnes and considered by Vardi [33]. However, here we prefer to stress the relation with the multi-linear zeta function ζ_m, and therefore we choose normalization in order to give a nicer Lerch formula. Our definition also allows us to introduce further multiple special functions related to the quadratic version of the multiple zeta functions z_m and ζ_m, as in corollary 2.10. We will give the main properties of all these new special functions at the end of this section.

We first introduce a natural algebraic relationship between the two sets of multiple zeta functions, which shows their common nature. Recall that the rising factorial [30]

$$R(x,k) = x(x+1)(x+2)\cdots(x+k-1)$$

[5]Note that $G_0(z+1) = 1/\Gamma(z+1)$.

is a polynomial in x of degree $k > 0$:

$$R(x,k) = \prod_{j=0}^{k-1}(x+j) = \sum_{j=0}^{k}(-1)^{k+j}s_{k,j}x^j,$$

whose coefficients $s_{k,j}$ are the Stirling numbers ($s_{k,0} = 0$, for all $k > 0$). Now

$$\zeta_m(s,a) = \sum_{n=1}^{\infty}\binom{n+m}{m}(n+a)^{-s}$$

and

$$\begin{aligned}\binom{x+m}{m} &= \frac{1}{m!}\prod_{j=0}^{m}(x+j) = \frac{1}{m!}\frac{1}{x}R(x,m+1)\\ &= \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m+j}s_{m+1,j+1}x^j = \frac{1}{m!}B_m(x),\end{aligned}$$

with the last equality defining the polynomial $B_m(x)$ of degree m. In particular,

$$B_0(x) = 1, \qquad B_1(x) = x+1, \qquad B_2(x) = x^2+3x+2,$$

and all the $B_m(x)$ have positive integer coefficients. It is also easy to see that the set $\{B_0(x),\dots,B_m(x)\}$ forms a basis for the algebra of the polynomial of maximum degree m, $\mathcal{P}_m(x)$, since the matrix A_m passing to the standard basis $\{1,x,x^2,\dots,x^m\}$ is upper triangular, with all '1's on the diagonal. Thus, $\det A_m = 1$ and, hence, actually, $A_m \in Sl_m(\mathbb{Z})$. This gives the following relation for our zeta functions. Introducing the variables[6] $z_{s,a} = z_0(s,a)$, $z'_{s,a} = z'_0(s,a)$, $\zeta_{s,a} = \zeta_0(s,a)$ and $\zeta'_{s,a} = \zeta'_0(s,a)$, and the formal powers $z^m_{s,a} = z_m(s,a)$, and similarly for the other variables, we can write

$$m!\zeta^m_{s,a} = B_m(z_{s,a}), \qquad m!(\zeta')^m_{s,a} = B_m(z'_{s,a}),$$

and these formulae can be inverted in $Sl_m(\mathbb{Z})$. For example,

$$\begin{aligned}\zeta^m_{s,0} = \zeta_m(s,0) &= \frac{1}{m!}B_m(z_{s,0})\\ &= \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m+j}s_{m+1,j+1}z^j_{s,0}\\ &= \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m+j}s_{m+1,j+1}z_j(s,0).\end{aligned}$$

We now study the zeta functions z_m using proposition 2.6. For $S_{1,a}$, $\alpha = p = m+1$, $z_m(s,0) = \zeta_{\mathrm{R}}(s-m)$ has a unique simple pole at $s = m+1$ with $R_0 = \gamma$ and $R_1 = 1$. The associated Fredholm determinant is given by

$$F(z,S_{1,0}) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{n^m}\exp\left\{n^m\sum_{j=1}^{m+1}\frac{(-1)^j}{j}\frac{z^j}{n^j}\right\},$$

[6]In particular, $z_0(s,a) = \zeta_{\mathrm{H}}(s,a+1)$.

and we can take

$$T(\lambda, S_{1,0}) = -\ln G_m(-\lambda+1) = \cdots - \frac{(-1)^m}{m+1} B_{m+1} \ln(-\lambda) - \zeta_R'(-m) + \cdots ,$$

where the expansion for large λ is given by [29]. Propositions 2.6 and 2.9 give

$$\begin{aligned} z_m(0,a) &= \frac{(-1)^{m+1}}{m+1}[a^{m+1} - B_{m+1}], \\ z_m'(0,a) &= \zeta_R'(-m) + \frac{(-1)^{m+1}}{m+1}[2\gamma + \psi(m+1)]a^{m+1} \\ &\quad + \sum_{j=1}^{m} \frac{(-1)^j}{j}\zeta_R(j-m)a^j - \ln F(a, S_{1,0}), \end{aligned}$$

where

$$z_m'(0,a) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \zeta_R'(j-m)a^j - \ln G_m(a+1). \tag{3.3}$$

is the Lerch formula.

Next, consider the sequence $S_{2,a}$. Again we have $\alpha = p = m+1$, but now the homogeneous zeta function has simple poles at $s = 1, 2, \ldots, m+1$. The residues are

$$\operatorname{res}_0(\zeta_m(s,0), s=k) = \frac{(-1)^{m+k+1}}{m!}\gamma s_{m+1,k} + \sum_{\substack{j=0, \\ j\neq k-1}}^{m} \frac{(-1)^{m+j}}{m!} s_{m+1,j+1}\zeta_R(k-j),$$

$$\operatorname{res}_1(\zeta_m(s,0), s=k) = \frac{(-1)^{m+k+1}}{m!} s_{m+1,k},$$

for $k = 1, \ldots, m+1$. The Fredholm determinant is given by

$$F(z, S_{2,0}) = \prod_{n\in\mathbb{N}_0^{m+1}} \left(1 + \frac{z}{n_0 + \cdots + n_m}\right) \exp\left\{ \sum_{j=1}^{m+1} \frac{(-1)^j}{j} \frac{z^j}{(n_0+\cdots+n_m)^j} \right\}.$$

We can calculate $\zeta_m(0,a)$ and $\zeta_m'(0,a)$ using the known values of the z_m functions and the relation introduced above; explicitly,

$$\zeta_m(0,0) = \frac{(-1)^m}{m!} \sum_{j=0}^{m} (-1)^j s_{m+1,j+1} \zeta_R(-j),$$

$$\zeta_m'(0,0) = \frac{(-1)^m}{m!} \sum_{j=0}^{m} (-1)^j s_{m+1,j+1} \zeta_R'(-j).$$

Then, using proposition 2.9,

$$\zeta_m(0,a) = \zeta_m(0,0) - \frac{(-1)^m}{m!} \sum_{j=1}^{m+1} \frac{1}{j} s_{m+1,j} a^j,$$

$$\begin{aligned}\zeta_m'(0,a)&\\ &= \zeta_m'(0,0) - \ln F(a, S_{2,0})\\ &\quad - \frac{(-1)^m}{m!}\sum_{j=1}^{m+1}\frac{1}{j}\biggl[(2\gamma+\psi(j))s_{m+1,j} - \sum_{\substack{k=0,\\ k\neq j-1}}^{m}(-1)^{k+j}s_{m+1,k+1}\zeta_{\mathrm{R}}(j-k)\biggr]a^j,\end{aligned}$$

where

$$\zeta_m'(0,a) = \zeta_m'(0,0) + \ln \mathit{\Gamma}_m(a+1) \tag{3.4}$$

is the Lerch formula.

We close the discussion on these linear multiple zeta functions by giving the functional equation for the homogeneous z_m. This can be deduced from the classical reflection formula for the Riemann zeta function (where $s_m = 2m+1$):

$$\pi^{(s-s_m)/2}\mathit{\Gamma}(\tfrac{1}{2}(s_m-s)-\tfrac{1}{2}m)z_m(s_m-s,0) = \pi^{-s/2}\mathit{\Gamma}(\tfrac{1}{2}s-\tfrac{1}{2}m)z_m(s,0).$$

The next step is to consider the related quadratic sequences:

$$L_{1,a^2} = \{n^m(n^2+a^2)\}_{n=1}^{\infty} \quad\text{and}\quad L_{2,a^2} = \{(n_0+\cdots+n_m)^2+a^2\}_{n\in\mathbb{N}_0^{m+1}}.$$

For the first sequence, $\alpha = \frac{1}{2}(m+1)$, $p = p_m = [\alpha]$ (the integer part) and the associated zeta function is

$$Z_m(s,a^2) = \zeta(s, L_{1,a^2}) = \sum_{n=1}^{\infty} n^m(n^2+a^2)^{-s}.$$

The homogeneous zeta function is $Z_m(s,0) = z_m(2s,0) = \zeta_{\mathrm{R}}(2s-m)$, with a simple pole at $s = \frac{1}{2}(m+1)$ with residues $R_0 = \gamma$ and $R_1 = \frac{1}{2}$, and

$$Z_m(0,0) = \zeta_{\mathrm{R}}(-m), \qquad Z_m'(0,0) = 2\zeta_{\mathrm{R}}'(-m),$$

$$F(a^2, L_{1,0}) = \prod_{n=1}^{\infty}\biggl(1+\frac{a^2}{n^2}\biggr)^{n^m}\exp\biggl\{n^m\sum_{j=1}^{p_m}\frac{(-1)^j}{j}\frac{a^{2j}}{n^{2j}}\biggr\}.$$

We can use either proposition 2.9 or corollary 2.10 to compute $Z_m'(0,a^2)$. Now

$$\begin{aligned}G_m(1+z)G_m(1-z) &= \exp\biggl\{\sum_{j=1}^{p_m-1}\biggl[2\binom{m}{2j}\zeta_{\mathrm{R}}'(2j-m) - \frac{1}{j}\zeta_{\mathrm{R}}(2j-m)\biggr]z^{2j}\biggr\}\\ &\quad\times\exp\biggl\{-2\frac{p(m)}{m+1}(2\gamma+\psi(m+1))z^{m+1}\biggr\}\\ &\quad\times\prod_{n=1}^{\infty}\biggl(1-\frac{z^2}{n^2}\biggr)^{n^m}\exp\biggl\{n^m\sum_{j=1}^{p_m}\frac{1}{j}\frac{z^{2j}}{n^{2j}}\biggr\},\end{aligned}$$

where $p(n) = 0$ (respectively, 1) if n is even (respectively, odd) suggests the introduction of the function (and its dual Sh_m of purely imaginary argument as usual)

$$S_m(\pi z) = \pi z\exp\biggl\{-2\frac{p(m)}{m+1}(2\gamma+\psi(m+1))z^{m+1}\biggr\}$$

$$\times \exp\left\{-\sum_{j=1}^{p_m-1}\frac{1}{j}\zeta_{\rm R}(2j-m)z^{2j}\right\}$$
$$\times\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)^{n^m}\exp\left\{n^m\sum_{j=1}^{p_m}\frac{1}{j}\frac{z^{2j}}{n^{2j}}\right\},$$

such that

$$G_m(1+z)G_m(1-z)=\exp\left\{2\sum_{j=1}^{p_m-1}\binom{m}{2j}\zeta_{\rm R}'(2j-m)z^{2j}\right\}\frac{S_m(\pi z)}{\pi z},$$

where

$$Z_m'(0,a^2)=Z_m'(0,0)-\ln\frac{Sh_m(\pi a)}{\pi a}-2\frac{(-1)^{p_m}p(m)}{m+1}\sum_{k=1}^{p_m}\frac{a^{m+1}}{2k-1} \tag{3.5}$$

is the Lerch formula.

The functions S_m and Sh_m can be thought as multiple sine and hyperbolic sine functions, and in fact they share some important properties with the classical sine and hyperbolic sine functions, as we will show at the end of this section (see also [19]). Before that we consider the last sequence, namely L_{2,a^2}. In this case, $\alpha=\frac{1}{2}(m+1)$, $p=p_m=[\alpha]$, and the associated zeta function is

$$\mathcal{Z}_m(s,a^2)=\zeta(s,L_{2,a^2})=\sum_{n\in\mathbb{N}_0^{m+1}}[(n_0+\cdots+n_m)^2+a^2]^{-s};$$

the function $\mathcal{Z}_m(s,0)=\zeta_m(2s,0)$ has simple poles at $s=\frac{1}{2},1,\frac{3}{2},\ldots,\frac{1}{2}(m+1)$ with

$$\begin{aligned}&{\rm res}_0(\mathcal{Z}_m(s,0),s=k)={\rm res}_0(\zeta_m(s,0),s=2k),\\&{\rm res}_1(\mathcal{Z}_m(s,0),s=k)=\tfrac{1}{2}{\rm res}_1(\zeta_m(s,0),s=2k),\end{aligned}$$

and

$$F(a^2,L_{2,0})=\prod_{n\in\mathbb{N}_0^{m+1}}\left(1+\frac{a^2}{(n_0+\cdots+n_m)^2}\right)\exp\left\{\sum_{j=1}^{p_m}\frac{(-1)^j}{j}\frac{a^{2j}}{(n_0+\cdots+n_m)^{2j}}\right\}.$$

As before, we can introduce the functions $\mathcal{S}_m$ and $\mathcal{S}h_m$ by

$$\begin{aligned}\mathcal{S}_m(\pi z)=\pi z\exp\Bigg\{\frac{(-1)^m}{m!}\sum_{j=1}^{p_m}\frac{1}{j}\Bigg[&\left(2\gamma+\psi(2j)-\sum_{k=1}^{j}\frac{1}{2k-1}\right)s_{m+1,2j}\\&+\sum_{k=0,k\neq 2j-1}^{m}s_{m+1,k+1}\zeta_{\rm R}(2j-k)\Bigg]z^{2j}\Bigg\}\\\times\prod_{n\in\mathbb{N}_0^{m+1}}\left(1-\frac{z^2}{(n_0+\cdots+n_m)^2}\right)&\exp\left\{\sum_{j=1}^{p_m}\frac{1}{j}\frac{z^{2j}}{(n_0+\cdots+n_m)^{2j}}\right\},\end{aligned}$$

so that

$$\frac{1}{\Gamma_m(1+z)\Gamma_m(1-z)} = \exp\left\{\frac{(-1)^m}{m!}\sum_{j=1}^{p_m}\frac{1}{j}\sum_{k=1}^{j}\frac{1}{2k-1}s_{m+1,2j}z^{2j}\right\}\frac{\mathcal{S}_m(\pi z)}{\pi z},$$

and

$$\mathcal{Z}'_m(0,a^2) = \mathcal{Z}'_m(0,0) - \ln\frac{\mathcal{S}h_m(\pi a)}{\pi a}, \tag{3.6}$$

where the Lerch formula (3.6) above should be compared with that obtained in (3.1) for the circle (see § 3.1).

We conclude this section by giving some properties of the multiple special functions. We start with the multiple Gamma function Γ_m. From the definition, we can see that $\Gamma_0(z) = \Gamma(z)$ and $\Gamma_m(1) = 1$ for all m. Next, since

$$\binom{m+n+1}{m+1} - \binom{m+n}{m} = \binom{m+n}{m+1},$$

we find that

$$\zeta_{m+1}(s,a+1) = \zeta_{m+1}(s,a) - \zeta_m(s,a) - (a+1)^{-s}.$$

This can be used to prove the following functional equation for the Γ_m.

PROPOSITION 3.1 (functional equation).

$$\Gamma_{m+1}(z+1) = \mathrm{e}^{-\zeta'_m(0,0)}\frac{z\Gamma_{m+1}(z)}{\Gamma_m(z)}. \tag{3.7}$$

Thanks to these properties, we find that the definition of the multiple Gamma function Γ_m actually coincides with that given by Vardi [33] up to a constant factor (depending on m); that is to say, it is the unique function satisfying the above properties[7] and some regularity assumption as shown by Vignèras [34] (see also [1] for the case $m = 0$). With our normalization, both the Lerch formula and the duplication formula (3.8), below, take a nicer form.

PROPOSITION 3.2 (duplication formula).

$$\Gamma_m(2z) = 2^{-\zeta_m(0,2z-1)}\mathrm{e}^{\zeta'_m(0,0)}\Gamma_m(z)\Gamma_m(z+\tfrac{1}{2}). \tag{3.8}$$

Proof. Proof of proposition 3.2 By definition,

$$\begin{aligned}\zeta_m(s,x) + \zeta_m(s,x-\tfrac{1}{2}) &= 2^s\sum_{n\in\mathbb{N}_0^{m+1}}(2n_0+\cdots+2n_m+2x)^{-s}\\ &\quad + 2^s\sum_{n\in\mathbb{N}_0^{m+1}}(2n_0+\cdots+2n_m+2x-1)^{-s}\\ &= 2^s\zeta_m(s,2x).\end{aligned} \tag{3.9}$$

[7] Here we also have the extra factor z due to the fact that we have excluded the zero modes in our sum.

Then

$$\zeta_m'(0,x)+\zeta_m'(0,x-\tfrac{1}{2})=\zeta_m(0,2x)\ln 2+\zeta_m'(0,2x),$$

and we can use the Lerch formula (3.4) with $2x+1=2z$ to obtain the thesis. □

The Taylor expansion of $\ln\Gamma_m(z+1)$ for small z can immediately be obtained from lemma 2.8 and the definition, while the asymptotic expansion can be obtained from [29] for the G_m function and using the fact that

$$-\ln\Gamma_m(z+1)=\frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j s_{m+1,j+1}\left[\ln G_j(z+1)-\sum_{k=1}^{j}(-1)^k\binom{j}{k}\zeta_{\mathrm{R}}'(k-j)z^k\right].$$

PROPOSITION 3.3. *For large z,*

$$\begin{aligned}\ln\Gamma_m(z+1)&=\frac{(-1)^{m+1}}{m!}\\&\quad\times\biggl[-\sum_{j=0}^{m}\frac{s_{m+1,j+1}}{j+1}z^{j+1}\ln z+\sum_{j=0}^{m}\frac{s_{m+1,j+1}}{j+1}(\gamma+\psi(j+2))z^{j+1}\\&\qquad+\sum_{j=0}^{m}(-1)^j s_{m+1,j+1}\zeta_{\mathrm{R}}'(-j)+\sum_{j=0}^{m}\frac{1}{j+1}s_{m+1,j+1}B_{j+1}\ln z\biggr]\\&\quad+O(z^{-1}).\end{aligned}$$

As a final remark, we look at a particular value of the multiple gamma function. Applying proposition 3.2 with $z=\frac{1}{2}$ and using the values calculated for the zeta functions, we obtain

$$\begin{aligned}\Gamma_m(\tfrac{1}{2})&=2^{\zeta_m(0,0)}\mathrm{e}^{-\zeta_m'(0,0)}\\&=2^{(-1)^m/m!\sum_{j=0}^{m}s_{m+1,j+1}\zeta_{\mathrm{R}}(-j)}\exp\biggl\{-\frac{(-1)^m}{m!}\sum_{j=0}^{m}s_{m+1,j+1}\zeta_{\mathrm{R}}'(-j)\biggr\}.\end{aligned}\tag{3.10}$$

This is a remarkable formula, since it express the value of the Γ_m at $\frac{1}{2}$ as a function of the dimension m; it seems the most natural generalization of the classical version

$$\Gamma(\tfrac{1}{2})=2^{\zeta_{\mathrm{R}}(0)}\mathrm{e}^{-\zeta_{\mathrm{R}}'(0)}=\sqrt{\pi},$$

and then gives another answer to the question considered in [33].

Finally, we give some properties of the multiple sine functions. It is clear that the two sets of functions (S and *Sh* versus $\mathcal{S}$ and $\mathcal{S}h$) are related by the same algebraic relation introduced above for the zeta functions; thus, it is sufficient to study one set of functions. It is easier to consider the functions $S_m(x)$, which are more explicitly related to the Barnes functions. Using the known expansions for the G_m functions, we make the following proposition.

Proposition 3.4. *For large real x,*

$$\begin{aligned}\ln Sh_m(\pi x) &= 2\frac{(-1)^{p_m}p(m)}{m+1}x^{m+1}\ln x \\ &\quad + \frac{(-1)^{p_m}}{m+1}[2p(m)(\gamma+\psi(m+2))+\pi p(m+1)]x^{m+1} \\ &\quad + \ln \pi x - \frac{2B_{m+1}}{m+1}\ln x + 2\zeta_{\mathrm{R}}'(-m) + O(x^{-1}).\end{aligned}$$

As for the function Γ_m, the Taylor expansion for $\ln Sh_m$ can be obtained from lemma 2.5. We prefer to conclude by considering explicitly the case $m=1$. In such a case we introduce the dual cosine double function that will turn out to be useful in the next section. With $m=1$,

$$S(z) = S_1(z) = z\exp\left\{-(\gamma+1)\frac{z^2}{\pi^2}\right\}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\pi^2 n^2}\right)^n \exp\left\{\frac{z^2}{\pi^2 n}\right\},$$

$$C(z) = C_1(z) = \exp\left\{-(\gamma+1)\frac{z^2}{\pi^2}\right\}\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{\pi^2(2n-1)^2}\right)^n \exp\left\{\frac{z^2}{\pi^2 n}\right\},$$

and the following relations hold:

$$S(\tfrac{1}{2}\pi \pm z) = \pi\Gamma\left(\frac{3}{2}\pm\frac{z}{\pi}\right)[G(\tfrac{1}{2})]^2 C(z),$$

$$G(1+z)G(1-z) = \frac{S(\pi z)}{\pi z}, \qquad G(\tfrac{1}{2}-z)G(\tfrac{1}{2}+z) = [G(\tfrac{1}{2})]^2 C(\pi z).$$

3.3. The zeta function on the 2-spheres

In this section we consider the sequence $S = \{(2n+1)[n(n+1)]^{-s}\}_{n=1}^{\infty}$, the spectrum of the standard Laplace operator on the 2-sphere of unit radius. Now $\alpha = p = 1$, and the associated zeta function is

$$\zeta_{S^2}(s,0) = \zeta(s,S) = \sum_{n=1}^{\infty}(2n+1)[n(n+1)]^{-s},$$

for $\mathrm{Re}(s) > 1$. The non-homogeneous zeta function $\zeta_{S^2}(s,a^2) = \zeta(s,S_{a^2})$ is given by

$$\zeta_{S^2}(s,a^2) = \sum_{n=1}^{\infty}(2n+1)[n(n+1)+a^2]^{-s} = \sum_{n=1}^{\infty}(2n+1)[(n+\tfrac{1}{2})^2+q]^{-s},$$

where $q = a^2 - \frac{1}{4}$. This suggests we instead consider the function [35]

$$\zeta_b(s) = \sum_{n=1}^{\infty}(2n+b)(n+\tfrac{1}{2}b)^{-2s} = 2\sum_{n=1}^{\infty}(n+\tfrac{1}{2}b)^{-2s+1},$$

with $b = 1$ as the (non-homogeneous) zeta function associated with the problem. In fact, $\zeta_1(s)$ factors through the Riemann zeta function

$$\zeta_1(s) = 2^{2s} \sum_{n=1}^{\infty} (2n+1)^{-2s+1} = (2^{2s} - 2)\zeta_{\mathrm{R}}(2s-1) - 2^{2s},$$

and in this way we get the zeta invariants for $\zeta_1(s) = \zeta_{S^2}(s, \frac{1}{4})$ and, using proposition 2.9, those for $\zeta_{S^2}(s, 0)$. In particular, this means that $\zeta_{S^2}(s, 0)$ has only one simple pole at $s = 1$ (see also [30]) with

$$\operatorname{res}_1(\zeta_{S^2}(s,0), s=1) = 1, \qquad \operatorname{res}_0(\zeta_{S^2}(s,0), s=1) = 2\gamma.$$

Despite this, the function $\zeta_1(s)$ seems completely unrelated to the geometry, as can be seen by considering that $\zeta_1(0) = -\frac{11}{12}$, while the value at $s = 0$ of the zeta function associated with the Laplace operator over a closed surface is

$$\zeta_M(0) = \tfrac{1}{6}\chi(M) - \dim\ker \Delta_M,$$

which gives $-\frac{2}{3}$ for $M = S^2$, where $\chi(M)$ is the Euler characteristic of M. For this reason, we investigate the zeta function $\zeta_{S^2}(s, 0)$, naturally associated with the geometry of the problem, directly. First, consider the heat function

$$f_{S^2}(t) = f(t, S) = \sum_{n=1}^{\infty} (2n+1)\mathrm{e}^{-n(n+1)t} + 1.$$

We give here a simple method of obtaining the full asymptotic expansion of $f_{S^2}(t)$ for small t using special functions. This result was originally obtained by Mulholland [23] by direct calculation, and is indeed a very useful result since it allows us to compute asymptotic expansions for the heat kernel not just for all spheres, but for all symmetric spaces of rank one [10]. Consider the function

$$\begin{aligned} f_b(t) &= \mathrm{e}^{\frac{1}{4}b^2 t} \sum_{n=1}^{\infty} (2n+b)\mathrm{e}^{-(n+\frac{1}{2}b)^2 t} \\ &= -\frac{2}{t}\mathrm{e}^{\frac{1}{4}b^2 t} \frac{\mathrm{d}}{\mathrm{d}b} \sum_{n=1}^{\infty} \mathrm{e}^{-(n+\frac{1}{2}b)^2 t}, \end{aligned}$$

where b is a real parameter. Then, $f_{S^2}(t) = f_1(t) + 1$, and we get the asymptotic of $f_{S^2}(t)$, giving that for $f_b(t)$. After some computation, we can write

$$f_b(t) = \frac{\mathrm{e}^{\frac{1}{4}b^2 t}}{2\pi \mathrm{i}} \int_{\Lambda_{c,\theta}} \mathrm{e}^{-\lambda t}[\psi(1+\tfrac{1}{2}b+\mathrm{i}\sqrt{-\lambda}) + \psi(1+\tfrac{1}{2}b-\mathrm{i}\sqrt{-\lambda}) - 2\psi(1+\tfrac{1}{2}b)]\,\mathrm{d}\lambda.$$

On changing the variable to λ/t and expanding the factors by using the knowm expansion for the digamma function, we get

$$f_{S^2}(t) = f_1(t) + 1 = \frac{1}{t} + \sum_{j=0}^{\infty} a_j t^j = \frac{1}{t} + \frac{1}{3} + \frac{t}{15} + \cdots,$$

$$a_j = \frac{1}{2^{2j+2}}\left[\frac{1}{(j+1)!} + \sum_{k=0}^{j} \frac{(-1)^k (2^{2k+2}-2)}{(k+1)!(j-k)!} B_{2k+2}\right].$$

Next, we give formulae for the other spectral functions on the sphere in terms of special functions. First, considering the expression $\psi(c)-\psi(a+b)+\psi(d)-\psi(a-b)$, we obtain

$$R(\lambda, S) = \psi(\tfrac{3}{2}+z)+\psi(\tfrac{3}{2}-z)-\psi(2)-\psi(1), \quad z=\sqrt{\lambda+\tfrac{1}{4}}.$$

The important point is that this expression can be integrated in terms of the Barnes G function. After some manipulation, we get

$$\begin{aligned} T(\lambda, S) &= -\ln F(-\lambda, S) \\ &= -\ln \prod_{n=1}^{\infty}\left(1+\frac{-\lambda}{n(n+1)}\right)^{2n+1} \mathrm{e}^{(2n+1)\lambda/n(n+1)} \\ &= -2\ln G(\tfrac{1}{2}+z)G(\tfrac{1}{2}-z) - \ln \Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z) + (2\gamma+1)(-\lambda) + \ln(-\lambda) \end{aligned}$$

and this can be written using the multiple cosine function introduced in § 3.2 as

$$\begin{aligned} &\vdots \\ &= -2\ln C(\pi z) + \ln\frac{\cos \pi z}{\pi} + (2\gamma+1)(-\lambda) + \ln(-\lambda) - 4\ln G(\tfrac{1}{2}) \\ &= \cdots + (B_1+\tfrac{1}{2})\ln(-\lambda) + \tfrac{1}{2} - 4\zeta_{\mathrm{R}}'(-1) + \cdots . \end{aligned}$$

This gives

$$\zeta_{S^2}(0) = -\tfrac{2}{3}, \qquad \zeta_{S^2}'(0) = 4\zeta_{\mathrm{R}}'(-1) - \tfrac{1}{2}.$$

The non-homogeneous case $\zeta_{S^2}(s, a^2)$ follows from proposition 2.9. We get (cf. [35])

$$\begin{aligned} \zeta_{S^2}(0, a^2) &= -\tfrac{2}{3} - a^2, \\ \zeta_{S^2}'(0, a^2) &= \zeta_{S^2}'(0, 0) - 2\gamma a^2 - \ln F(a^2, S) \\ &= 4\zeta_{\mathrm{R}}'(-1) - \tfrac{1}{2} + a^2 + \ln a^2 \\ &\qquad - 2\ln G\Big(\tfrac{1}{2}+\sqrt{\tfrac{1}{4}-a^2}\Big)G\Big(\tfrac{1}{2}-\sqrt{\tfrac{1}{4}-a^2}\Big) \\ &\qquad - \ln \Gamma\Big(\tfrac{1}{2}+\sqrt{\tfrac{1}{4}-a^2}\Big)\Gamma\Big(\tfrac{1}{2}-\sqrt{\tfrac{1}{4}-a^2}\Big), \end{aligned}$$

where

$$\begin{aligned} \zeta_{S^2}'(0, a^2) = \zeta_{S^2}'(0,0) - 4\ln G(\tfrac{1}{2}) + a^2 + \ln a^2 \\ - 2\ln C\Big(\pi\sqrt{\tfrac{1}{4}-a^2}\Big) + \ln\frac{\cos \pi\sqrt{\tfrac{1}{4}-a^2}}{\pi} \end{aligned} \tag{3.11}$$

is a further generalization of the Lerch formula and gives another solution in terms of special functions, to the problem of the Laplacian coupled with a constant potential, as considered in the final remark of [30, § 5].

References

1 E. Artin. *The Gamma function* (New York: Holt, Rinhehart and Winston, 1964).

2 M. Atiyah, R. Bott and V. K. Patodi. On the heat equation and the index theorem. *Invent. Math.* **19** (1973), 279–330.

3 E. W. Barnes. The theory of the G function. *Q. J. Math.* **31** (1899), 264–314.

4 E. W. Barnes. The theory of the multiple Gamma function. *Trans. Camb. Phil. Soc.* **19** (1904), 374–425.

5 J. Billingham and A. C. King. Uniform asymptotic expansions for the Barnes double gamma function. *Proc. R. Soc. Lond.* A **453** (1997), 1817–1829.

6 T. Bransom and P. Gilkey. The asymptotics of the Laplacian on a manifold with boundary. *Commun. PDEs* **15** (1990), 245–272.

7 J. Brüning and R. Seeley. The resolvent expansion for second-order regular singular operators. *J. Funct. Analysis* **73** (1987), 369–429.

8 D. Burghelea, L. Friedlander and T. Kappeler. On the determinant of ellptic boundary value problems on a line segment. *Proc. Am. Math. Soc.* **123** (1995), 3027–3028.

9 A. A. Bytsenko, E. Elizalde, S. D. Odintsov, A. Romeo and S. Zerbini. *Zeta regularization techniques with applications* (World Scientific, 1994).

10 R. S. Cahn and J. A. Wolf. Zeta functions and their asymptotic expansions for compact symmetric spaces of rank one. *Comment. Math. Helv.* **51** (1976), 1–21.

11 J. Cheeger. Analytic torsion and the heat equation. *Ann. Math.* **109** (1979), 259–322.

12 J. Cheeger. Spectral geometry of singular Riemannian spaces. *J. Diff. Geom.* **18** (1984), 575–657.

13 J. Choi and J. R. Quine. Zeta regularized products and functional determinants on spheres. *Rocky Mt. J. Math.* **26** (1996), 719–729.

14 P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah–Singer index theorem.* Studies in Advanced Mathematics (CRC Press, 1995).

15 I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products* (Academic, 1980).

16 S. W. Hawking. Zeta function regularization of path integrals in curved space time. *Commun. Math. Phys.* **55** (1977), 139–170.

17 L. Hörmader. *The analysis of linear partial differential operators. III.* (Springer, 1984).

18 M. Kontsevich and S. Vishik. Geometry of determinants of elliptic operators. In *Functional analysis on the eve of the 21st century.* Progress in Mathematics, vol. 131, pp. 173–197 (Birkhäuser, 1995).

19 N. Kurokawa. Multiple sine functions and Selberg zeta functions. *Proc. Jpn Acad.* A **67** (1991), 61–64.

20 H. A. Lauwrier. *Asymptotic analysis* (Amsterdam: Mathematisch Centrum, 1977).

21 M. Lesch. Determinants of regular singular Sturm–Liouville operators. *Math. Nachr.* **194** (1998), 139–170.

22 R. B. Melrose. Weyl's conjecture for manifolds with concave boundary. In *The geometry of the Laplace operator.* Proceedings of Symposia in Pure Mathematics, vol. 36, pp. 257–274 (Providence, RI: American Mathematical Society, 1980).

23 H. P. Mulholland. An asymptotic expansion for $\sum_{n=1}^{\infty}(2n+1)\mathrm{e}^{(n+1/2)^2 t}$. *Proc. Camb. Phil. Soc.* **24** (1928), 280–289.

24 F. W. J. Olver. *Asymptotics and special functions*, AKP Classics Series (Wellesley, MA: Peters, 1997).

25 D. B. Ray and I. M. Singer. *R*-torsion and the Laplacian on Riemannian manifolds. *Adv. Math.* **7** (1974), 145–210.

26 S. Rosenberg. *The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds.* London Mathematical Society Student Texts, vol. 31 (Cambridge University Press, 1997).

27 Y. Safarov and D. Vassiliev. *The asymptotic distribution of eigenvalues of partial differential operators.* Translations of Mathematical Monographs, vol. 155 (American Mathematical Society, 1997).

28 P. Sarnak. Determinants of Laplacians. *Commun. Math. Phys.* **110** (1987), 113–120.

29 R. Shuster. A generalized Barnes G-function. *Z. Analysis Anwend.* **11** (1992), 229–236.

30 M. Spreafico. Zeta function and regularized determinant on projective spaces. *Rocky Mt. J. Math.* **33** (2003), 1499–1512.

31 M. Spreafico. A generalization of the Euler Gamma function. *Funct. Analysis Applic.* **39** (2005), 87–91.

32 M. Spreafico. On the non-homogeneous Bessel zeta function. *Mathematika* **51** (2004), 123–130.

33 I. Vardi. Determinants of Laplacians and multiple Gamma functions. *SIAM J. Math. Analysis* **19** (1988), 493–507.

34 M. F. Vignéras. L'équation functionelle de la function zéta de Selberg du group modulaire $SL(2,\mathbb{Z})$. *Astérisque* **61** (1979), 235–249.

35 A. Voros. Spectral functions, special functions and the Selberg zeta function. *Commun. Math. Phys.* **110** (1987), 439–465.

36 N. Weiner. Tauberian theorems. *Ann. Math.* **33** (1932), 461–472.

37 E. T. Whittaker and G. N. Watson. *A course in modern analysis* (Cambridge University Press, 1946).

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