FINITE TEMPERATURE QUANTUM FIELD THEORY ON NONCOMPACT DOMAINS AND APPLICATION TO DELTA INTERACTIONS

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We use relative zeta functions technique of W. Muller to investigate the regularized partition function of a finite temperature quantum field theory on a ultrastatic space-time with noncompact spatial section. As an application, we study the case of massless scalar field with singular delta-like potential, as described in [1].

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1. Introduction

To begin with, we recall that the partition function for a finite temperature quantum field theory on a ultrastatic space-time with compact spatial section is constructed as follows. Let M be a compact Riemannian manifold of dimension n, and consider the product $N = S^1_r \times M$, where S^1_r is the circle of radius $r = \beta/2\pi$, and $\beta = 1/T$ is the inverse of the temperature. Let L be some nonnegative self-adjoint operator (typically the Laplacian) acting on some function space (we shall deal with scalar fields) defined on M and $H = -\partial_u^2 + L$. The canonical partition function at temperature T of this model may be formally written as $Z = \det^{-\frac{1}{2}}(\ell^2 H)$, where ℓ is some renormalization constant. It is well known that a rigorous interpretation to this functional determinant can be given using zeta function regularization. The zeta function regularization technique was first introduced by Ray and Singer [22] to define the regularized determinant for the Laplacian on forms, and used by Hawking

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[17] in order to regularize Gaussian path integrals on a curved space time, and soon became a fundamental tool in mathematical physics and may provide a way for regularizing the partition function of a quantum field theory at finite temperature on compact domains. Recall that the zeta function of a nonnegative self-adjoint operator A is defined by (where $\mathrm{Sp}^{+}A$ denotes the positive part of the spectrum of A) $\zeta(s;A)=\sum_{\lambda\in Sp^{+}A}\lambda^{-s}$, when $\mathrm{Re}\,(s)>s_0$ (for some suitable s_0), and by analytic continuation elsewhere. Since N is compact, zero is not a pole of $\zeta(s;H)$, and using the zeta function, the regularized functional determinant of H is defined by

$$\det H = \mathrm{e}^{-\frac{d}{ds}\zeta(s;H)\big|_{s=0}},$$

and the partition function is

$$\log Z = \frac{1}{2}\zeta'(0;H)-\frac{1}{2}\zeta(0;H)\log \ell^2.$$

Introducing the geometric zeta function, namely the zeta function of the restriction L of H to M, the following equations hold (we assume here for simplicity that $\ker L=\emptyset$):

$$\zeta(0;H) = -\beta \operatorname*{Res_1}_{s=-\frac{1}{2}}\zeta(s;L), \tag{1}$$

$$\zeta'(0;H) = -\beta \operatorname*{Res_0}_{s=-\frac{1}{2}}\zeta(s;L) - 2\beta(1-\log 2)\operatorname*{Res_1}_{s=-\frac{1}{2}}\zeta(s;L) - 2\log\eta\,(\beta;L)\,, \tag{2}$$

where the generalized Dedekind eta function for a positive self-adjoint operator A in some Hilbert space $\mathcal{H}(M)$, where M is compact, is defined by [20]

$$\eta(\tau;A) = \prod_{\lambda\in \mathrm{Sp}\,A}\left(1-\mathrm{e}^{-\tau\sqrt{\lambda}}\right),$$

and $\mathrm{Res}_{\mathrm{k}s=s_0} f(s)$ denotes the coefficient of the term $(s-s_0)^{-k}$ of the Laurent expansion of $f(s)$ at $s=s_0$ (see for example [5] p. 420). This is a classical and well-known result (see for example [14, 11]), and we have used the formulation of [20] (see also [6, 10] and [13] for an extension), and it leads to the natural question of a suitable generalization for noncompact domains.

In this paper we will try to answer rigorously this question and we prove a generalization of equations (1) and (2) that holds for a quantum scalar field on a noncompact domain. In this case, we shall consider operators such that the spectrum involved is no longer only discrete and a continuous contribution appears. With regard to the treatment of the continuous spectrum, we will follow the approach of Müller [19]. However, we should mention that the introduction of relative traces appeared in the seminal paper [2], where the so-called second virial coefficient, proportional to the relative trace $\mathrm{Tr}\,(e^{-\beta(H_O+V)}-e^{-\beta H_O})$ (here H_0 is free Hamiltonian), was expressed in terms of the trace of scattering matrix, the Beth–Uhlenbeck formula. More recently, functional determinants in quantum field theory have been investigated with relative zeta functions (see for example [12]). The mathematical counterpart of these approaches in the physical literature makes use of Krein formula (see [4]). Beside the natural interest of the generalization itself, we would like to note that on a more general ground, scattering methods have been applied in the physical literature (see for example the review [18]) in order to study quantum vacuum effects between material bodies and our result provides a rigorous justification of these formal approaches. Furthermore, a motivation is also given by the recently growing interest in delta interactions, namely a theory described by a scalar field in a flat space-time perturbed by pointlike (uncharged) "impurities", modelled by delta-like potentials. Since these are solvable quantum models, it is quite natural to analyze explicitly these examples, as illustrative application of our results. Actually, the case of one delta interaction turns out to be particularly interesting, since it can be completely solved, and thus plays the role of the leading example, as the Laplacian on the circle is the leading example for the compact case.

2. Relative determinants

We introduce in this section the mathematical tools necessary in order to state our main results. This is essentially based on the work of Müller [19], however, we will reformulate the approach of Müller in terms of the resolvent rather than of the heat semigroup, because in specific applications we have an explicit expression for the resolvent function instead than that for the heat kernel. Anyway, it is well known that one may investigate equivalently the resolvent of an elliptic operator instead of heat semigroup.

Let $\mathcal{H}$ be a separable Hilbert space, and let A and A_0 be two self-adjoint nonnegative linear operators in $\mathcal{H}$. Suppose that $\mathrm{Sp}\, A = \mathrm{Sp}_c A \cup \mathrm{Sp}_p A$, where Sp_p is the point spectrum, and Sp_c is the continuous spectrum, and we assume both 0 and ∞ are accumulation points of $\mathrm{Sp}\, A$. It is convenient to split the point spectrum in the null part, $\mathrm{Sp}_p^0 A = \{\lambda_0 = 0\}$, that has finite multiplicity, plus the positive part, $\mathrm{Sp}_p^+ A = \{\lambda_j\}_{j=1}^J$, where each eigenvalue is counted according to multiplicity. Let $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_p$ be the orthogonal decomposition into the subspaces that correspond to the continuous and the point spectrum of A, respectively, and let A_c and A_p denote the restrictions of A to $\mathcal{H}_c$ and $\mathcal{H}_p$, respectively. Let $R(\lambda, T) = (\lambda I - T)^{-1}$ denote the resolvent of the operator T, and $\rho(T)$ the resolvent set. Then, we introduce the following two sets of conditions. First, we assume that the sequence $\mathrm{Sp}_p^+ A$ is a totally regular sequence of spectral type with finite exponent s_0, as defined in [25]. This implies that the following conditions hold:

(A.1) The operator $R(\lambda, A_p)$ is of trace class for all $\lambda \in \rho(A_p)$;

(A.2) as $\lambda \to \infty$ in $\rho(A_p)$, there exists an asymptotic expansion of the form

$$\mathrm{Tr}\, R(\lambda, A_p) - \dim\ker A_p \frac{1}{\lambda} \sim \sum_{j=0}^{\infty} \sum_{k=0}^{K_j} a'_{j,k} (-\lambda)^{\alpha'_j} \log^k(-\lambda),$$

where $-\infty < \cdots < \alpha'_1 < \alpha'_0 \leq s_0 - 1$, and $\alpha'_j \to -\infty$, for large j, and $a'_{j,k} = 0$ for $k > 0$;

(A.3) as $\lambda \to 0$, there exists an asymptotic expansion of the form

$$\mathrm{Tr}\, R(\lambda, A_p) - \dim\ker A_p \frac{1}{\lambda} \sim \sum_{j=0}^{\infty} b'_j (-\lambda)^{\beta'_j},$$

where $0 = \beta'_0 < \beta'_1 < \dots$, and $\beta'_j \to +\infty$, for large j.

Second, we assume the following conditions on the pair (A_c, A_0):

(B.1) the operator $R(\lambda, A_c) - R(\lambda, A_0)$ is of trace class for all $\lambda \in \rho(A_c) \cap \rho(A_0)$;

(B.2) as $\lambda \to \infty$ in $\rho(A_c) \cap \rho(A_0)$, there exists an asymptotic expansion of the form

$$\mathrm{Tr}\,(R(\lambda, A_c) - R(\lambda, A_0)) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{K_j} a_{j,k} (-\lambda)^{\alpha_j} \log^k(-\lambda),$$

where $-\infty < \cdots < \alpha_1 < \alpha_0$, $\alpha_j \to -\infty$, for large j, and $a_{j,k} = 0$ for $k > 0$;

(B.3) as $\lambda \to 0$, there exists an asymptotic expansion of the form

$$\mathrm{Tr}\,(R(\lambda, A_c) - R(\lambda, A_0)) \sim \sum_{j=0}^{\infty} b_j (-\lambda)^{\beta_j},$$

where $-1 \le \beta_0 < \beta_1 < \dots$, and $\beta_j \to +\infty$, for large j.

We introduce the further consistency condition (that will be always tacitely assumed)

(C) $\alpha_0 < \beta_0$.

By results of [25], it follows that the zeta function of the operator A_p is well defined by the uniformly convergent series

$$\zeta(s; A_p) = \sum_{j=1}^{\infty} \lambda_j^{-s},$$

when $\mathrm{Re}\,(s) > s_0 \ge \alpha'_0 + 1$, and by analytic continuation elsewhere. In particular, the heat semigroup e^{-tA_p} is of trace class, and the following equations hold:

$$\mathrm{Tr}\,\mathrm{e}^{-A_p t} - \dim\ker A_p = \frac{1}{2\pi i} \int_{\Lambda_{\theta,-a}} \mathrm{e}^{-\lambda t} \left(\mathrm{Tr}\, R(\lambda, A_p) - \dim\ker A_p \frac{1}{\lambda} \right) d\lambda, \tag{3}$$

where the Hankel type contour is $\Lambda_{\theta,-a} = \left\{ \lambda \in \mathbb{C} \mid |\arg(\lambda + a) = \frac{\theta}{2} \right\}$, oriented counter clockwise, with some fixed $a > 0$, $0 < \theta < \pi$,

$$\zeta(s; A_p) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\mathrm{Tr}\,\mathrm{e}^{-tA_p} - \dim\ker A_p \right) dt. \tag{4}$$

We can prove similar results for the relative heat semigroup and the relative zeta function, using [19]. We introduce the following lemma.

LEMMA 2.1. *If the pair of nonnegative self-adjoint operators* (T, T_0) *satisfies conditions* (B.1)–(B.3)*, then it satisfies the conditions* (1.1)–(1.3) *of* [19].

Proof: The proof that conditions (B.1) and (B.2) imply conditions (1.1) and (1.2) of [19] follows from the equation

$$\mathrm{e}^{-Tt} - \mathrm{e}^{-T_0 t} = \frac{1}{2\pi i} \int_{\Lambda_{\theta,-a}} \mathrm{e}^{-\lambda t} \left(R(\lambda, T) - R(\lambda, T_0) \right) d\lambda, \tag{5}$$

and 2.2 of [25], respectively. Next, assume (B.3). Then, for any fixed β_J,

$$\left| \mathrm{Tr}\, (R(\lambda, T) - R(\lambda, T_0)) - b_j (-\lambda)^{\beta_J - 1} \right| \leq K | - \lambda|^{\beta_J}.$$

We can use this bound for the remainder in order to obtain (1.3) of [19]. Using the expansion given by condition (B.3) of the difference of the resolvents in Eq. (5), the remainder is

$$r_J(t) = \frac{1}{2\pi i} \int_{\Lambda_{\theta,-a}} \mathrm{e}^{-\lambda t} \Big(\mathrm{Tr}\, (R(\lambda, T) - R(\lambda, T_0)) - \sum_{j=0}^{J} b_j (-\lambda)^{-\beta_j} \Big) d\lambda,$$

and thus it satisfies the bound

$$|r_J(t)| \leq K \int_{|\Lambda_{\theta,-a}|} |\mathrm{e}^{-\lambda}| |(-\lambda)^{\beta_J}| |d\lambda| t^{-\beta_J - 1},$$

where the integral is a finite constant. □

Therefore, assuming conditions (B.1)-(B.3) for the pair of nonnegative self-adjoint operators (A_c, A_0), all the results of [19] hold, and in particular we can define the relative zeta function for the pair (A_c, A_0) by the following equation

$$\zeta(s; A_c, A_0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}\, \left(\mathrm{e}^{-tA_c} - \mathrm{e}^{-tA_0} \right) dt, \tag{6}$$

when $\alpha_0 + 1 < \mathrm{Re}\,(s) < \beta_0 + 1$, and by analytic continuation elsewhere. Back to the pair (A, A_0), note that

$$\mathrm{Tr}\, \left(\mathrm{e}^{-tA} - \mathrm{e}^{-tA_0} \right) = \mathrm{Tr}\, \mathrm{e}^{-tA_p} + \mathrm{Tr}\, \left(\mathrm{e}^{-tA_c} - \mathrm{e}^{-tA_0} \right),$$

and the problem decomposes additively into the two terms arising from the pure point and the continuous spectrum, namely $\zeta(s; A, A_0) = \zeta(s; A_p) + \zeta(s; A_c, A_0)$. Thus, if we define the regularized relative determinant of the pair of operators (A, A_0) by

$$\det\, (A, A_0) = \mathrm{e}^{-\frac{d}{ds} \zeta(s; A, A_0) \big|_{s=0}},$$

then we have the decomposition

$$\det\, (A, A_0) = \det\, (A_p) \det\, (A_c, A_0),$$

and the two regularizations can be treated independently. This suggests to introduce the following definition for the zeta regularized partition function of a model described by the operator A, under the assumption that there exists a second operator A_0

such that (using the above decomposition) the operator A_p satisfies assumptions (A.1)–(A.2) and the pair of operators (A_c, A_0) satisfies assumptions (B.1)–(B.3),

$$\log Z = \frac{1}{2}\zeta'(0; A_p) - \frac{1}{2}\zeta(0; A_p)\log \ell^2 + \frac{1}{2}\zeta'(0; A_c, A_0) - \frac{1}{2}\zeta(0; A_c, A_0)\log \ell^2. \quad (7)$$

This is the natural generalization of the classical zeta regularization technique to the relative case. We conclude this section with a technical result.

LEMMA 2.2. *Assume that the nonnegative self-adjoint operator T decomposes additively as sum of two nonnegative self-adjoint commuting operators T_1 and T_2, where e^{-tT_1} is compact trace class and satisfies an expansion for small t as $\sum_{j=0}^{\infty}\sum_{k=0}^{K_j} c_{j,k} t^{\gamma_j}\log^k t$, with $-\infty < \gamma_0 < \gamma_1 < \dots$, $\gamma_j \to +\infty$, for large j, and $c_{j,k} = 0$ for $k > 0$. Then, if there exists an operator T_0 such that conditions* (B.1)–(B.3) *hold for the pair (T_2, T_0), then conditions* (1.1)–(1.3) *of* [19] *hold for the pair $(T, T_1 + T_0)$, and viceversa. In particular, the following equation holds*

$$\mathrm{Tr}\left(e^{-tT} - e^{-t(T_1+T_0)}\right) = \mathrm{Tr}e^{-tT_1}\mathrm{Tr}\left(e^{-tT_2} - e^{-tT_0}\right).$$

Proof: By standard properties of the heat semigroup

$$e^{-tT} - e^{-t(T_1+T_0)} = e^{-tT_1}\left(e^{-tT_2} - e^{-tT_0}\right).$$

Suppose that (B.1) holds for (T_2, T_0). Then, $e^{-tT_2} - e^{-tT_0}$ is of trace class by Lemma 2.1, and the above equation implies that $e^{-tT} - e^{-t(T_1+T_0)}$ is of trace class. Therefore, the equation given in the statement of the lemma holds. This implies that, if (B.2) and (B.3) hold for (T_2, T_0) and (A.2) holds for T_1, then (1.2)–(1.3) of [19] hold for $(T, T_1 + T_0)$. The proof of the converse is similar. □

3. Relative partition function

Let M be a smooth Riemannian manifold of dimension n, and consider the product $N = S^1_{\frac{\beta}{2\pi}} \times M$, where S^1_r is the circle of radius r. Let ξ be a complex line bundle over N, and L a self-adjoint nonnegative linear operator on the Hilbert space $\mathcal{H}(M)$ of the L^2 sections of the restriction of ξ onto M, with respect to some fixed metric g on M. Let H be the self-adjoint nonnegative operator $H = -\partial_u^2 + L$, on the Hilbert space $\mathcal{H}(N)$ of the L^2 sections of ξ, with respect to the product metric $du^2 \oplus g$ on N, and with periodic boundary conditions on the circle. Assume that there exists a second operator L_0 defined on $\mathcal{H}(M)$, such that the pair (L, L_0) satisfies the assumptions (B.1)–(B.3) of Section 2 (since we have seen that the problem decomposes additively in point and continuous part, we assume here without loss of generality that the point spectrum is empty). Then, by Lemma 2.2, it follows that there exists a second operator H_0 defined in $\mathcal{H}(N)$, such that the pair (H, H_0) satisfies those assumptions too. Under these requirements, we introduce the relative zeta regularized partition function of the model described by the pair of operators (H, H_0) using equation (7), and we can prove the following result.

PROPOSITION 3.1. *Let L be a nonnegative self-adjoint operator on M, and $H = -\partial_u^2 + L$, on $S_r^1 \times M$ as defined above. Assume that there exists an operator L_0 such that the pair (L, L_0) satisfies conditions* (B.1)–(B.3)*. Then,*

$$\begin{aligned}
\zeta(0; H, H_0) &= -\beta \operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s; L, L_0),\\
\zeta'(0; H, H_0) &= -\beta \operatorname*{Res_0}_{s=-\frac{1}{2}} \zeta(s; L; L_0) - 2\beta(1 - \log 2) \operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s; L, L_0)\\
&\quad - 2\log \eta(\beta; L, L_0),
\end{aligned}$$

where $H_0 = -\partial_u^2 + L_0$ and the relative Dedeckind eta function is defined by

$$\begin{aligned}
\log \eta(\tau; L, L_0) &= \int_0^\infty \log\left(1 - \mathrm{e}^{-\tau v}\right) e(v; L, L_0) dv,\\
e(v; L, L_0) &= \frac{v}{\pi i} \lim_{\epsilon \to 0^+} \left(r(v^2 \mathrm{e}^{2i\pi - i\epsilon}; L, L_0) - r(v^2 \mathrm{e}^{i\epsilon}; L, L_0) \right),\\
r(\lambda; L, L_0) &= \mathrm{Tr}\,(R(\lambda, L) - R(\lambda, L_0)).
\end{aligned}$$

Proof: Since (L, L_0) satisfies (B.1)–(B.3), by Lemma 2.2 (H, H_0) satisfies (1.1)–(1.3) of [19] and the zeta function is defined by

$$\zeta(s; H, H_0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}\left(\mathrm{e}^{-tH} - \mathrm{e}^{-tH_0}\right) dt,$$

when $\alpha_0 + 1 < \mathrm{Re}\,(s) < \beta_0 + 1$. By Lemma 2.2

$$\mathrm{Tr}\left(\mathrm{e}^{-Ht} - \mathrm{e}^{-H_0 t}\right) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\frac{n^2}{r^2} t} \mathrm{Tr}\left(\mathrm{e}^{-tL} - \mathrm{e}^{-tL_0}\right),$$

and hence, using the well-known Jacobi summation formula we obtain

$$\begin{aligned}
\zeta(s; H, H_0) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\frac{n^2}{r^2} t} \mathrm{Tr}\left(\mathrm{e}^{-tL} - \mathrm{e}^{-tL_0}\right) dt\\
&= \frac{\sqrt{\pi} r}{\Gamma(s)} \int_0^\infty t^{s - \frac{1}{2} - 1} \mathrm{Tr}\left(\mathrm{e}^{-tL} - \mathrm{e}^{-tL_0}\right) dt\\
&\quad + \frac{2\sqrt{\pi} r}{\Gamma(s)} \int_0^\infty t^{s - \frac{1}{2} - 1} \sum_{n=1}^\infty \mathrm{e}^{-\frac{\pi^2 r^2 n^2}{t}} \mathrm{Tr}\left(\mathrm{e}^{-tL} - \mathrm{e}^{-tL_0}\right) dt\\
&= \frac{\sqrt{\pi} r}{\Gamma(s)} \Gamma\left(s - \frac{1}{2}\right) \zeta\left(s - \frac{1}{2}; L, L_0\right)\\
&\quad + \frac{2\sqrt{\pi} r}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty t^{s - \frac{1}{2} - 1} \mathrm{e}^{-\frac{\pi^2 r^2 n^2}{t}} \mathrm{Tr}\left(\mathrm{e}^{-tL} - \mathrm{e}^{-tL_0}\right) dt\\
&= z_1(s) + z_2(s).
\end{aligned} \tag{8}$$

The first term, $z_1(s)$, can be expanded near $s = 0$, and this gives the result stated. In fact, by Proposition 1.1 of [19], $\zeta(s; H, H_0)$ is regular at $s = 0$, and this implies that the pole of $\zeta(s; L, L_0)$ at $s = -\frac{1}{2}$ is simple. To deal with the second term, since (L, L_0) satisfies (B.1)–(B.3), we can write

$$\mathrm{Tr}\left(\mathrm{e}^{-tL} - \mathrm{e}^{-tL_0}\right) = \frac{1}{2\pi i}\int_{\Lambda_{\theta,-a}} \mathrm{e}^{-\lambda t}\mathrm{Tr}\,(R(\lambda, L) - R(\lambda, L_0))d\lambda.$$

Now, it is convenient to change the spectral variable to $k = \lambda^{\frac{1}{2}}$, with the principal value of the square root, i.e. with $0 < \arg k < \pi$. Then,

$$\mathrm{Tr}\left(\mathrm{e}^{-tL} - \mathrm{e}^{-tL_0}\right) = \frac{1}{\pi i}\int_{\gamma} \mathrm{e}^{-k^2 t}\mathrm{Tr}\,(R(k^2, L) - R(k^2, L_0))kdk,$$

where γ is the line $k = -ic$, for some $c > 0$. Writing $k = v\mathrm{e}^{i\theta}$, and $r(\lambda; L, L_0) = \mathrm{Tr}\,(R(\lambda, L) - R(\lambda, L_0))$, a standard computation leads to

$$\mathrm{Tr}\left(\mathrm{e}^{-tL} - \mathrm{e}^{-tL_0}\right) = \int_0^\infty \mathrm{e}^{-v^2 t} e(v; L, L_0)dv, \tag{9}$$

$$\zeta(s; L, L_0) = \int_0^\infty v^{-2s} e(v; L, L_0)dv, \tag{10}$$

where we have introduced the trace of the relative spectral measure

$$e(v; L, L_0) = \lim_{\epsilon\to 0^+} \frac{v}{\pi i}(r((v^2\mathrm{e}^{2i\pi - i\epsilon}; L, L_0) - r(v^2\mathrm{e}^{i\epsilon}; L, L_0)), \tag{11}$$

associated to the pair of operators (L, L_0).

As a result, the second term, $z_2(s)$, of Eq. (8) becomes

$$z_2(s) = \frac{2\sqrt{\pi} r}{\Gamma(s)}\sum_{n=1}^\infty \int_0^\infty t^{s-\frac{1}{2}-1}\mathrm{e}^{-\frac{\pi^2 n^2 r^2}{t}} \int_0^\infty \mathrm{e}^{-v^2 t} e(v; L, L_0)dvdt,$$

and we can do the t integral using for example [16] 3.471.9. We obtain

$$z_2(s) = \frac{4\sqrt{\pi} r}{\Gamma(s)}\sum_{n=1}^\infty \int_0^\infty \left(\frac{\pi n r}{v}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi n r v)dv. \tag{12}$$

Since the Bessel function is analytic in its parameter, regular at $-\frac{1}{2}$, and $K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}$, Eq. (12) gives the formula for the analytic extension of the zeta function $\zeta(s; H, H_0)$ near $s = 0$. We obtain

$$z_2(0) = 0\,, \quad z_2'(0) = -2\int_0^\infty \log\left(1 - \mathrm{e}^{-2\pi r v}\right) e(v; L, L_0)dv, \tag{13}$$

and the integral converges by assumptions (B.2) and (B.3), and Eq. (11) for the trace of the spectral measure. This completes the proof. □

Note that Eq. (7) is the natural generalization of the zeta function technique from the absolute (compact) to the relative (noncompact) case, and that Proposition 3.1 extends the main result of the absolute case (see Eqs. (1) and (2)) to the relative one. Namely, the partition function of the model described above satisfies the equation stated in Corollary 3.2, that follows from Eq. (7) and Proposition 3.1.

COROLLARY 3.2.

$$\log Z = \beta\,(\log 2\ell - 1) \operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s; L, L_0) - \frac{\beta}{2} \operatorname*{Res_0}_{s=-\frac{1}{2}} \zeta(s; L, L_0) - \log \eta\,(\beta; L, L_0)\,.$$

Next, we show that a further feature of the compact case, namely the behaviour for small temperature, extends to the noncompact case.

COROLLARY 3.3. *For large* β

$$\log Z = -E_{\text{vacuum}}\beta + O\left(\beta^{-\epsilon}\right),$$

with some $\epsilon > 0$*, and where* E_{vacuum} *is the vacuum energy*

$$E_{\text{vacuum}} = -\,(\log 2\ell - 1) \operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s; L, L_0) + \frac{1}{2} \operatorname*{Res_0}_{s=-\frac{1}{2}} \zeta(s; L, L_0).$$

Proof: By definition

$$E_{\text{vacuum}} = -\lim_{\beta\to+\infty} \partial_\beta \log Z.$$

The result follows from the equation given in Corollary 3.2, once we show that $\log \eta\,(\beta; L, L_0) = O(\beta^{-\epsilon})$, for large β. Recall the definition of the Dedekind eta function

$$\log \eta(\tau; L, L_0) = \int_0^\infty \log\left(1 - \mathrm{e}^{-\tau v}\right) e(v; L, L_0)\,dv.$$

We split the integral at $v = 1$. Since the above integral converges uniformly for large v, the $\int_1^\infty$ is $O(\mathrm{e}^{-\tau})$ for large τ. For the other integral, we use the expansion of the trace of the relative resolvent $r(\lambda; L, L_0)$ for small λ assumed by condition (B.3), to obtain from the definition of the relative spectral measure, Eq. (11), that $e(v; L, L_0) = O(v^{1+2\beta_j})$, where β_j is the first noninteger exponent in (B.3). □

REMARK 3.4. *Eq.* (10) *defines the relative zeta function when* $\alpha_0 + 1 < \mathrm{Re}\,(s) < \beta_0 + 1$*. In general, the meromorphic continuation of this quantity has a simple pole at* $s = -1/2$ *and the formal definition of the vacuum energy*

$$E_{\text{vacuum}} = \int_0^\infty v e(v; L, L_0)\,dv,$$

is meaningless. Our formula in Corollary 3.3 *provides a rigorous regularization scheme for this formal definition.*

4. Zeta regularized partition function for delta interactions

We analyze in this section two natural applications of the method presented in Section 2. The geometry of our model is given by a scalar field in the three-dimensional flat space interacting with one or two external fields described by delta-like potentials, thus the geometric operator describing our model is formally $L = -\Delta - \mu_0\delta(0) - \mu_1\delta(a)$, where Δ is the Laplace operator in $\mathbb{R}^3$, μ_j are real constants (the strength of the interactions), and a is a fixed point in $\mathbb{R}^3$. Models of this type have been studied by different authors (see [8, 9]). In particular, in the case of a one-point interaction, a rigorous definition has been also obtained by Green's function approach, and formulae for the heat kernel have been given [23, 21, 24]. However, a unified approach valid for finitely many points interaction, was presented by Albeverio et al. in [1], using Fourier transform, a method first used in [3]. We will use this approach.

4.1. One-point interaction in three dimensions

The concrete geometric operator describing our model is $L = -\Delta_\alpha$, where $-\Delta_\alpha$ is defined in Theorem I.1.1.2 of [1] by the resolvent with the following kernel

$$\ker(x, x', (\lambda I + \Delta_\alpha)^{-1}) = -G_k(x - x') - \frac{1}{\alpha - \frac{ik}{4\pi}} G_k^2(x), \tag{14}$$

with $\lambda = k^2 \in \rho(-\Delta_{\alpha,a})$, $\operatorname{Im} k > 0$, α is a real parameter related to the strength μ_0 (we have $\mu_1 = 0$ in the present case) [1] II.1.1.30, and the free Green function is

$$G_k(x) = \frac{e^{ik|x|}}{4\pi|x|}.$$

Note that the case $\alpha = \infty$ corresponds to the negative free Laplace operator $-\Delta = -\Delta_\infty$. By [1] Theorem I.1.1.4 the spectrum of $-\Delta_\alpha$ is purely absolutely continuous $\mathrm{Sp}(-\Delta_\alpha) = [0, \infty)$, if $\alpha \geq 0$, while has one negative eigenvalue, $\lambda = -(4\pi\alpha)^2$, if $\alpha < 0$.

The complete operator describing our model is $H = -\partial_u^2 - \Delta_\alpha$, and, because of the above result on the spectrum of $-\Delta_\alpha$, we assume $\alpha \geq 0$. Proceeding as in Section 3, we introduce the unperturbed operator $H_0 = -\partial_u^2 - \Delta$, and we consider the pair of operators (H, H_0). The partition function of our model is given by Eq. (7), without the part arising from the point spectrum, and we need to study the analytic continuation of the relative zeta function $\zeta(s; H, H_0)$. We first check that the requirements (B.1)–(B.3) of Section 2 are satisfied by the pair of operators $(L, L_0) = (-\Delta_\alpha, -\Delta)$, and this is true as a particular instance of a general class of pairs of operators considered in Section 4.1 of [19] (and references therein for this particular type of potential) or in Section 1.6 of [4]. However, note that we will be able to verify directly conditions (B.1)–(B.3). Next, using Eq. (14), the difference of the kernels of the resolvents of the geometric operators is

$$\ker(x, x', R(\lambda, -\Delta_\alpha)) - \ker(x, x', R(\lambda, -\Delta)) = -\frac{e^{2ik|x|}}{4\pi|x|^2(4\pi\alpha - ik)},$$

and is class trace, since it follows

$$\mathrm{Tr}(R(\lambda, -\Delta_\alpha) - R(\lambda, -\Delta)) = \frac{1}{2ik(4\pi\alpha - ik)}.$$

A further simple computation gives the trace of the relative spectral measure

$$e(v; -\Delta_\alpha, -\Delta) = \frac{4\alpha}{(4\pi\alpha)^2 + v^2}, \tag{15}$$

and the following formulae for the main geometric spectral functions:

$$\mathrm{Tr}\Big(e^{-t(-\Delta_\alpha)} - e^{-t(-\Delta)}\Big) = \frac{e^{(4\pi\alpha)^2 t}}{2}\Big(1 - \Phi(4\pi\alpha\sqrt{t})\Big), \tag{16}$$

$$\zeta(s; -\Delta_\alpha, -\Delta) = \frac{1}{2}\frac{(4\pi\alpha)^{-2s}}{\cos\pi s}, \tag{17}$$

$$\eta(\tau; -\Delta_\alpha, -\Delta) = \log\Gamma(2\alpha\tau) + \frac{1}{2}\log 2\alpha\tau - 2\alpha\tau(\log 2\alpha\tau - 1) - \frac{1}{2}\log 2\pi. \tag{18}$$

The formula in Eq. (16) follows from the definition, since using Eq. (9) and (15),

$$\mathrm{Tr}\Big(e^{-(-\Delta_\alpha)t} - e^{-(-\Delta)t}\Big) = 4\alpha\int_0^\infty \frac{e^{-v^2 t}}{(4\pi\alpha)^2 + v^2}dv,$$

and next we can apply [16] 3.363.2 (the probability integral function is defined accordingly to [16] 8.250—recall that α is nonnegative). Note that the integral representation for the trace of the difference of the heat operators given in Eq. (16) satisfies the conditions (1.1)–(1.3) of [19] for the pair of operators $(-\Delta_\alpha, -\Delta)$. The formula in Eq. (17) follows using Eqs. (10), and (15). The same result also follows using the formula in Eq. (9), and the previous result for the trace of the difference of the heat semigroups, under the condition that $\mathrm{Re}(s) > 0$, and using [16] 6.286.1. The formula in Eq. (18) follows by definition and [16] 4.319.1.

Expanding the formula in Eq. (17) near $s = 0$, we obtain

$$\operatorname*{Res}_{1}_{s=-\frac{1}{2}} \zeta(s; -\Delta_\alpha, -\Delta) = 2\alpha, \qquad \operatorname*{Res}_{0}_{s=-\frac{1}{2}} \zeta(s; -\Delta_\alpha, -\Delta) = -4\alpha\log 4\pi\alpha.$$

Using these results in Corollary 3.2, we obtain the explicit formula for the partition function

$$\log Z = 2(\log 4\pi\alpha\ell - 1)\alpha\beta - \log\eta(\beta; -\Delta_\alpha, -\Delta)$$
$$= 2(\log 4\pi\alpha\ell - 1)\alpha\beta - \log\Gamma(2\alpha\beta) - \frac{1}{2}\log 2\alpha\beta + 2\alpha\beta(\log 2\alpha\beta - 1)$$
$$+ \frac{1}{2}\log 2\pi.$$

Note that, using the classical expansion for the Gamma function, this result is consistent with Corollary 3.3. The regularized vacuum energy follows immediately from the above expression.

4.2. Two-point interactions in three dimensions

The concrete geometric operator describing our model is $L = -\Delta_{\alpha,a}$, where $-\Delta_{\alpha,a}$ is defined in Theorem II.1.1.1 of [1], by the resolvent with the following integral kernel

$$\ker(x, x', (\lambda I + \Delta_{\alpha,a})^{-1}) = -G_k(x - x') - \sum_{j,l=0}^{1} \Gamma^{-1}_{\alpha,a}(k)_{j,l} G_k(x - a_j) G_k(x' - a_l),$$

with $\lambda = k^2 \in \rho(-\Delta_{\alpha,a})$, $\mathrm{Im}\, k > 0$, and where the α_j are real parameters (see [1] II.(1.1.25)), and

$$\Gamma_{\alpha,a}(k) = \begin{pmatrix} \alpha_0 - \frac{ik}{4\pi} & -G_k(a) \\ -G_k(a) & \alpha_1 - \frac{ik}{4\pi} \end{pmatrix}.$$

Note that the case $\alpha_j = \infty$ corresponds to the negative free Laplace operator $-\Delta = -\Delta_{\infty,a}$, and $\alpha_1 = \infty$ to the case considered in Section 4.1. By [1] Theorem I.1.1.4 the spectrum of $-\Delta_{\alpha,a}$ is purely absolutely continuous $\mathrm{Sp}\,(-\Delta_{\alpha,a}) = [0, \infty)$, plus at most two negative eigenvalues. The eigenvalues are present if $\det \Gamma_{\alpha,a}(k) = 0$ for $\mathrm{Im}\, k > 0$. An explicit analysis (see also the end of Section II.1.1 of [1]) shows that the condition necessary in order to have a purely continuous spectrum is $4\pi^2\alpha_0\alpha_1 a^2 \geq 1$. We will proceed assuming this condition.

The unperturbed geometric operator is $-\Delta$, and the fact that the pair $(-\Delta_{\alpha,a}, -\Delta)$ satisfies conditions (B.1)–(B.3) follows as in Section 4.1. The difference of the resolvents has trace

$$\mathrm{Tr}(R(k^2, -\Delta_{\alpha,a}) - R(k^2, -\Delta)) = \frac{a^2}{ika} \frac{2\pi(\alpha_0 + \alpha_1)a - ika + \mathrm{e}^{2ika}}{(4\pi\alpha_0 a - ika)(4\pi\alpha_1 a - ika) - \mathrm{e}^{2ika}}.$$

This allows to write a formula for the trace of the relative spectral measure.

Using the definition in Eq. (11), we obtain

$$e(v; -\Delta_{\alpha,a}, -\Delta)$$
$$= \frac{a}{\pi}\left(\frac{2\pi(\alpha_0+\alpha_1)a - iav + \mathrm{e}^{2iav}}{a^2(4\pi\alpha_0 - iv)(4\pi\alpha_1 - iv) - \mathrm{e}^{2iav}} + \frac{2\pi(\alpha_0+\alpha_1)a + iav + \mathrm{e}^{-2iav}}{a^2(4\pi\alpha_0 a + iv)(4\pi\alpha_1 a + iv) - \mathrm{e}^{-2iav}}\right).$$

Note that in the limit case $\alpha_1 \to \infty$ the relative spectral measure $e(v; -\Delta_{\alpha,a}, -\Delta)$ reduces smoothly to the one $e(v; -\Delta_{\alpha_0}, -\Delta)$, considered in Section 4.1.

The formula for the trace of the relative spectral measure allows to compute all the quantities appearing in Proposition 3.1, and therefore to obtain an explicit result for the partition function using Corollary 3.2. For, note that the function $e(v; -\Delta_{\alpha,a}, -\Delta)$ is a smooth function, as it is the quotient of powers and trigonometric functions. To compute the values of the residue and of the finite part of the zeta function $\zeta(s; -\Delta_{\alpha,a}, -\Delta)$ at $s = -\frac{1}{2}$, we use the expansions of $e(v; -\Delta_{\alpha,a}, -\Delta)$ for small and large v. For small v,

$$e(v; -\Delta_{\alpha,a}, -\Delta) = \frac{a}{\pi}\frac{4\pi(\alpha_0+\alpha_1)a + 2}{16\pi^2\alpha_0\alpha_1 a^2 - 1} + O(v),$$

while for large v,

$$e(v; -\Delta_{\alpha,a}, -\Delta) = \frac{-2\cos(2av) + 4\pi(\alpha_0+\alpha_1)a}{\pi a v^2} + O(v^{-3}).$$

Using the integral representation for the zeta function given in Eq. (10), we can split the integral at $x = 1$,

$$\zeta(s; -\Delta_{\alpha,a}, -\Delta) = \zeta_0(s; a) + \zeta_\infty(s; a)$$
$$= \int_0^1 v^{-2s} e(v; -\Delta_{\alpha,a}, -\Delta)dv + \int_1^\infty v^{-2s} e(v; -\Delta_{\alpha,a}, -\Delta)dv.$$

Making use of the above expansion of the function $e(v; -\Delta_{\alpha,a}, -\Delta)$ for small v, we see that $\zeta_0(s; a)$ is regular near $s = -\frac{1}{2}$, and its value is

$$\zeta_0\left(-\frac{1}{2}; a\right) = \int_0^1 v e(v; -\Delta_{\alpha,a}, -\Delta)dv.$$

Next, $\zeta_\infty(s; a)$ is not regular near $s = -\frac{1}{2}$. However, using the asymptotic expansion given above

$$\zeta_\infty(s; a) = z_A(s; a) + z_B(s; a)$$
$$= \int_1^\infty v^{-2s}\left(e(v; -\Delta_{\alpha,a}, -\Delta) + \frac{2\cos(2av) - 4\pi(\alpha_0+\alpha_1)a}{\pi a v^2}\right)dv$$
$$- \int_1^\infty v^{-2s}\frac{2\cos(2av) - 4\pi(\alpha_0+\alpha_1)a}{\pi a v^2}dv.$$

Using the expansion of the function $e(v; -\Delta_{\alpha,a}, -\Delta)$ for large v, we see that that $z_A(s; a)$ is regular at $s = -\frac{1}{2}$ and its value is

$$z_A\Big(-\frac{1}{2}; a\Big) = \int_1^\infty v\left(e(v; -\Delta_{\alpha,a}, -\Delta) + \frac{2\cos(2av) - 4\pi(\alpha_0 + \alpha_1)a}{\pi a v^2}\right) dv.$$

The last term $z_B(s; a)$ is not regular at $s = -\frac{1}{2}$. However, we can deal with this term exactly:

$$\begin{aligned} z_B(s; a) &= -\int_1^\infty v^{-2s} \frac{2\cos(2av) - 4\pi(\alpha_0 + \alpha_1)a}{\pi a v^2} dv \\ &= -\frac{2}{\pi a}\int_1^\infty v^{-2s}\frac{\cos(2av)}{v^2} dv + 4(\alpha_0 + \alpha_1)\frac{1}{2s+1}, \end{aligned}$$

and therefore

$$\operatorname*{Res_1}_{s=-\frac{1}{2}} z_B(s; a) = 2(\alpha_0 + \alpha_1), \qquad \operatorname*{Res_0}_{s=-\frac{1}{2}} z_B(s; a) = \frac{2\mathrm{ci}(2a)}{\pi a}.$$

As a consequence, the partition function of our model in the range $4\pi^2\alpha_0\alpha_1 a^2 \geq 1$ is

$$\begin{aligned} \log Z = &-\frac{\beta}{2}\int_1^\infty v\left(e(v; -\Delta_{\alpha,a}, -\Delta) + \frac{2\cos(2av) - 4\pi(\alpha_0 + \alpha_1)a}{\pi a v^2}\right) dv \\ &+ 2(\alpha_0 + \alpha_1)(\log 2\ell - 1)\beta - \frac{\mathrm{ci}(2a)}{\pi a}\beta - \frac{\beta}{2}\int_0^1 v e(v; -\Delta_{\alpha,a}, -\Delta) dv \\ &- \int_0^\infty \log\big(1 - \mathrm{e}^{-v\beta}\big) e(v; -\Delta_{\alpha,a}, -\Delta) dv. \end{aligned}$$

Note that the zeta-function regularization used implies the presence of the renormalization scale ℓ in the final expression for the canonical partition function. In the present case, the dependence drops out as soon as one is interested in evaluation of physical quantities, as for example the Casimir force, defined as the derivative of the regularized vacuum energy with respect to external parameter a.

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